MATH 286 – WINTER 2008
HOMEWORK SET 4

What you need to know:

- Homogeneous ODEs with constant coefficients
- A bit on complex numbers (in particular, Euler's formula)
- Repeated roots and reduction of order
- Non-homogeneous equations and the Method of Undetermined Coefficients
- Sections 3.1, 3.4, 3.5, 3.6

What you shouldn't forget:

- Existence and Uniqueness Theorem
- Linear, separable, and exact ODEs
- Linear dependence and Wronskians

Ex # 1. i) Consider the second-order ODE

$$y'' - (1+\kappa)y' + \kappa y = 0, \qquad \kappa > 1.$$

Determine the solutions ϕ_κ and ψ_κ that satisfy the initial conditions

$$\begin{cases} \phi_\kappa(0) = 1, \\ \phi_\kappa'(0) = 0, \end{cases} \qquad \text{and} \qquad \begin{cases} \psi_\kappa(0) = 0, \\ \psi_\kappa'(0) = 1 \end{cases}$$

and show that the two solutions are linearly independent.

ii) Let ϕ_κ and ψ_κ be the solutions above. Determine the limits

$$\phi(x) := \lim_{\kappa \to 1^+} \phi_\kappa(x) \qquad \text{and} \qquad \psi(x) := \lim_{\kappa \to 1^+} \psi_\kappa(x).$$

Are ϕ and ψ linearly independent solutions of the limit ODE

$$y'' - 2y' + y = 0?$$

Ex # 2. Consider the second-order ODE

$$y'' + y = 0. \qquad (\spadesuit)$$

i) Show that $y_1(t) = \cos t$ and $y_2(t) = \sin t$ are linearly independent solutions of ($\spadesuit$), and find the general solution.

ii) Using the previous result, find (formally) the particular solution satisfying the initial conditions $y(0) = 1$ and $y'(0) = i$. Show (formally) that $y = e^{it}$ is also a solution of the same initial value problem. What can you conclude?

Ex # 3. i) Consider the ODE

$$y'' + 2\kappa y' + \kappa^2 y = 0.$$

Find a solution ϕ_1 by solving the characteristic equation.

ii) Show that the Wronskian of any two solutions of the given equation is

$$W(y_1, y_2)(t) = ce^{-2\kappa t},$$

where c is a real constant.

iii) Using the explicit formula for ϕ_1 found before, find a solution ϕ_2 which is linearly independent from ϕ_1.

Ex # 4. Consider the ODE

$$xy'' - y' + 4x^3 y = 0, \qquad x > 0.$$

i) Show that $y_1(x) = \sin x^2$ is a solution.

ii) Using the method of reduction of order find a solution y_2 which is linearly independent from y_1.

Ex # 5. Determine the general solution of

$$y'' + \lambda^2 y = \sum_{m=1}^{N} a_m \sin m\pi t,$$

where $\lambda > 0$ and $\lambda \neq m\pi$ for $m = 1, \dots, N$.

Ex # 6. Determine the values of κ, if any, for which all solutions of

$$y'' - (2\kappa - 1)y' + \kappa(\kappa - 1)y = 0$$

tend to zero as $t \to +\infty$; and the values, if any, for which all nonzero solutions become unbounded as $t \to +\infty$.

Ex # 7. Find the solution of

$$9y'' + 36y' + 52y = 0$$

satisfying $y(0) = 1$, $y'(0) = 0$.

Ex # 8. i) If the solutions of the equation

$$x^2 + 2bx + c = 0$$

are the complex numbers $p \pm iq$, $(p, q \in \mathbb{R})$, find expressions for b, c in terms of p and q.

ii) $(\star)$ Compute $\sqrt{i}$.