Example: The Jacobian

The Jacobian matrix can be used in many ways to describe the characteristics of a system such as its manipulability, or the relationship between forces and torques. This example shows how the Jacobian can be used to evaluate center of mass motion with respect to individual joint rotations.

Consider a four-segment model of the human body in the sagittal plane, at right, with reference frames fixed to each segment as shown. Suppose the length of segment $i$ and the distance from its bottom-most end to its center of mass are given by $l_i$ and $l_{ci}$, respectively.

The positions of the mass center’s of the segments are given in vector form by

\[
\begin{align*}
\mathbf{p}_{ca} &= l_{c1}a_1, \\
\mathbf{p}_{cb} &= l_1a_1 + l_2b_1, \\
\mathbf{p}_{cc} &= l_1a_1 + l_2b_1 + l_3c_1, \\
\mathbf{p}_{cd} &= l_1a_1 + l_2b_1 + l_3c_1 + l_{c4}d_1,
\end{align*}
\]

which can be broken into x and y coordinates:

\[
\begin{align*}
x_{c1} &= -l_{c1} \cos(\theta_1) \\
y_{c1} &= l_{c1} \sin(\theta_1) \\
x_{c2} &= -l_1 \cos(\theta_1) + l_{c2} \cos(\theta_2 - \theta_1) \\
y_{c2} &= l_1 \sin(\theta_1) + l_{c2} \sin(\theta_2 - \theta_1) \\
x_{c3} &= -l_1 \cos(\theta_1) + l_2 \cos(\theta_2 - \theta_1) - l_{c3} \cos(\theta_3 - \theta_2 + \theta_1) \\
y_{c3} &= l_1 \sin(\theta_1) + l_2 \sin(\theta_2 - \theta_1) + l_{c3} \sin(\theta_3 - \theta_2 + \theta_1) \\
x_{c4} &= -l_1 \cos(\theta_1) + l_2 \cos(\theta_2 - \theta_1) - l_3 \cos(\theta_3 - \theta_2 + \theta_1) + l_{c4} \cos(\theta_4 - \theta_3 + \theta_2 - \theta_1) \\
y_{c4} &= l_1 \sin(\theta_1) + l_2 \sin(\theta_2 - \theta_1) + l_3 \sin(\theta_3 - \theta_2 + \theta_1) + l_{c4} \sin(\theta_4 - \theta_3 + \theta_2 - \theta_1)
\end{align*}
\]

The whole body center of mass is easily found using the quantities above:

\[
\begin{align*}
x_c &= \frac{1}{(m_1 + m_2 + m_3 + m_4)} \left[ -m_1l_{c1} \cos(\theta_1) - m_2l_1 \cos(\theta_1) + m_1l_{c2} \cos(\theta_2 - \theta_1) - m_2l_1 \cos(\theta_1) + m_2l_2 \cos(\theta_2 - \theta_1) - m_1l_{c3} \cos(\theta_3 - \theta_2 + \theta_1) - m_2l_1 \cos(\theta_1) + m_2l_2 \cos(\theta_2 - \theta_1) - m_1l_{c4} \cos(\theta_4 - \theta_3 + \theta_2 - \theta_1) + m_2l_1 \cos(\theta_1) + m_1l_{c4} \sin(\theta_1) + m_2l_{c1} \sin(\theta_1) + m_2l_{c2} \sin(\theta_2 - \theta_1) + m_1l_{c4} \sin(\theta_1) + m_2l_2 \sin(\theta_2 - \theta_1) + m_1l_{c3} \sin(\theta_3 - \theta_2 + \theta_1) + m_2l_1 \sin(\theta_1) + m_2l_2 \sin(\theta_2 - \theta_1) + m_1l_{c3} \sin(\theta_3 - \theta_2 + \theta_1) + m_2l_{c4} \sin(\theta_4 - \theta_3 + \theta_2 - \theta_1) \right] \\
y_c &= \frac{1}{(m_1 + m_2 + m_3 + m_4)} \left[ -m_1l_{c1} \sin(\theta_1) - m_2l_1 \sin(\theta_1) + m_1l_{c2} \sin(\theta_2 - \theta_1) - m_2l_1 \sin(\theta_1) + m_2l_2 \sin(\theta_2 - \theta_1) - m_1l_{c3} \sin(\theta_3 - \theta_2 + \theta_1) - m_2l_1 \sin(\theta_1) + m_2l_2 \sin(\theta_2 - \theta_1) - m_1l_{c4} \sin(\theta_4 - \theta_3 + \theta_2 - \theta_1) + m_2l_1 \sin(\theta_1) + m_1l_{c4} \cos(\theta_1) + m_2l_{c1} \cos(\theta_1) + m_2l_{c2} \cos(\theta_2 - \theta_1) + m_1l_{c4} \cos(\theta_1) + m_2l_2 \cos(\theta_2 - \theta_1) + m_1l_{c3} \cos(\theta_3 - \theta_2 + \theta_1) + m_2l_1 \cos(\theta_1) + m_2l_2 \cos(\theta_2 - \theta_1) + m_1l_{c3} \cos(\theta_3 - \theta_2 + \theta_1) + m_2l_{c4} \cos(\theta_4 - \theta_3 + \theta_2 - \theta_1) \right]
\end{align*}
\]

The equations above can be assembled into a vector-valued center-of-mass position function,
\[
    \chi_c = \ell(q)
\]
where the input positions are assembled into a vector:

\[
    q = [\theta_1 \ \theta_2 \ \theta_3 \ \theta_4]^T.
\]

Taking the Jacobian of the function,

\[
    J = \frac{\partial \ell(q)}{\partial q} = \begin{bmatrix}
        J_{11} & J_{12} & J_{13} & J_{14} \\
        J_{21} & J_{22} & J_{23} & J_{24}
    \end{bmatrix},
\]
yields the following entries:

\[
    J_{11} = \frac{1}{(m_1 + m_2 + m_3 + m_4)} \begin{bmatrix}
        (m_1 l_{c1} + m_2 l_{c1} + m_3 l_{c1} + m_4 l_{c1}) \sin(\theta_1) + (m_2 l_{c2} + m_3 l_{c2} + m_4 l_{c2}) \sin(\theta_2 - \theta_1) + \\
        (m_2 l_{c3} + m_4 l_{c3}) \sin(\theta_3 - \theta_2 + \theta_1) + (m_4 l_{c4}) \sin(\theta_4 - \theta_3 + \theta_2 - \theta_1)
    \end{bmatrix}
\]

\[
    J_{12} = \frac{1}{(m_1 + m_2 + m_3 + m_4)} \begin{bmatrix}
        -(m_2 l_{c2} + m_3 l_{c2} + m_4 l_{c2}) \sin(\theta_2 - \theta_1) - \\
        (m_3 l_{c3} + m_4 l_{c3}) \sin(\theta_3 - \theta_2 + \theta_1) - (m_4 l_{c4}) \sin(\theta_4 - \theta_3 + \theta_2 - \theta_1)
    \end{bmatrix}
\]

\[
    J_{13} = \frac{1}{(m_1 + m_2 + m_3 + m_4)} \begin{bmatrix}
        (m_2 l_{c3} + m_4 l_{c3}) \sin(\theta_3 - \theta_2 + \theta_1) + (m_4 l_{c4}) \sin(\theta_4 - \theta_3 + \theta_2 - \theta_1)
    \end{bmatrix}
\]

\[
    J_{14} = \frac{1}{(m_1 + m_2 + m_3 + m_4)} \begin{bmatrix}
        -(m_4 l_{c4}) \sin(\theta_4 - \theta_3 + \theta_2 - \theta_1)
    \end{bmatrix}
\]

\[
    J_{21} = \frac{1}{(m_1 + m_2 + m_3 + m_4)} \begin{bmatrix}
        (m_1 l_{c1} + m_2 l_{c1} + m_3 l_{c1} + m_4 l_{c1}) \cos(\theta_1) - (m_2 l_{c2} + m_3 l_{c2} + m_4 l_{c2}) \cos(\theta_2 - \theta_1) + \\
        (m_3 l_{c3} + m_4 l_{c3}) \cos(\theta_3 - \theta_2 + \theta_1) - (m_4 l_{c4}) \cos(\theta_4 - \theta_3 + \theta_2 - \theta_1)
    \end{bmatrix}
\]

\[
    J_{22} = \frac{1}{(m_1 + m_2 + m_3 + m_4)} \begin{bmatrix}
        (m_2 l_{c2} + m_3 l_{c2} + m_4 l_{c2}) \cos(\theta_2 - \theta_1) - \\
        (m_3 l_{c3} + m_4 l_{c3}) \cos(\theta_3 - \theta_2 + \theta_1) + (m_4 l_{c4}) \cos(\theta_4 - \theta_3 + \theta_2 - \theta_1)
    \end{bmatrix}
\]

\[
    J_{23} = \frac{1}{(m_1 + m_2 + m_3 + m_4)} \begin{bmatrix}
        (m_3 l_{c3} + m_4 l_{c3}) \cos(\theta_3 - \theta_2 + \theta_1) - (m_4 l_{c4}) \cos(\theta_4 - \theta_3 + \theta_2 - \theta_1)
    \end{bmatrix}
\]

\[
    J_{24} = \frac{1}{(m_1 + m_2 + m_3 + m_4)} \begin{bmatrix}
        (m_4 l_{c4}) \cos(\theta_4 - \theta_3 + \theta_2 - \theta_1)
    \end{bmatrix}
\]

There are several ways this Jacobian can be used to describe how changes in the joint angles produce changes in the center-of-mass position. For example, the columns of the Jacobian illustrate the magnitude and direction of c.o.m. change for a unit change in each joint angle, as plotted to the right, for a body in various degrees of crouch. Four vectors, each corresponding to a column of the Jacobian, are plotted with origin at the center of mass. They show that when the body is crouched, a unit change in any joint angle will result in a specific motion of the center of mass, in one of four possible directions. As the body stands up, these directions tend to align more horizontally,
meaning that larger joint angle changes are required to achieve the same amount of vertical displacement of the c.o.m. This effect is offset by the fact that the horizontal motions become larger.

A numerical way to summarize the above interpretation is to plot the maximum singular values and the product of singular values, vs. the crouching positions. The maximum singular value describes the largest motion in the c.o.m. which can take place for a given change in the joint angle vector. It does not, however, describe the trade-off between large motions in one direction and smaller motions in other directions. This is summarized by the product of singular values, which gives an overall measure of manipulability: