Constructing Quot Schemes

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1 Introduction

In this expository article we explain the construction of a very important class of geometric objects called Quot schemes. Ubiquitous in algebraic geometry today, Quot schemes were first constructed by Grothendieck as schemes representing the functor \( \text{Quot}_{E/X/S} : \text{Sch}_S \rightarrow \text{Set} \), where \( S \) is a noetherian scheme, \( X \rightarrow S \) is a morphism of finite type, \( E \) is a coherent sheaf on \( X \), \( \text{Sch}_S \) is the category of locally noetherian \( S \)-schemes and \( \text{Set} \) is the category of sets. Define a family of quotients of \( E \) parametrized by \( T \) for \( T \in \text{Sch}_S \) as a pair \((F, q)\) such that

- \( F \) is a coherent sheaf on \( X_T := X \times_S T \), flat over \( T \) and supported on a subscheme proper over \( T \).
- \( q \) is a surjective \( O_{X_T} \)-linear morphism of sheaves \( q : E_T \rightarrow F \) where \( E_T \) is the pullback of \( E \) along \( X_T \rightarrow X \).

Two families \((F, q)\) and \((F', q')\) are said to be equivalent if \( \ker(q) = \ker(q') \). Since properness and flatness are preserved under base-change and tensor products are right-exact, the pullback of an equivalence class of families under a morphism of \( S \)-schemes is well-defined. We can thus define the functor \( \text{Quot}_{E/X/S} \) as the contravariant functor taking \( T \in \text{Sch}_S \) to the set of equivalence classes of families of quotients of \( E \) parametrized by \( T \).

Some examples of \( \text{Quot}^{\Phi,L}_{E/X/S} \) include various functors of points for schemes such as projective space over any noetherian base, the Grassmanian as a scheme over \( \mathbb{Z} \), the Hilbert scheme of points and others. The Grassmanian scheme in particular turns out to be of great important in the construction of the Quot-scheme.

Grothendieck’s achievement in [2] was to prove that the functor \( \text{Quot}_{E/X/S} \) is representable when \( E, X \) and \( S \) are suitably constrained. This exposition will follow that of Nitsure in [3] which we recommend for its clarity and attention to detail.

2 Proof of Main Theorem

We explain the proof of the following theorem of Grothendieck:

**Theorem 2.1** (Grothendieck, [2]). Let \( S \) be a noetherian scheme, \( \pi : X \rightarrow S \) a projective morphism, and \( E \) a coherent \( O_X \)-module. The functor \( \text{Quot}_{E/X/S} \) is represented by a projective \( S \)-scheme \( \text{Quot}_{E/X/S} \).

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1 I pledge that this article represents my own work in accordance with University regulations and the Honour Code

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2 My thanks are due to Prof. Nicholas M. Katz for suggesting the main reference and to Kevin Wilson for answering several of my questions about base-change and semi-continuity
Let \( \mathcal{F} \) be a coherent sheaf on \( X \) whose support is proper over \( S \) and \( L \) be a relatively very-ample line bundle on \( X \). For \( s \in S \), recall that the Hilbert polynomial of \( \mathcal{F}_s := \mathcal{F}|_X \) calculated relative to \( L_s := L|_X \) where \( X_s := \pi^{-1}(s) \) is given by

\[
\Phi_s(m) := \sum_{i=0}^{n} (-1)^i \dim_{k_s} H^i(X_s, \mathcal{F}_s \otimes L_s^m).
\]

The dimensions of the cohomology groups are finite because of the coherence and properness conditions. We know from [3.2] that \( s \mapsto \Phi_s \) is locally constant on \( S \). We can thus decompose the functor \( \text{Quot}_{E/X/S} \) as the coproduct of the functors \( \text{Quot}_{E/X/S}^{\Phi,L} \) over all \( \Phi \in \mathbb{Q}[x] \), where \( \text{Quot}_{E/X/S}^{\Phi,L} \) associates to any \( T \), the set of all equivalence classes of families of quotients parametrized by \( T \) such that at each \( t \in T \), the Hilbert polynomial of \( \mathcal{F}_t \) calculated using the pull-back of \( L \) is \( \Phi \). Thus if \( \text{Quot}_{E/X/S}^{\Phi,L} \) is representable by \( \text{Quot}_{E/X/S}^{\Phi,L} \) for all \( \Phi \in \mathbb{Q}[x] \) we have that

\[
\text{Quot}_{E/X/S} = \coprod_{\Phi \in \mathbb{Q}[x]} \text{Quot}_{E/X/S}^{\Phi,L}.
\]

(See Part 1, [3] for abstract generalities about representable functors.)

**Remark 2.2.** We note that it is crucial that \( L \) be very ample, else the result is no longer true!

We thus have to show that \( \text{Quot}_{E/X/S}^{\Phi,L} \) is representable for all \( \Phi \in \mathbb{Q}[x] \). We do this by first proving the following simpler theorem of Altman and Kleiman and then observe the modifications necessary to prove Grothendieck’s theorem.

**Theorem 2.3** (Altman-Kleiman, [1]). Let \( S \) be a noetherian scheme, \( X \) a closed subscheme of \( \mathbb{P}(V) \) for some vector bundle \( V \) on \( S \), \( \pi \) the morphism \( X \to S \), \( L := \mathcal{O}_X(V)(1) \), \( E \) a coherent quotient sheaf of \( \pi^*(W)(\nu) \) where \( W \) is a vector bundle on \( S \) and \( \nu \in \mathbb{Z} \), and \( \Phi \in \mathbb{Q}[x] \). Then the functor \( \text{Quot}_{E/X/S}^{\Phi,L} \) is represented by a scheme \( \text{Quot}_{E/X/S}^{\Phi,L} \) which can be embedded over \( S \) as a closed subscheme of \( \mathbb{P}(F) \) for some vector bundle \( F \) on \( S \).

**Remark 2.4.** The hypothesis here is that of strong quasi-projectivity where \( X \) admits a closed embedding into \( \mathbb{P}(V) \) for some vector bundle \( V \) on \( S \). This is distinguished from the hypothesis of Grothendieck where \( \pi : X \to S \) is said to be projective if it is proper and there exists a relatively very ample line bundle \( L \) on \( X \) over \( S \), or equivalently there exists coherent sheaf \( E \) on \( S \) such that \( X \) admits a closed embedding into \( \mathbb{P}(E) \).

The following lemma reduces the proof of Theorem 2.3 to the case where \( X = \mathbb{P}(V) \) and \( E = \pi^*(W) \) where \( V \) and \( W \) are vector bundles on \( S \).

**Lemma 2.5.** For \( \nu \in \mathbb{Z} \), tensoring by \( L' \) gives an isomorphism of functors from \( \text{Quot}_{E/X/S}^{\Phi,L} \) to \( \text{Quot}_{E(\nu)/X/S}^{\Phi,L} \) where \( \Psi(x) = \Phi(x + \nu) \). Furthermore, if \( \varphi : E \to G \) is a surjective morphism of coherent sheaves on \( X \), the corresponding natural transformation \( \tilde{\varphi} : \text{Quot}_{E/X/S}^{\Phi,L} \to \text{Quot}_{G/X/S}^{\Phi,L} \) is a closed embedding.

**Proof.** The first part of the lemma is obvious. Let \( T' \) be the vanishing scheme of \( \ker(\varphi) \to E \xrightarrow{\varphi} G \). Since \( \ker(\varphi) \) and \( \mathcal{F} \) are coherent and \( \mathcal{F} \) is flat over \( T \) (See Remark 3.9), if \( U \) is a locally noetherian \( S \)-scheme, \( X_U := X \times_S U \) and \( f : U \to T \), the pull-back of \( q : E \to F \) along \( p_2 : X_U \to X \) denoted by \( q_U \) factors via the pull-back of \( \varphi \) along \( p_2 \) if and only if \( f \) factors through \( T' \hookrightarrow T \). This implies the second part of the lemma.

\[ \square \]
Proof of Theorem 2.3. Thus if $\text{Quot}^{F,L}_{\pi^*W/\pi^*P(V)/S}$ exists, we can take $\text{Quot}^{F,L}_{E/X/S}$ to be a closed subscheme of it. We thus assume henceforth that $X = \mathbb{P}(V)$ and $E = \pi^*W$ where $V$ and $W$ are vector bundles on $S$. Assume further that $L = \mathcal{O}_{\mathbb{P}(V)}(1)$. Fixing $s \in S$ having residue field $k := k(s)$, we have that $\mathbb{P}(V)_s \cong \mathbb{P}_k^n$ where $n = \text{rank}(V) - 1$ and $E_s \cong \oplus^p \mathcal{O}_{E_s}$ where $p = \text{rank}(W)$. It follows from Theorem 3.5 that fixing $\Phi \in F$, the sheaf $E_s$ has Hilbert polynomial $\Phi$, the sheaf $G$ and its associated kernel sheaf $\mathcal{G}$ are both $m$-regular. Lemma 3.3 implies that for $r \geq m$, the cohomology groups $H^i(X_s, E_s(r))$, $H^i(X_s, F(r))$ and $H^i(X_s, \mathcal{G}(r))$ are trivial for $i \geq 1$ and the corresponding zeroeth cohomologies are generated by the global sections of the sheaves $E_s(r), F(r)$ and $\mathcal{G}(r)$ respectively.

If $T$ is an $S$-scheme and $q : E_T \to F$ is a $T$-flat coherent quotient with Hilbert polynomial $\Phi$, and $\ker(q) := G$, we pick $m$ as above for $F' := F_t$ and $G' := G_t$ where $t \in T$. Since the $m$ produced by Theorem 3.5 is independent of the field $k$, it is independent of the choice of $t \in T$. It follows by Theorem 3.10 that the sheaves $\pi_T^*E_T(r), \pi_T^*F(r), \pi_T^*G(r)$ are locally free of ranks determined by $n, p, r, \Phi$ and the map $\pi_T^*E_T \to \pi_T^*F$ (resp. for $F, G$) is surjective. Here, we use the fact that properness and flatness are preserved under base-change. This induces the following commutative diagram where rows are exact and vertical maps are surjective:

$$
\begin{array}{cccc}
0 & \to & \pi_T^*\pi_T^*G(r) & \to & \pi_T^*\pi_T^*E_T(r) & \to & \pi_T^*\pi_T^*F(r) & \to & 0 \\
0 & \to & G(r) & \to & E_T(r) & \to & F(r) & \to & 0 \\
\end{array}
$$

Fix $r \geq m$ and note that the rank of $\pi_T^*F(r)$ is $\Phi(r)$ while $\pi_T^*E(r) = W \otimes \mathcal{O}_S \text{Sym}^rV$. Therefore, the map $\pi_T^*E_T(r) \to \pi_T^*F(r)$ defines an element of $\text{Grass}(W \otimes \mathcal{O}_S \text{Sym}^rV, \Phi(r))(T)$ which gives us a natural transformation

$$
\alpha : \text{Quot}^{F,L}_{E/X/S} \to \text{Grass}(W \otimes \mathcal{O}_S \text{Sym}^rV, \Phi(r)),
$$

which associates to $q : E_T \to F$ the quotient $\pi_T^*(q(r)) : \pi_T^*E_T(r) \to \pi_T^*F(r)$. We prove that $\alpha$ is injective by recovering $q$ from $\pi_T^*(q(r))$. Let $G = \text{Grass}(W \otimes \mathcal{O}_S \text{Sym}^rV, \Phi(r)), p_G : G \to S$ and $u : p_G^*E \to U$ the universal quotient on $G$ with kernel $G_0$. We find that we can recover $\pi_T^*\pi_T^*G(r) \to \pi_T^*\pi_T^*G$ by pulling $v : G_0 \to p_G^*E$ back. Let $h$ be the composite map $\pi_T^*\pi_T^* \to E_T(r)$. From the diagram, the following is right exact:

$$
\pi_T^*\pi_T^*G(r) \to E_T(r) \to F \to 0
$$

and the map from $E_T(r) \to F$ is just the map of $\pi_T^*\pi_T^*G(r) \to \pi_T^*F(r)$. Finally twisting by $-r$, we recover $q$ — thus the natural transformation $\alpha$ is injective.

We now show that $\alpha$ is relatively representable; equivalently, that $\text{Quot}^{F,L}_{E/X/S}$ is represented by a locally closed subscheme of $\text{Grass}(W \otimes \mathcal{O}_S \text{Sym}^rV, \Phi(r))$. The key input in constructing this subscheme is the flattening stratification of Section 3.4. Given $T \in \text{Sch}_S$ and surjective morphism $f : W_T \otimes_{\mathcal{O}_T} \text{Sym}^rV_T \to \mathcal{F}$ where $\mathcal{F}$ is a locally free $\mathcal{O}_T$-module of rank $\Phi(r)$, there exists locally closed subscheme $T'$ with the following universal property:

**Property:** If $Y \in \text{Sch}_S$ and $\phi : Y \to T$ is an $S$-morphism, $f_Y$ is the pull-back of $f$ and $K_Y := \ker(f_Y) = \phi^*\ker(f)$. Denote the projection map $\pi_Y : X_Y \to Y$, let $h : \pi_Y^*K_Y \to E_Y$ be the composite map

$$
\pi_Y^*K_Y \to \pi_Y^*(W \otimes \mathcal{O}_S \text{Sym}^rV) = \pi_Y^*\pi_Y^*E_Y \to E_Y
$$
and let \( \mathcal{F} := \text{coker}(h) \) be equipped with the map \( q : E_Y \to \mathcal{F} \). Then \( \mathcal{F} \) is flat over \( Y \) with its Hilbert polynomials on all its fibers equal to \( \Phi \) if and only if \( \phi : T \to T \) factors through \( Y' \hookrightarrow Y \).

Theorem 3.12 tells us that for \( T = \text{Grass}(W \otimes_{S} \text{Sym}^{r}V, \Phi(r)) \) with universal quotient \( u : p_{G}^{*}E \to U \), the subscheme \( T' \) in question is the stratum \( S_{\Phi} \). Thus we see that \( \text{Quot}_{E/X/S}^{\Phi,L} \) is proper over \( S \) and therefore our locally closed embedding becomes an actual embedding – thus is \( \text{Quot}_{E/X/S}^{\Phi,L} \) a quasi-projective \( S \)-scheme.

Finally we observe that in order to generalize the above proof to the setting of Theorem 2.1, we need to make the following two main changes:

- Since \( S \) is noetherian, we can find sufficiently large \( m \) so that given any \( k \)-valued point \( s \in S \) and coherent quotient \( q : E_{s} \to F_{s} \) on \( X_{s} \) with Hilbert polynomial \( \Phi \), the sheaves \( E_{s}(r), F_{s}(r) \) and \( G(r) := \ker(q)(r) \) are generated by global sections and have no higher cohomology for \( r \geq m \).

- From this \( m \), we get as before an injective map into the Grassmanian functor \( \text{Grass}(\pi s E(r), \Phi(r)) \) – here the sheaf \( \pi s E(r) \) is coherent but not necessarily the quotient of a vector bundle on \( S \). This leads to the change in conditions on projectivity.

3 Background for proof

In the proof of Theorem 2.1, several techniques and background results turn out to play crucial roles. In this section, we elaborate on some of these, giving proofs or suitable references when necessary. Specifically, we discuss the construction of the Grassmanian as a scheme, the important notion of \( m \)-regularity of Castelnuovo-Mumford, the existence of flattening stratifications by Hilbert polynomials, flat base-change and the semi-continuity theorem. We give full proofs for the first two topics, a sketch of the proof for the third, and a reference for the fourth.

3.1 The Grassmanian as a \( \mathbb{Z} \)-scheme

In this section, we describe the construction of the Grassmanian, denoted \( \text{Grass}(r, d) \) where \( r \) and \( d \) are integer parameters and \( r < d \). This scheme plays a vital role in showing that \( \text{Quot}_{E/X/S}^{\Phi,L} \) is representable – we embed the Quot functor into the hitherto-undefined Grassmanian functor which is known to be representable by \( \text{Grass}(r, d) \).

We prove that the Grassmanian is a proper scheme over \( \mathbb{Z} \) and construct a natural locally free sheaf \( \mathcal{U} \) called the universal quotient on it. The determinant of the universal quotient is then used to produce a projective embedding. Furthermore, we show that the pair \((\text{Grass}(r, d), \mathcal{U})\) represents the functor \( \mathfrak{Grass} : \text{Sch}_{S} \to \text{Set} \) which associates to an \( S \)-scheme \( T \), the set of all equivalence classes of quotients \((F, q)\) where \( q : \oplus^{r}O_{T} \to F \) and \( F \) is locally free on \( T \) of rank \( d \).

**Construction:** Let \( M \) be a \( d \times r \) matrix where \( 1 \leq d \leq r \) are integers. If \( I, J \subset \{1, \ldots, r\} \) have cardinality \( d \), the \( I \)-th minor \( M_{I} \) of \( M \) denotes the \( d \times d \) minor of \( M \) whose columns are indexed
by \( I \). We consider the \( d \times r \) matrix \( X^I \) whose \( I \)-th minor \( X^I_I \) is the \( d \times d \) identity matrix, while the other entries are free variables \( X^I_{p,q} \) over \( \mathbb{Z} \) which generate a polynomial ring denoted by \( \mathbb{Z}[X^I] \). Let \( U^I \) be defined as \( \text{Spec} \mathbb{Z}[X^I] \) – these are the affine pieces which we now glue together.

Define the maps gluing the affine pieces together as follows: Consider the open subscheme \( U^I_r := \text{Spec} \mathbb{Z}[X^I_r, 1/P^I_r] \subset U^I \) as the scheme defined by the invertibility of \( P^I_r := \det(X^I_r) \). Define the morphism of rings \( \theta_{I,J} : \mathbb{Z}[X^I_r, 1/P^I_r] \to \mathbb{Z}[X^J_r, 1/P^J_r] \) to be the map such that \( \theta_{I,J}(X^I_r) = (X^I_J)^{-1}X^I_{r,d} \) – note that this defines the images of \( x^I_{p,q} \) and hence fixes the map \( \theta_{I,J} \). This map on rings induces \( \phi_{I,J} \), a morphism of schemes from \( U^I_r \to U^J_r \). It’s easy to see that \( \phi_{I,K} = \phi_{I,J} \phi_{J,K} \) where \( I, J, K \) have cardinality \( d \) and so the schemes \( U^I \) can be glued along the morphisms \( \phi_{I,J} \) to form a scheme of finite type over \( \mathbb{Z} \). The scheme thus obtained is the Grassmanian, denoted by \( \text{Grass}(r,d) \). Since \( U^I \) is clearly isomorphic to \( \mathbb{A}^d_{\mathbb{Z}}^{(r,d)} \), we see that the Grassmanian is smooth over \( \mathbb{Z} \) and has relative dimension \( d(r-d) \).

**Proposition 3.1.** \( \text{Grass}(r,d) \) is proper as a scheme over \( \text{Spec} \mathbb{Z} \).

**Proof.** We first show that the Grassmanian is separated over \( \mathbb{Z} \). Cover \( \text{Grass}(r,d) \times_{\mathbb{Z}} \text{Grass}(r,d) \) with \( G_{I,J} := U^I \times_{\mathbb{Z}} U^J \) and consider the intersection of the diagonal embedding \( \Delta(\text{Grass}(r,d)) \subset \text{Grass}(r,d) \times_{\mathbb{Z}} \text{Grass}(r,d) \) with \( G_{I,J} \). Clearly this is defined by the equation \( X^I_r X^I_{r,d} - X^J_r = 0 \); thus the diagonal embedding is closed and the Grassmanian is separated over \( \mathbb{Z} \).

To pass from separatedness to properness, we show that the map from \( \text{Grass}(r,d) \) to \( \mathbb{Z} \) satisfies the valuative criterion of properness. We use the version for discrete valuation rings which is a stronger statement than the theorem in Hartshorne’s book, but is discussed as an exercise. Let \( R \) be a discrete valuation ring having \( \mathbb{Z} \) as its field of fractions. Let \( \varphi : \text{Spec} \mathbb{K} \to \text{Grass}(r,d) \) be a morphism, induced by the map of rings \( f : \mathbb{Z}[X^I] \to \mathbb{K} \) for some \( I \). Choose \( J \) such that \( \nu(f(P^I_r)) \) is minimized – where \( \nu \) is the discrete valuation corresponding to \( I \). Choosing \( J = I \) would give us \( 0 \), so clearly \( \nu(f(P^I_r)) \leq 0 \) i.e. \( f(X^I_r) \in \text{GL}_d(\mathbb{K}) \). Now define a map \( g : \mathbb{Z}[X^J] \to \mathbb{K} \) as \( g(X^J) := f((X^J_I)^{-1}X^I_{r,d}) \) noting once more that this completely fixes \( g \). Clearly \( g \) induces the map \( \varphi : \text{Spec} \mathbb{K} \to \text{Grass}(r,d) \) and so the \( g \)-values of \( P^I_r \) for any \( L \) have positive valuation at \( \nu \). Since \( X^J_r \) is the identity, we conclude that in fact \( \nu(g(x^I_{p,q})) \geq 0 \) and so \( g \) factors through \( R \). Since \( \text{Grass}(r,d) \) is already known to be separated, this map must be unique by the valuative criterion of separatedness and thus by the valuative criterion of properness, the Grassmanian is proper as a scheme over \( \mathbb{Z} \).

Define the universal quotient, a locally free sheaf \( U \) on \( \text{Grass}(r,d) \) as follows: On each \( U^I \), we define \( u^I : \oplus^d \mathcal{O}_{\text{Grass}(r,d)} \to \oplus^d \mathcal{O}_{U^I} \) as the map given by matrix \( X^I \). We can glue the trivial vector bundles \( \oplus^d \mathcal{O}_{U^I} \) using the gluing data \( g_{I,J} := (X^I_J)^{-1} \) which is compatible with \( \phi_{I,J} \) for gluing the schemes and in fact with the \( u^I \). We thus get a locally free sheaf \( U \) of rank \( d \) on \( \text{Grass}(r,d) \) and a surjective homomorphism \( u : \oplus^d \mathcal{O}_{\text{Grass}(r,d)} \to U \).

We use the line bundle \( \mathcal{L} := \det(U) \) to embed \( \text{Grass}(r,d) \) into \( \mathbb{P}^{d-1}_{\mathbb{Z}} \). For each \( I \), we define a global section \( \sigma_I \) of \( \mathcal{L} \) by gluing the sections \( P^I_r \) on \( U^J \), which works since \( \mathcal{L}|_{U^J} = \mathcal{O}_{U^J} \). Let \( \mathfrak{d} \) denote the linear system spanned by the sections \( \sigma_I \). If \( Q \) is a point of \( \text{Grass}(r,d) \) such that \( Q \in U^I \), we see that the section \( \sigma_I \notin \mathfrak{m}_Q \mathcal{L} \), making the linear system base-point free. It can be checked that \( \mathfrak{d} \) separates points and tangent vectors implying that \( \mathcal{L} \) is very ample over \( \mathbb{Z} \) and induces the embedding of \( \text{Grass}(r,d) \) into \( \mathbb{P}^{d-1}_{\mathbb{Z}} \).

We end this section by making two observations without proof:

- The Grassmanian along with its universal quotient represent the contravariant Grassmanian functor \( \text{Grass}(r,d) := \text{Quot}^d_{\oplus^d \mathcal{O}_{\mathbb{Z}}/\mathcal{O}_{\mathbb{Z}}} \) which associates to any \( T \in \text{Sch}_S \) the set of all equiv-
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The first term vanishes by the inductive hypothesis for \( r \). We use this fact in the proof of Theorem 2.3 when we construct the embedding \( \alpha \) of a general Quot-scheme into the Grassmanian scheme.

- All the results about Grassmanians have analogues for the case of vector bundles that we use freely without proof. For more details, see [2].

### 3.2 \( m \)-Regularity

In this section, we discuss the notion of \( m \)-regularity for coherent sheaves on projective space over a field. This notion is crucial for the construction of Quot-schemes; it's used in conjunction with the semi-continuity theorem for flat sheaves to twist certain coherent sheaves by a precise \( m \) at which point they have no higher cohomology.

Let \( m \) be an integer and \( \mathcal{F} \) be a coherent sheaf on \( \mathbb{P}_k^n \) where \( k \) is an infinite field. We say that \( \mathcal{F} \) is \( m \)-regular if for all \( i \geq 1 \), we have \( H^i(\mathcal{F}(m-i)) = 0 \). Since \( \mathcal{F} \) is coherent and \( \mathbb{P}_k^n \) is noetherian, we know that the set of associated points of \( \mathcal{F} \) is finite. Since \( k \) is infinite, we can thus choose a hyperplane \( H \) not passing through any of these associated points. Defining \( \mathcal{F}_H \) to be the restriction of \( \mathcal{F} \) to \( H \), we have the exact sequence

\[
0 \to \mathcal{F}(m-i-1) \xrightarrow{\alpha} \mathcal{F}(m-i) \to \mathcal{F}_H(m-i) \to 0
\]

where the map \( \alpha \) is locally defined by multiplication with the linear form locally defining \( H \). Taking the long exact sequence of cohomology, we get

\[
\ldots \to H^i(\mathcal{F}(m-i)) \to H^i(\mathcal{F}_H(m-i)) \to H^{i+1}(\mathcal{F}(m-i-1)) \to \ldots
\]

and by the vanishing of the first and third terms, we see that \( \mathcal{F}_H \) is also \( m \)-regular.

**Remark 3.2.** The result desired holds for a finite field as well, since cohomologies base-change correctly under a field extension, but we pass to an extension of infinite cardinality for convenience.

The following important lemma is true for \( m \)-regular sheaves:

**Lemma 3.3** (Castelnuovo). The following three facts are true for \( m \)-regular sheaf \( \mathcal{F} \) on \( \mathbb{P}_k^n \).

1. The map \( H^0(\mathbb{P}_n, \mathcal{O}_{\mathbb{P}_n}(1)) \otimes H^0(\mathbb{P}_n, \mathcal{F}(r)) \to H^0(\mathbb{P}_n, \mathcal{F}(r+1)) \) is surjective for \( r \geq m \).

2. If \( \mathcal{F} \) is \( m \)-regular, it is \( m' \)-regular for \( m' \geq m \).

3. For \( r \geq m \), the sheaf \( \mathcal{F}(r) \) is generated by its global sections and has no higher cohomology.

**Proof.** Fixing \( m \), we prove this lemma by induction on \( n \). For \( n = 0 \), all the results are clearly true, so assume their truth for some \( n_0 \geq 1 \) and let \( n := n_0 + 1 \). As argued above, we can pick hyperplane \( H \) for which \( \mathcal{F}_H \) is \( m \)-regular and since \( H \cong \mathbb{P}_k^{n-1} \), the lemma holds for \( \mathcal{F}_H \).

Fixing \( i \), we induct on \( r \geq m-i \) (the base case follows from the def. of \( m \)-regularity). We first prove the second statement. From the long exact-sequence of cohomology from above, we have

\[
H^i(\mathbb{P}_n, \mathcal{F}(r-1)) \to H^i(\mathbb{P}_n, \mathcal{F}(r)) \to H^i(\mathbb{P}_n, \mathcal{F}_H(r)).
\]

The first term vanishes by the inductive hypothesis for \( r \) and the third term vanishes by inductive hypothesis for \( n \) since \( H \cong \mathbb{P}_k^{n-1} \), so the middle term also vanishes. Thus if \( m' \geq m \), we have that \( H^i(\mathbb{P}_n, \mathcal{F}(m'-i)) = H^i(\mathbb{P}_n, \mathcal{F}(r)) = 0 \) since \( r \geq m-i \). This shows the second statement.

Consider
the following commutative diagram:

\[
\begin{array}{ccc}
H^0(\mathbb{P}^n, \mathcal{F}(r)) \otimes H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) & \xrightarrow{\sigma} & H^0(H, \mathcal{F}_H(r)) \otimes H^0(H, \mathcal{O}_H(1)) \\
\downarrow \mu & & \downarrow \tau \\
H^0(\mathbb{P}^n, \mathcal{F}(r)) & \xrightarrow{\alpha} & H^0(\mathbb{P}^n, \mathcal{F}(r+1)) & \xrightarrow{\nu_{r+1}} & H^0(H, \mathcal{F}_H(r+1))
\end{array}
\]

For \( r \geq m \), we know by the second statement that \( \mathcal{F} \) is \( r \)-regular and so we have that \( H^1(\mathbb{P}^n, \mathcal{F}(r-1)) = 0 \). As \( H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) \) is well-known to be trivial, we find by taking the long exact sequence of cohomology of the exact sequences

\[
0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1) \to \mathcal{O}_H(1) \to 0
\]

and

\[
0 \to \mathcal{F}(r-1) \to \mathcal{F}(r) \to \mathcal{F}_H(r) \to 0
\]

that \( \sigma \) is surjective. By the inductive hypothesis for \( n-1 \) we have that \( \tau \) must be surjective, since \( H \cong \mathbb{P}^{n-1} \). Hence \( \tau \circ \sigma = \nu_{r+1} \circ \mu \) is surjective, whence we must have that

\[
H^0(\mathbb{P}^n, \mathcal{F}(r+1)) = \text{im}(\mu) + \text{ker}(\nu_{r+1}) = \text{im}(\mu) + \text{im}(\alpha),
\]

since \( \text{im}(\alpha) = \text{ker}(\nu_{r+1}) \) by exactness of the lower row. However it’s clear that \( \text{im}(\alpha) \subseteq \text{im}(\mu) \) from the definiton of \( \alpha \) whence we derive that \( \mu \) must be surjective. Thus we have the first statement of the lemma, by induction.

The third statement follows since by repeated application of the first, we have that

\[
H^0(\mathbb{P}^n, \mathcal{F}(r)) \otimes H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(p)) \to H^0(\mathbb{P}^n, \mathcal{F}(r+p))
\]

is surjective for \( r \geq m \). Furthermore for \( p \gg 0 \), \( H^0(\mathbb{P}^n, \mathcal{F}(r+p)) \) is generated by its global sections whence we derive that \( H^0(\mathbb{P}^n, \mathcal{F}(r)) \) is also generated by its global sections for \( r \geq m \). We know from the second statement that \( H^1(\mathbb{P}^n, \mathcal{F}(r)) = 0 \) for \( i > 0 \) and \( r \geq m \). This shows the third statement. \( \Box \)

**Remark 3.4.** We note that if \( \mathcal{F}_H \) is \( m \)-regular, it is \( r \)-regular for \( r \geq m \) by Lemma 3.3. Assuming further that the restriction map \( \nu_r \) is surjective, by the above lemma \( H^0(H, \mathcal{F}_H(r)) \otimes H^0(H, \mathcal{O}_H(1)) \to H^0(H, \mathcal{F}_H(r+1)) \) is surjective and so Diagram 3.2 implies that \( \nu_{r+1} \) is surjective. Inductively we get \( \nu_p \) is surjective for all \( p \geq r \).

The following theorem of Mumford plays an important role in the construction of Quot schemes. Given a suitable coherent sheaf \( \mathcal{F} \) on projective space over a field, it produces a value of \( m \) for which \( \mathcal{F} \) is \( m \)-regular.

**Theorem 3.5 (Mumford).** Let \( \mathcal{F} \) be a coherent sheaf on \( \mathbb{P}^n_k \) such that \( \mathcal{F} \subset \oplus^p \mathcal{O}_{\mathbb{P}^n} \). Picking the binomial basis for \( \mathbb{Q}[x] \), the Hilbert polynomial of \( \mathcal{F} \) can be expressed as

\[
\Phi(x) = \sum_{i=1}^n a_i \binom{x}{i},
\]

where \( a_0, a_1, \ldots, a_n \in \mathbb{Z} \). There exists a polynomial \( F_{p,n} \in \mathbb{Z}[x_0, \ldots, x_n] \) such that \( \mathcal{F} \) is \( m \)-regular where \( m := F_{p,n}(a_0, a_1, \ldots, a_n) \).
Proof. We proceed by induction on $n$ — clearly for $n = 0$, any polynomial works. Now let $n \geq 1$ and let $H$ be a hyperplane which contains none of the finitely many associated points of the coherent sheaf $\oplus^p \mathcal{O}_{\mathbb{P}^n}/\mathcal{F}$. By the choice of $H$, we have that the sheaf $\mathcal{I}_{\mathcal{O}_1}^{O_{\mathbb{P}^n}}(\mathcal{O}_H, \oplus^p \mathcal{O}_{\mathbb{P}^n}/\mathcal{F}) = 0$. Therefore restricting to $H$ gives the exact sequence

$$0 \to \mathcal{F}_H \to \oplus^p \mathcal{O}_H \to \oplus^p \mathcal{O}_H/\mathcal{F}_H \to 0.$$ 

This tells us that $\mathcal{F}_H$ is isomorphic to a subsheaf of $\oplus^p \mathcal{O}_{\mathbb{P}^n}$ when $H$ is identified with $\mathbb{P}^{n-1}_k$.

We consider the exact sequence

$$0 \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{F}_H \to 0$$

which exists because $\mathcal{F}$ is torsion-free. Twisting by $r$ and taking Euler characteristics, we get that

$$\chi(\mathcal{F}_H(r)) = \chi(\mathcal{F}(r)) - \chi(\mathcal{F}(r-1)) = \sum_{i=0}^n a_i \left( \binom{r}{i} - \binom{r-1}{i} \right) = \sum_{i=0}^n a_i \binom{r-1}{i-1}$$

by Pascal’s identity. By a change of basis write

$$\chi(\mathcal{F}_H(r)) = \sum_{j=0}^{n-1} b_j \binom{r}{j},$$

where $b_j = q_j(a_0, \ldots, a_n)$ and the polynomials $q_j$ have coefficients independent of $\mathcal{F}$. Now applying the inductive hypothesis, we have that there exists a polynomial $F_{p,n-1}$ such that for $m_0 := F_{p,n-1}(b_0, \ldots, b_{n-1})$ the sheaf $\mathcal{F}_H$ is $m_0$-regular and by substituting $b_j = q_j(a_0, \ldots, a_n)$ we get $m_0 = G(a_0, \ldots, a_n)$ where $G \in \mathbb{Z}[x_0, \ldots, x_n]$ with coefficients independent of $k$ and $\mathcal{F}$. Thus for $m \geq m_0 - 1$, we have the long exact sequence of cohomology

$$0 \to H^0(\mathcal{F}(m-1)) \to H^0(\mathcal{F}(m)) \to H^0(\mathcal{F}_H(m)) \to H^1(\mathcal{F}(m-1)) \to H^1(\mathcal{F}(m)) \to H^1(\mathcal{F}(m-1)) = 0 \to \ldots$$

which for $i \geq 2$ gives us $H^i(\mathcal{F}(m-1)) \cong H^i(\mathcal{F}(m))$. Since $H^i(\mathcal{F}(N)) = 0$ for $N \gg 0$, we have that $H^i(\mathcal{F}(m)) = 0$ for $i \geq 2$ and $m \geq m_0 - 2$.

We have that $H^1(\mathcal{F}(m-1))$ surjects onto $H^1(\mathcal{F}(m))$ for all $m \geq m_0$, and moreover equality of dimension holds for a given $m$ only if the restriction $\nu_m : H^0(\mathcal{F}(m)) \to H^0(\mathcal{F}_H(m))$ is surjective. Since $\mathcal{F}_H$ is $m$-regular, Remark 3.3 tells us that $\nu_j$ is surjective for all $j \geq m$ nd in particular $h^1(\mathcal{F}(j-1)) = h^1(\mathcal{F}(j))$ for all $j \geq m$. Since $h^1(\mathcal{F}(j)) = 0$ for $j \gg 0$, we can’t have the desired equality and $h^1(\mathcal{F}(m))$ must be strictly decreasing for $m \geq m_0$. Therefore $H^1(\mathcal{F}(m)) = 0$ for $m \geq m_0 + h^1(\mathcal{F}(m_0))$.

Since $\mathcal{F} \subset \oplus^p \mathcal{O}_{\mathbb{P}^n}$, we have $h^0(\mathcal{F}(r)) \leq p\binom{n+r}{n}$ and since $h^i(\mathcal{F}(m)) = 0$ for $i \geq 2$ and $m \geq m_0 - 2$ we have

$$h^1(\mathcal{F}(m_0)) = h^0(\mathcal{F}(m_0)) - \chi(\mathcal{F}(m_0)) \leq p\binom{n+m_0}{m} - \sum_{i=0}^n a_i \binom{m_0}{i} = P(a_0, a_1, \ldots, a_n)$$

using the fact that $m_0 = G(a_0, \ldots, a_n)$. Furthermore $P(a_0, \ldots, a_n) \geq 0$.

Therefore $H^1(\mathcal{F}(m)) = 0$ for $m \geq (G+P)(a_0, \ldots, a_n)$ and if $F_{p,n} := G+P$, since $P(a_0, \ldots, a_n) \geq 0$, we see that $\mathcal{F}$ is $F_{p,n}(a_0, \ldots, a_n)$-regular where $F_{p,n}$ is independent of $\mathcal{F}$ and the field $k$. Hence the theorem.

\qed
3.3 Base-change and semi-continuity

In this sub-section, we state without proof, some general theorems on base-change and semi-continuity. All proofs can be found in Section 5.3, [3]. We begin with the following lemma on base-change without flatness.

**Lemma 3.6.** Let \( \varphi : T \to S \) be a morphism of noetherian schemes, \( F \) a coherent sheaf on \( \mathbb{P}_S^r \) and \( F_T := \varphi^*(F) \) where \( \varphi' : \mathbb{P}_T^r \to \mathbb{P}_S^r \). Denote \( \pi_S \) and \( \pi_T \) to be projections from \( \mathbb{P}_S^r \to S \) and \( \mathbb{P}_T^r \to T \) respectively. There exists integer \( r_0 \) such that the base-change homomorphism

\[
\varphi^*\pi_S^* F(r) \to \pi_T^* F_T(r)
\]

is an isomorphism for \( r \geq r_0 \).

Note that for the above lemma, the absence of a flatness condition results in \( r_0 \) depending on the choice of \( \varphi \). The following result gives us the equivalence of flatness of sheaf \( F \) on \( \mathbb{P}_S^r \) and local freedom of \( \pi_S^* F(r) \) for all \( r \geq N \).

**Lemma 3.7.** Let \( S \) be a noetherian scheme and \( F \) a coherent sheaf on \( \mathbb{P}_S^r \). There exists integer \( N \) such that for all \( r \geq N \), \( \pi_S^* F(r) \) is locally free if and only if \( F \) is flat over \( S \).

To state the following result, we need to make some definitions. Define a linear scheme \( V \to S \) to be a scheme of the form \( \text{SpecSym}_{\mathcal{O}_S} Q \) where \( Q \) is a coherent sheaf on \( S \). Its zero section \( V_0 \subset V \) is the closed subscheme of \( V \) defined by the ideal generated by \( Q \). In particular the map \( V_0 \to S \) is an isomorphism.

**Theorem 3.8.** Let \( S \) be noetherian, \( \pi : S \to S \) be a projective morphism and let \( E \) and \( F \) be coherent sheaves on \( X \). The functor \( \text{hom}(E, F) \), which assigns \( T \in \text{Sch}_S \) to the set of all \( \mathcal{O}_{X_T} \)-linear homomorphisms between the pullbacks \( E_T \) and \( F_T \) from \( X \) to \( X_T := X \times_ST \), is representable by a linear scheme \( V \) over \( S \) if \( F \) is flat over \( S \).

**Remark 3.9.** As a corollary, if \( f : E_T \to F_T \) is an \( \mathcal{O}_{X_T} \)-linear morphism and corresponds to \( \varphi_f : T \to V \), the inverse image of the zero section in \( T \) is a closed subscheme of \( T' \) with the universal property that if \( U \to T \) is a morphism of schemes such that the pullback of \( f \) is zero, then \( U \to T \) factors via \( T' \).

Finally when \( F \) is flat, we have the following theorem:

**Theorem 3.10** (Semi-Continuity and Flat Base-Change). Let \( \pi : X \to S \) be a proper morphism of noetherian schemes and \( F \) be a coherent \( \mathcal{O}_X \)-module which is flat over \( S \). Then the following statements are true:

1. For integer \( i \), the function \( s \to \dim_k(s) H^i(X_s, F_s) \) is upper-semicontinuous on \( S \).
2. The function \( s \to \sum_i (-1)^i \dim_k(s) H^i(X_s, F_s) \) is locally constant on \( S \).
3. If for some integer \( i \), there exists integer \( d \geq 0 \) such that for all \( s \in S \), we have \( \dim_k(s) H^i(X_s, F_s) = d \), then \( R^i\pi_* \mathcal{F} \) is locally free of rank \( d \) and \( (R^i\pi_* \mathcal{F})_s \to H^{i-1}(X_s, F_s) \) is an isomorphism for all \( s \in S \).
4. If for some integer \( i \) and point \( s \in S \), the map \( (R^i\pi_* \mathcal{F})_s \to H^i(X_s, F_s) \) is surjective, there exists an open subscheme \( U \subset S \) containing \( s \) such that for any quasi-coherent \( \mathcal{O}_U \)-module \( \mathcal{G} \), the natural homomorphism

\[
R^i\pi_{U*} \mathcal{F}_U \otimes_X U \mathcal{G} \to R^i\pi_{U*}(\mathcal{F}_U \otimes_{\mathcal{O}_U} \pi_U^* \mathcal{G})
\]

is an isomorphism where \( X_u := \pi^{-1}(U) \) and \( \pi_U : X_U \to U \) is induced by \( \pi \). In particular, \( (R^i\pi_* \mathcal{F})_{s'} \to H^i(X_{s'}, F_{s'}) \) is an isomorphism for \( s' \in U \).
3.4 Flattening Stratifications

In this sub-section, we state and sketch a proof of the existence of flattening stratifications. We need the following lemma from commutative algebra:

**Lemma 3.11 (Lemma on Generic Flatness, [X]).** Let \( A \) be a noetherian domain, \( B \) an \( A \)-algebra of finite type and \( M \) a finite \( B \)-module. There exists non-zero \( f \in A \) such that \( M_f \) is a free \( A_f \)-module.

The following is a sketch of the proof of the existence of flattening stratifications. The full details of the proof can be found in [X] – we have attempted to distill the essence of the proof into the following summary so as to better illuminate the sequence of the arguments.

**Theorem 3.12 (Existence of Flattening Stratifications, [X]).** Let \( S \) be a noetherian scheme, and let \( \mathcal{F} \) be a coherent sheaf on \( \mathbb{P}^n_S \). Then the set \( I \) of Hilbert polynomials of restrictions of \( \mathcal{F} \) to fibres over \( \mathbb{P}^n_S \to S \) is finite. Moreover, for \( f \in I \), there exists a locally closed subscheme \( S_f \) of \( S \), such that:

1. The underlying set of \( S_f \), denoted by \( |S_f| \), consists precisely of the points \( s \in S \) where the Hilbert polynomial of the restriction of \( \mathcal{F} \) to \( \mathbb{P}^n_s \) is \( f \).

2. Letting \( S' := \bigcap_{f \in I} S_f \), and let \( i : S' \to S \) be the morphisms induced by \( S_f \hookrightarrow S \). The sheaf \( i^*(\mathcal{F}) \) is flat over \( S' \) and is universal in the sense that for a morphism \( \varphi : T \to S \), the pullback \( \varphi^*(\mathcal{F}) \) on \( \mathbb{P}^n_T \) is flat over \( T \) iff \( \varphi \) factors through \( i \).

3. Let \( I \) be given a total ordering as follows: \( f < g \) if \( f(n) < g(n) \) for all \( n \gg 0 \). Then the closure of \( |S_f| \) in \( S \) is contained in \( |S_g| \) where \( f \leq g \).

**Summary of Proof:** The proof of the above result proceeds in the following steps:

1. First we prove all three statements in the case where \( n = 0 \) and thus \( \mathbb{P}^0_S = S \). In this case, the scheme \( S \) is stratified by the dimensions of \( \mathcal{F}_s \otimes k_s \) as a \( k_s \)-vector space, where \( k_s \) is the residue field of \( \mathcal{O}_S \) at \( s \), and flatness is equivalent to local freedom.

2. An easy noetherian induction and Lemma 3.11 give us a finite set of disjoint, reduced, locally closed, mutually disjoint subschemes \( V_i \) of \( S \) such that the underlying set \( |S| \) is the disjoint union of the sets \( |V_i| \) and the restriction of \( \mathcal{F} \) to \( \mathbb{P}^n_{V_i} \) is flat over \( \mathcal{O}_{V_i} \). Proving the desired result for these subschemes is observed to be sufficient, since the resulting strata can be glued together using the universal property.

3. For each of the above \( V_i \), there are finitely many Hilbert polynomials of \( \mathcal{F}_s \) as \( s \) varies over points of \( V_i \) since the Hilbert polynomial is locally constant, by Theorem 3.10. Applying the theorems of base-change and semi-continuity from Section 3.3, we derive the following consequences:

   (a) There are finitely many Hilbert polynomials for \( \mathcal{F}_s \) as \( s \) varies over points of \( S \).

   (b) There exists integer \( N_1 \) such that \( R^r \pi_* \mathcal{F}(m) = 0 \) where \( \pi : \mathbb{P}^n_S \to S \), \( r \geq 1 \) and \( m \geq N_1 \). In particular, \( H^r(\mathbb{P}^n_s, \mathcal{F}(m)) = 0 \) for all \( s \in S \).

   (c) There exists integer \( N \geq N_1 \) such that the base-change map \( (\pi_s \mathcal{F}(m))|_s \to H^0(\mathbb{P}^n_s, \mathcal{F}_s(m)) \) is an isomorphism.
4. We consider the coherent sheaves $E_i := \pi_* F(N + i)$ on $S$ for $i \in \{0, \ldots, n\}$. For the sheaf $E_0$, we apply the case for $n = 0$ to get a stratification $W_{e_0}$ of $S$ indexed by integers $e_0$, such that for morphism $f : T \to S$, the sheaf $f^* E_0$ is locally free of rank $e_0$ iff $f$ factors through $W_{e_0} \hookrightarrow S$. Inductively we apply the same to $E_i|_{W_{e_0}, \ldots, e_{i-1}}$ to get flattening stratification $W_{e_0}, \ldots, e_i$. This gives us $W_{e_0, \ldots, e_n}$ such that the pull-back along $f : T \to S$ of $E_i$ gives a locally free $O_T$-module iff $f$ factors through $W_{e_0, \ldots, e_n} \hookrightarrow S$. Tuples $(e_0, \ldots, e_n)$ for $e_i \in \mathbb{Z}$ are in bijection with polynomials of degree $\leq n$ with integer coefficients using the bijection $f \mapsto (e_0, \ldots, e_n)$ where $e_i = f(N + i)$. Therefore, we can re-name $W_{e_0, \ldots, e_n}$ as $W_f$. This is especially note-worthy since for $s \in W_f$, the Hilbert polynomial of $F_s$ is seen to be $f$.

5. Having suitably picked the underlying sets $|W_f|$ for $S_f$, we put the scheme structure $S_f \subset W_f$ on $|W_f|$. We do this by finding ideal sheaves $I_i$ that define subschemes $W_f^{(i)}$ of $W_f$ for any $i \geq 1$ having the following properties:

(a) $\pi_* F(N + i)$ is locally free of rank $f(N + i)$ when restricted to $W_f^{(i)}$.

(b) Any base-change $T \to S$ for which $\pi_S F(N + i)$ pulls-back to a locally-free sheaf of rank $f(N + i)$ factors through $W_f^{(i)} \hookrightarrow S$.

Since $S$ is noetherian, the ascending chain $I_0 \subset I_0 + I_1 \subset \ldots$ terminates whence the sheaf $I := I_0 + \ldots$ is a coherent ideal sheaf. Let $S_f$ be the sub-scheme of $W_f$ defined by the ideal sheaf $I$. We check that it satisfies the properties desired and is in fact the flattening stratification needed.

\[\square\]

References

