The period-index problem

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(joint work with Aise Johan de Jong and with James Hotchkiss)

The complexity of a Brauer class $\alpha \in \mathrm{Br}(K)$ over a field K can be measured by two integers: the *period* $\mathrm{per}(\alpha)$, which is simply the order of α in $\mathrm{Br}(K)$; and the *index* $\mathrm{ind}(\alpha)$, equal to $\sqrt{\dim_K(D)}$ where D is the unique central division algebra of class α. The integers $\mathrm{per}(\alpha)$ and $\mathrm{ind}(\alpha)$ share the same prime factors, and $\mathrm{per}(\alpha)$ divides $\mathrm{ind}(\alpha)$. The *period-index problem* is to determine an integer e such that $\mathrm{ind}(\alpha)$ divides $\mathrm{per}(\alpha)^e$. The following longstanding conjecture, first raised in print by Colliot-Thélène [2], predicts a precise value of e for function fields of varieties.

Conjecture 1 (Period-index conjecture)**.** *Let K be a field of finite transcendence degree d over an algebraically closed field k. For all $\alpha \in \mathrm{Br}(K)$, $\mathrm{ind}(\alpha)$ divides $\mathrm{per}(\alpha)^{d-1}$.*

This is vacuously true for $d \leq 1$, as then $\mathrm{Br}(K) = 0$ by Tsen's theorem. It is also true for $d = 2$ by de Jong [3] when $\mathrm{per}(\alpha)$ is prime to the characteristic of k, and by [9, 5] in general. In higher dimensions, the conjecture is wide open: it is not even known for a single field K of transcendence degree $d \geq 3$ that there exists an integer e such that $\mathrm{ind}(\alpha)$ divides $\mathrm{per}(\alpha)^e$ for all $\alpha \in \mathrm{Br}(K)$.

If X is a smooth projective model for K, then the restriction $\mathrm{Br}(X) \to \mathrm{Br}(K)$ is an isomorphism onto the subgroup of unramified Brauer classes. By [5], Conjecture 1 would follow in general if it were known for all unramified Brauer classes on all K. In this way, the conjecture can be viewed as a global problem. I reported on recent progress on the period-index problem from this perspective, restricting for simplicity to the case where the base field $k = \mathbf{C}$ is the complex numbers. The first result, joint with de Jong, gives evidence that the integer e in the period-index problem can be chosen uniformly in α.

Theorem 2 ([4])**.** *Let $X \to S$ be a smooth proper morphism of complex varieties. Assume that the very general fiber of $X \to S$ is projective and satisfies the Lefschetz standard conjecture in degree 2. Then there exists a positive integer e such that for all $s \in S(\mathbf{C})$ and $\alpha \in \mathrm{Br}(X_s)$, $\mathrm{ind}(\alpha)$ divides $\mathrm{per}(\alpha)^e$.*

Recall that for a smooth projective d-dimensional variety Y with an ample divisor h, the Lefschetz standard conjecture in degree 2 says that the inverse of the hard Lefschetz isomorphism $(-) \cup h^{d-2} \colon \mathrm{H}^2(Y, \mathbf{Q}) \xrightarrow{\sim} \mathrm{H}^{2d-2}(Y, \mathbf{Q})$ is algebraic. This conjecture is known for some interesting classes of varieties, including threefolds of Kodaira dimension less than 3 [11, 12] and holomorphic symplectic varieties of $\mathrm{K3}^{[n]}$ or Kummer type [1, 6].

The idea behind the proof of Theorem 2 is to use the algebraicity of the inverse Lefschetz isomorphism to reduce to studying classes in the image of a correspondence from a surface, and to use the known period-index conjecture for surfaces to bound the index of such classes.

The second result, joint with Hotchkiss, establishes the first nontrivial case of the unramified period-index conjecture in dimension greater than 2.

Theorem 3 ([8])**.** *Let X be a complex abelian threefold. For all $\alpha \in \mathrm{Br}(X)$, $\mathrm{ind}(\alpha)$ divides $\mathrm{per}(\alpha)^2$.*

The proof relies on the Hodge theory and enumerative geometry of categories. Suppose that $\mathcal{C}$ is an enhanced triangulated category that admits an embedding as a semiorthogonal component into the derived category of a smooth proper variety. Then by [10] there is an associated finitely generated abelian group $\mathrm{K}_0^{\mathrm{top}}(\mathcal{C})$ equipped with a weight 0 Hodge structure and a natural map $\mathrm{K}_0(\mathcal{C}) \to \mathrm{K}_0^{\mathrm{top}}(\mathcal{C})$ from the Grothendieck group factoring through the subgroup $\mathrm{Hdg}(\mathcal{C}) \subset \mathrm{K}_0^{\mathrm{top}}(\mathcal{C})$ of integral Hodge classes. When $\mathcal{C}$ is the derived category of α-twisted sheaves for a class $\alpha \in \mathrm{Br}(X)$ on a smooth proper variety, we write $\mathrm{K}_0^{\mathrm{top}}(X, \alpha)$, $\mathrm{Hdg}(X, \alpha)$, and $\mathrm{K}_0(X, \alpha)$ for these invariants. Since $\mathrm{ind}(\alpha)$ can be computed as the minimal positive rank of an element of $\mathrm{K}_0(X, \alpha)$, the period-index conjecture for α factors into two steps:

(1) Construct a Hodge class $v \in \mathrm{Hdg}(X, \alpha)$ of rank $\mathrm{per}(\alpha)^{\dim(X)-1}$.
(2) Show that v is algebraic, i.e. in the image of $\mathrm{K}_0(X, \alpha) \to \mathrm{Hdg}(X, \alpha)$.

Step (1) was solved by Hotchkiss [7] when $\mathrm{per}(\alpha)$ is prime to $(\dim X - 1)!$. The key ingredient is an explicit description of the Hodge structure $\mathrm{K}_0^{\mathrm{top}}(X, \alpha)$ when α is topologically trivial, in terms of a twist by a B-field. In special cases, like when X is an abelian variety, this also allows Step (1) to be solved for all $\alpha \in \mathrm{Br}(X)$.

Step (2) can be regarded as a case of the integral Hodge conjecture for categories. In [8] we develop a theory of reduced Donaldson–Thomas invariants for CY3 categories, with the feature that the variational integral Hodge conjecture holds for classes with nonvanishing invariant. Theorem 3 is then proved by specializing (X, α) within the Hodge locus for the class v from Step (1) to an untwisted abelian threefold with nonvanishing invariant.

References

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