**ME 599-1 Handout 4**

*Vertical mass-spring system*

The system pictured at right is configured such that when the spring is slack, \( y = 0 \). The equation of motion for the **stance** phase is

\[
m\ddot{y} + k_{\text{vert}} y = -mg.
\]

This equation may be made dimensionless by defining \( \tilde{y} \equiv (k_{\text{vert}}/mg) \cdot y \) and \( \tilde{t} \equiv \omega_0 \cdot t \), where \( \omega_0 = \sqrt{k_{\text{vert}}/m} \), the resonant frequency of the mass-spring system. The result is

\[
\ddot{\tilde{y}} = -\tilde{y} + 1
\]

which may be simulated by defining initial conditions

\[
\tilde{y}(0) = 0 \\
\dot{\tilde{y}}(0) = d\tilde{y}(0)/d\tilde{t} = -\nu \omega_0 / g
\]

with \( \nu \) denoting the actual downward vertical velocity at impact. The solution to this equation is

\[
\tilde{y}(\tilde{t}) = (\nu \omega_0 / g) \cdot \sin \tilde{t} - \left( 1 - \cos \tilde{t} \right)
\]

and we can also determine the vertical force in the spring,

\[
f_y = -k_{\text{vert}} y = -mg \tilde{y}
\]

which is normalized to

\[
\tilde{f}_y = f_y/mg = -\tilde{y} = (\nu \omega_0 / g) \cdot \sin \tilde{t} + \left( 1 - \cos \tilde{t} \right).
\]

This force reaches a maximum at time

\[
\tilde{t}_{\text{maxforce}} = \tan^{-1} \left( \nu \omega_0 / g \right) = \pi - \tan^{-1} \left( \nu \omega_0 / g \right)
\]

with a value

\[
\tilde{f}_{y\text{max}} = (\nu \omega_0 / g) \sin \tilde{t}_{\text{maxforce}} + 1 - \cos \tilde{t}_{\text{maxforce}}.
\]

The ground contact time is simply double the time of maximum force,

\[
\tilde{t}_c = 2\tilde{t}_{\text{maxforce}} = 2\pi - 2\tan^{-1} \left( \nu \omega_0 / g \right).
\]

The **aerial** phase is described by the equation

\[
\ddot{\tilde{y}} = -1
\]

with a total aerial time of

\[
\tilde{t}_a = 2\nu \omega_0 / g.
\]

The stride frequency is

\[
f_s = 1/(t_c + t_a)
\]

which is normalized to

\[
\tilde{f}_s = f_s/\omega_0 = 1/(\tilde{t}_c + \tilde{t}_a).
\]

The Groucho number, which is dimensionless form of the vertical contact velocity, is \( \ddot{\tilde{y}} = -1 \).
Mass-spring system with forward motion

The system at right incorporates forward motion in the mass-spring. The horizontal and vertical speeds at ground contact are $u$ and $v$, respectively. The angle of the spring is $\theta$ measured clockwise from vertical, and the initial leg angle is $\theta = -\theta_0$. The spring has stiffness $k_{\text{leg}}$, and slack length $l_0$, which is also the initial value for the leg length $l$.

The dimensionless equations of motion are

\[
\begin{align*}
\ddot{x} &= k_{\text{leg}} (1 - \tilde{l}) \sin \theta \\
\ddot{y} &= k_{\text{leg}} (1 - \tilde{l}) \cos \theta - 1
\end{align*}
\]

where the mass position is normalized to leg length, $\tilde{x} \equiv x/l_0$ and $\tilde{y} \equiv y/l_0$. Time is made dimensionless by dividing by the period of the inverted pendulum, $\tilde{t} \equiv t/l_0/g$. In addition, the leg stiffness has been made dimensionless, $\tilde{k}_{\text{leg}} \equiv k_{\text{leg}} l_0/mg$. This quantity may be thought of as the square of the ratio of the natural frequency of the mass-spring to that of the pendulum. This has the effect of making the minimum acceptable value to be $\tilde{k}_{\text{leg}} = 1$, because at that point the spring force when the leg is compressed by the full leg length, $\tilde{k}_{\text{leg}} l_0$, is equal to the force of gravity, $mg$. A value of $\tilde{k}_{\text{leg}} < 1$ is therefore not tolerated, because the mass would bottom out the spring and strike the ground.

The horizontal and vertical contact speeds are normalized into Froude numbers, $\tilde{u} = u\sqrt{gl_0}$ and $\tilde{v} = v\sqrt{gl_0}$. The initial and final conditions are

\[
\begin{align*}
\theta (0) &= -\theta_0, \quad \theta (\tilde{t}) = \theta_0 \\
\tilde{l} (0) &= \tilde{l} (\tilde{t}) = 1 \\
\tilde{x} (0) &= \tilde{x} (\tilde{t}) = \tilde{u} \\
\tilde{y} (0) &= -\tilde{v}, \quad \tilde{y} (\tilde{t}) = \tilde{v}
\end{align*}
\]

The behavior of this system is dependent on values of $\tilde{u}$, $\tilde{v}$, $\theta_0$, and $\tilde{k}_{\text{leg}}$. The horizontal and vertical Froude numbers are determined by the speed of the animal being studied. $\theta_0$ is set by the stride length, and $\tilde{k}_{\text{leg}}$ is found by trying to get this to match.

The fundamental dimensionless group determining the behavior of this system is the dimensionless stiffness, $\tilde{k}_{\text{leg}} \equiv k_{\text{leg}} l_0/mg$.

Note that when $\theta = 0$ and $\tilde{u} = 0$, the system reduces to the vertical mass-spring system, which has characteristics determined by the Groucho number, $\nu\omega_0/g = \sqrt{k_{\text{vert}} \tilde{v}}$. 