Intersections of two Grassmannians in $\mathbf{P}^9$

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(joint work with Lev Borisov and Andrei Căldăraru)

We work over an algebraically closed field k of characteristic 0. Let V be a 5-dimensional vector space over k, and let $W = \wedge^2 V$. We consider intersections of the form

$$X = g_1(\mathrm{Gr}(2,V)) \cap g_2(\mathrm{Gr}(2,V)) \subset \mathbf{P}(W),$$

where $g_1, g_2 \in \mathrm{PGL}(W)$ and $\mathrm{Gr}(2,V) \subset \mathbf{P}(W)$ via the Plücker embedding. When smooth of expected dimension, X is a Calabi–Yau threefold with Hodge numbers

$$h^{1,1}(X) = 1, \quad h^{1,2}(X) = 51.$$

These varieties were previously studied in works of Gross–Popescu [4], G. Kapustka [6], M. Kapustka [7], and Kanazawa [5], after whom we call X a *GPK*^{3} *threefold.*

The elements $g_1, g_2 \in \mathrm{PGL}(W)$ determine another intersection of the same type, in the dual projective space:

$$Y = g_1^{-T}(\mathrm{Gr}(2,V^\vee)) \cap g_2^{-T}(\mathrm{Gr}(2,V^\vee)) \subset \mathbf{P}(W^\vee),$$

where $g_i^{-T} = (g_i^{-1})^\vee \colon \mathbf{P}(W^\vee) \to \mathbf{P}(W^\vee)$ is the inverse transpose of g_i. The variety X is a smooth threefold if and only if Y is. In this case, X and Y are smooth deformation equivalent Calabi–Yau threefolds, which we call *GPK*^{3} *double mirrors.* This terminology is motivated by the following result, which appears as an example in forthcoming joint work with Alexander Kuznetsov.

Theorem 1 ([8])**.** *If X and Y are GPK^3 double mirrors, then there is an equivalence $\mathrm{D^b}(X) \simeq \mathrm{D^b}(Y)$ of bounded derived categories of coherent sheaves.*

Our main result says that, nonetheless, X and Y are typically not birational.

Theorem 2 ([3])**.** *For generic $g_1, g_2 \in \mathrm{PGL}(W)$, the varieties X and Y are not birational.*

Theorem 2 was also independently proved by John Ottem and Jørgen Rennemo [9]. Before explaining the main idea of our proof, we discuss some applications and auxiliary results.

Applications. Generic GPK^3 double mirrors give the first example of deformation equivalent, derived equivalent, but non-birational Calabi–Yau threefolds. By an observation from [1], a derived equivalence of complex Calabi–Yau threefolds induces an isomorphism of integral polarized Hodge structures on third cohomology. Thus we obtain:

Corollary 3 ([3])**.** *Generic complex GPK^3 double mirrors give a counterexample to the birational Torelli problem for Calabi–Yau threefolds.*

Previously, Szendrői [10] showed the usual Torelli problem fails for Calabi–Yau threefolds, but the birational version was open until our result.

As a second application of Theorem 2, we prove the following.

Theorem 4 ([3])**.** *If X and Y are GPK^3 double mirrors, then:*

(1) *In the Grothendieck ring* $\mathrm{K}_0(\mathrm{Var}/k)$ *of k-varieties, we have*

$$([X]-[Y])\mathbf{L}^4 = 0,$$

where $\mathbf{L} = [\mathbf{A}^1]$ *is the class of the affine line.*

(2) *If the elements* $g_1, g_2 \in \mathrm{PGL}(W)$ *defining X and Y are generic, then*

$$[X] \neq [Y].$$

This adds to the growing list of examples, begun by [2], of derived equivalent varieties whose difference in the Grothendieck ring is annihilated by a power of $\mathbf{L}$. Part (1) is proved by studying a certain incidence correspondence, and part (2) is an easy consequence of Theorem 2.

Geometry and moduli of GPK^3 threefolds. Our proof of Theorem 2 involves several independently interesting results on the geometry and moduli of GPK^3 threefolds. The main result about the geometry of these threefolds is the following.

Proposition 5 ([3])**.** *The two* $\mathrm{Gr}(2, V)$ *translates containing a GPK^3 threefold X are unique.*

This is proved by studying the restriction to X of the normal bundles of the translates $g_i(\mathrm{Gr}(2, V)) \subset \mathbf{P}(W)$; the key insight is that these are slope stable vector bundles on X, whose isomorphism class determines $g_i(\mathrm{Gr}(2, V)) \subset \mathbf{P}(W)$. Using Proposition 5, we obtain an explicit description of the automorphism group of X.

In terms of moduli, we consider two spaces: the *moduli stack $\mathcal{N}$ of GPK^3 data*, defined as a $\mathbf{Z}/2 \times \mathrm{PGL}(W)$-quotient of the space of pairs of $\mathrm{Gr}(2, V)$ translates in $\mathbf{P}(W)$ whose intersection is a smooth threefold (where $\mathbf{Z}/2$ swaps the two translates); and the *moduli stack $\mathcal{M}$ of GPK^3 threefolds*, defined as a $\mathrm{PGL}(W)$-quotient of an open subscheme of the appropriate Hilbert scheme. There is a natural morphism $f : \mathcal{N} \to \mathcal{M}$ given pointwise by intersecting the two $\mathrm{Gr}(2, V)$ translates.

Theorem 6 ([3])**.** *The morphism $f : \mathcal{N} \to \mathcal{M}$ is an open immersion of smooth separated Deligne–Mumford stacks of finite type over k.*

For this, the main step is showing that the derivative of f at any point is an isomorphism.

Theorem 7 ([3])**.** *The automorphism group of any geometric point $s \in \mathcal{N}$ acts faithfully on the tangent space $\mathrm{T}_s\mathcal{N}$. Moreover, if $1 \neq \gamma \in \mathrm{Aut}_{\mathcal{N}}(s)$ is an involution, then the trace of the induced element $\gamma_* \in \mathrm{GL}(\mathrm{T}_s\mathcal{N})$ satisfies*

$$\mathrm{tr}(\gamma_*) \in \{3, 1, -3, -5, -13, -15, -35\}.$$

This is proved by a careful analysis of the eigenvalues of the action on $\mathrm{T}_s\mathcal{N}$, which uses our description of the automorphism groups of GPK^3 threefolds.

The involution of $\mathrm{PGL}(W) \times \mathrm{PGL}(W)$ given by $(g_1, g_2) \mapsto (g_1^{-T}, g_2^{-T})$ descends to the *double mirror involution* $\tau : \mathcal{N} \to \mathcal{N}$. In these terms, our proof of Theorem 2 boils down to the following infinitesimal claim: there exists a fixed point $s \in \mathcal{N}$

of τ such that the derivative $d_s\tau \in GL(T_s\mathcal{N})$ is not contained in the image of the homomorphism $\mathrm{Aut}_{\mathcal{N}}(s) \to GL(T_s\mathcal{N})$. For this, we exhibit an explicit fixed point s such that $\mathrm{tr}(d_s\tau)$ does not occur in the list of traces from Theorem 7.

References

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