Math 592: Algebraic Topology, Winter 2023

Problem Set 9

Due Monday, March 27, 2023 at 11:59pm

1. (a) Let $f, g\colon (X, A) \to (Y, B)$ be maps of pairs. As we saw in class, there are induced morphisms of singular chain complexes $f_\sharp, g_\sharp\colon C_\bullet(X, A) \to C_\bullet(Y, B)$. Show that $f_\sharp$ and $g_\sharp$ are homotopic as maps of chain complexes if f and g are *homotopic as maps of pairs*, i.e. if there exists a homotopy $H\colon X \times I \to Y$ from f to g such that $H_t(A) \subset B$ for all $t \in I$. Conclude that if f and g are homotopic as maps of pairs, then the induced maps $f_*, g_*\colon \mathrm{H}_n(X, A) \to \mathrm{H}_n(Y, B)$ on relative homology are equal.

   (b) Let $B \subset A$ be a subspace such that B is a strong deformation retract of A. Show that $\mathrm{H}_n(A, B) = 0$ for all n.

   (c) Let X be a space with subspaces $B \subset A \subset X$. Show that there is a long exact sequence associated to the triple (X, A, B), relating their relative homologies, of the following form:
   $$\cdots \to \mathrm{H}_n(A, B) \to \mathrm{H}_n(X, B) \to \mathrm{H}_n(X, A) \xrightarrow{\delta} \mathrm{H}_{n-1}(A, B) \to \cdots$$
   such that when $B = \varnothing$ this recovers the long exact sequence of the pair (X, A).

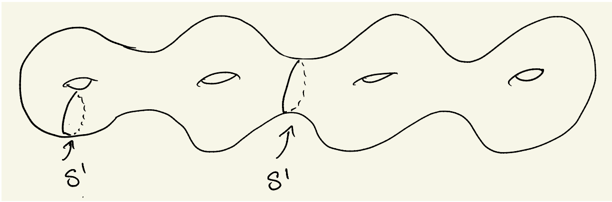
   (d) Continue to assume that (X, A, B) is a triple with $B \subset A \subset X$. Prove that if B is a strong deformation retract of A, then the map $\mathrm{H}_n(X, B) \to \mathrm{H}_n(X, A)$ induced by the inclusion $(X, B) \to (X, A)$ is an isomorphism for all n.

   Remark. In class, we used the above isomorphism in our proof that when (X, A) is a good pair then $\mathrm{H}_n(X, A) \cong \widetilde{\mathrm{H}}_n(X/A)$.

2. Let
$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_\bullet & \longrightarrow & B_\bullet & \longrightarrow & C_\bullet & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle\alpha} & & \big\downarrow{\scriptstyle\beta} & & \big\downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & A'_\bullet & \longrightarrow & B'_\bullet & \longrightarrow & C'_\bullet & \longrightarrow & 0 \end{array}$$
be a commutative diagram of morphisms of chain complexes, where the rows are short exact sequences. Show that there is a commutative diagram relating the long exact sequences associated to the rows:
$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & \mathrm{H}_n(A_\bullet) & \longrightarrow & \mathrm{H}_n(B_\bullet) & \longrightarrow & \mathrm{H}_n(C_\bullet) & \xrightarrow{\delta} & \mathrm{H}_{n-1}(A_\bullet) & \longrightarrow & \cdots \\ & & {\scriptstyle\alpha_*}\big\downarrow & & {\scriptstyle\beta_*}\big\downarrow & & {\scriptstyle\gamma_*}\big\downarrow & & {\scriptstyle\alpha_*}\big\downarrow & & \\ \cdots & \longrightarrow & \mathrm{H}_n(A'_\bullet) & \longrightarrow & \mathrm{H}_n(B'_\bullet) & \longrightarrow & \mathrm{H}_n(C'_\bullet) & \xrightarrow{\delta} & \mathrm{H}_{n-1}(A'_\bullet) & \longrightarrow & \cdots \end{array}$$

3. Let Σ_g be a compact orientable genus g surface. By choosing a suitable cover $\Sigma_g = A \cup B$ by open sets and using excision and a long exact sequence, compute the homology groups $\mathrm{H}_n(\Sigma_g)$ for all n.

4. Let X be the space obtained from a compact orientable genus 4 surface by attaching a copy of D^2 along the identity map $\partial D^2 = S^1 \to S^1$ for each of the copies of S^1 shown below:



Compute $\mathrm{H}_n(X)$ for all n.

5. (a) Show that $X = S^1 \times S^1$ and $Y = S^1 \vee S^1 \vee S^2$ satisfy $\mathrm{H}_n(X) \cong \mathrm{H}_n(Y)$ for all n.

   (b) Show that X and Y are *not* homotopy equivalent.

   (c) Let $\tilde{X}$ and $\tilde{Y}$ be the universal covers of X and Y. Compute the homology groups $\mathrm{H}_n(\tilde{X})$ and $\mathrm{H}_n(\tilde{Y})$. (In particular, you will see that homology groups of $\tilde{X}$ and $\tilde{Y}$ are not the same, despite this being true for X and Y.)

6. Note that $\mathrm{H}_n(\Delta^n, \partial\Delta^n) \cong \widetilde{\mathrm{H}}_n(\Delta^n/\partial\Delta^n) \cong \mathbf{Z}$, since $\Delta^n/\partial\Delta^n \cong S^n$. Let $\sigma \in C_n(\Delta^n, \partial\Delta^n)$ be the n-cycle given by the identity map $\sigma = \mathrm{id}\colon \Delta^n \to \Delta^n$. Read the proof from Example 2.23 in Hatcher that its class in homology $[\sigma] \in \mathrm{H}_n(\Delta^n, \partial\Delta^n) \cong \mathbf{Z}$ is a generator.

   Convince yourself that this fills in the omitted detail in our proof from class that singular and simplicial homology are isomorphic; namely, in the notation from class, show that the map $\mathrm{H}_n^{\Delta}(X^k, X^{k-1}) \to \mathrm{H}_n(X^k, X^{k-1})$ is an isomorphism, by using the previous paragraph and the descriptions of $\mathrm{H}_n^{\Delta}(X^k, X^{k-1})$ and $\mathrm{H}_n(X^k, X^{k-1})$ from class.

   You do not need to submit anything for this problem.

7. Read the proof of Proposition 2.21 in Hatcher, which was left as a blackbox in our proof of excision in class.

   You do not need to submit anything for this problem, and it is optional in the sense that accepting it as a blackbox should not affect your understanding of the rest of the material in the class.