Derived categories of Gushel–Mukai varieties

Alexander Perry

(joint work with Alexander Kuznetsov)

Background. We work over an algebraically closed field $\mathbf{k}$ of characteristic 0. Let V_5 be a 5-dimensional vector space. A *Gushel–Mukai (GM) variety* is a smooth n-dimensional intersection

$$X = \mathrm{CGr}(2, V_5) \cap \mathbf{P}^{n+4} \cap Q, \qquad 2 \leq n \leq 6,$$

where $\mathrm{CGr}(2, V_5) \subset \mathbf{P}^{10}$ is the cone over the Grassmannian $\mathrm{Gr}(2, V_5) \subset \mathbf{P}(\wedge^2 V_5)$ in its Plücker embedding, $\mathbf{P}^{n+4} \subset \mathbf{P}^{10}$ is a linear subspace, and $Q \subset \mathbf{P}^{n+4}$ is a quadric hypersurface. Results of Gushel [3] and Mukai [7], recently revisited and generalized in [2], show that this class of varieties coincides with the class of all smooth Fano varieties of Picard number 1, coindex 3, and degree 10, together with the Brill–Noether general polarized K3 surfaces of degree 10.

In dimension four, GM varieties behave very similarly to cubic fourfolds. For instance, both types of fourfolds are unirational, but conjecturally irrational if very general. Further, for fourfolds of either type, there are certain Noether–Lefschetz loci where the "non-special cohomology" is isomorphic to (a Tate twist of) the primitive cohomology of a polarized K3 surface (see [4], [1]). In fact, this condition is conjecturally necessary for rationality.

For a cubic fourfold X', rationality is also connected to the structure of the bounded derived category of coherent sheaves on X', denoted $\mathrm{D}^{\mathrm{b}}(X')$. Namely, Kuznetsov [5] showed there is a semiorthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}(X') = \langle \mathcal{A}_{X'}, \mathcal{O}_{X'}, \mathcal{O}_{X'}(1), \mathcal{O}_{X'}(2) \rangle$$

where $\mathcal{A}_{X'}$ is a "K3 category", i.e. has Serre functor $\mathrm{S}_{\mathcal{A}_{X'}} = [2]$ given by the shift-by-2 functor. Kuznetsov conjectured that if X' is rational, then $\mathcal{A}_{X'}$ is equivalent to the derived category of a K3 surface. Further, he proved that this condition holds for the known families of rational cubic fourfolds, but does not hold for a very general cubic.

Results. Our work extends the parallel between GM and cubic fourfolds to the level of derived categories. In fact, for any GM variety X (not necessarily a fourfold) we defined a special subcategory of $\mathrm{D}^{\mathrm{b}}(X)$ as follows. Projection from the vertex of $\mathrm{CGr}(2, V_5)$ gives a morphism $f \colon X \to \mathrm{Gr}(2, V_5)$, and pulling back the ample generator of the Picard group and the rank 2 tautological subbundle on $\mathrm{Gr}(2, V_5)$ gives bundles $\mathcal{O}_X(1)$ and $\mathcal{U}_X$ on X. We showed there is a semiorthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{U}_X^\vee, \mathcal{O}_X(1), \mathcal{U}_X^\vee(1), \dots, \mathcal{O}_X(n-3), \mathcal{U}_X^\vee(n-3) \rangle,$$

where $n = \dim(X)$. The following result gathers some of the basic properties of the category $\mathcal{A}_X$.

Theorem 1 ([6])**.** *Let X be an n-dimensional GM variety.*

(1) *The Serre functor of* $\mathcal{A}_X$ *is given by* $\mathrm{S}_{\mathcal{A}_X} = [2]$ *if* n *is even, and by* $\mathrm{S}_{\mathcal{A}_X} = \sigma \circ [2]$ *for an involutive autoequivalence* σ *of* $\mathcal{A}_X$ *if* n *is odd.*

(2) *The Hochschild homology of* $\mathcal{A}_X$ *is given by*

$$\mathrm{HH}_\bullet(\mathcal{A}_X) \cong \begin{cases} \mathbf{k}[2] \oplus \mathbf{k}^{22}[0] \oplus \mathbf{k}[-2] & \text{if } n \text{ is even,} \\ \mathbf{k}^{10}[1] \oplus \mathbf{k}^{2}[0] \oplus \mathbf{k}^{10}[-1] & \text{if } n \text{ is odd.} \end{cases}$$

(3) *Suppose either that* $n \geq 4$ *is even and* X *is very general, or that* n *is odd. Then* $\mathcal{A}_X$ *is not equivalent to the derived category of any variety.*

Part (1) of the theorem shows that, in terms of its Serre functor, $\mathcal{A}_X$ behaves like the derived category of a K3 or Enriques surface according to whether n is even or odd. Part (2) shows that this analogy persists at the level of Hochschild homology if n is even, but breaks down if n is odd (the Hochschild homology agrees with that of a K3 surface if n is even, but with that of a genus 10 curve if n is odd). This discrepancy when n is odd implies part (3) for such n, whereas the result for even n follows by relating it to the existence of algebraic cycles on X and using the period map.

In dimension four, however, we proved that $\mathcal{A}_X$ does sometimes coincide with the derived category of a K3 surface. To explain this, we need some terminology. A GM variety X is called *ordinary* if the linear space $\mathbf{P}^{n+4}$ appearing in the intersection defining X does not contain the vertex of $\mathrm{CGr}(2, V_5)$. Equivalently, X can be expressed as $X = \mathrm{Gr}(2, V_5) \cap Q$, where $Q \subset \mathbf{P}^{n+4}$ is a quadric hypersurface inside a linear subspace $\mathbf{P}^{n+4} \subset \mathbf{P}(\wedge^2 V_5)$.

Theorem 2 ([6])**.** *Let* X *be an ordinary GM fourfold which can be expressed as* $X = \mathrm{Gr}(2, V_5) \cap Q$ *for a rank* 6 *quadric* $Q \subset \mathbf{P}^8 \subset \mathbf{P}(\wedge^2 V_5)$*. Then there is a K3 surface* Y *such that* $\mathcal{A}_X \simeq \mathrm{D^b}(Y)$*.*

The fourfolds in the theorem form a 23-dimensional family (a divisor in moduli), and can be characterized as those ordinary GM fourfolds containing a quintic del Pezzo surface. The K3 surface Y is actually a GM surface, given explicitly by $Y = \mathrm{Gr}(2, V_5^\vee) \cap Q^\vee$, where $Q^\vee$ is the quadric in $\mathbf{P}^6 = \mathbf{P}(\ker(Q)^\perp) \subset \mathbf{P}(\wedge^2 V_5^\vee)$ projectively dual to $Q \subset \mathbf{P}(\wedge^2 V_5)$. In fact, Theorem 2 verifies a special case of a duality conjecture that we formulated, which gives equivalences between the categories $\mathcal{A}_X$ for GM varieties of possibly different dimensions. Further, we note that fourfolds as in the theorem are rational. Hence the result can be considered as evidence for the GM analogue of Kuznetsov's rationality conjecture for cubic fourfolds.

Our final result directly connects the K3 categories of GM and cubic fourfolds.

Theorem 3 ([6])**.** *Let* X *be a generic GM fourfold containing a plane of the form* $\mathrm{Gr}(2, V_3) \subset \mathrm{Gr}(2, V_5)$ *for a* 3*-dimensional subspace* $V_3 \subset V_5$*. Then there is a cubic fourfold* X' *such that* $\mathcal{A}_X \simeq \mathcal{A}_{X'}$*.*

The GM fourfolds in the theorem form a 21-dimensional family (codimension 3 in moduli). Projection from the plane $\mathrm{Gr}(2, V_3)$ maps the GM fourfold X birationally onto the cubic fourfold X', and we use the structure of this birational isomorphism to establish the result. We remark that if X is a very general fourfold satisfying the assumption of the theorem, then $\mathcal{A}_X$ is *not* equivalent to the derived category of a K3 surface; thus Theorem 3 is of an essentially different nature than Theorem 2.

References

[1] Olivier Debarre, Atanas Iliev, and Laurent Manivel, *Special prime Fano fourfolds of degree 10 and index 2*, Recent advances in algebraic geometry, London Math. Soc. Lecture Note Ser., vol. 417, Cambridge Univ. Press, Cambridge, 2015, pp. 123–155.

[2] Olivier Debarre and Alexander Kuznetsov, *Gushel–Mukai varieties: classification and birationalities*, arXiv preprint arXiv:1510.05448 (2016).

[3] N. P. Gushel', *On Fano varieties of genus 6*, Izv. Math. **21** (1983), no. 3, 445–459.

[4] Brendan Hassett, *Special cubic fourfolds*, Compositio Math. **120** (2000), no. 1, 1–23.

[5] Alexander Kuznetsov, *Derived categories of cubic fourfolds*, Cohomological and geometric approaches to rationality problems, Progr. Math., vol. 282, Birkhäuser Boston, Inc., Boston, MA, 2010, pp. 219–243.

[6] Alexander Kuznetsov and Alexander Perry, *Derived categories of Gushel–Mukai varieties*, arXiv preprint arXiv:1605.06568 (2016).

[7] Shigeru Mukai, *Biregular classification of Fano* 3*-folds and Fano manifolds of coindex* 3, Proc. Nat. Acad. Sci. U.S.A. **86** (1989), no. 9, 3000–3002.