Math 592: Algebraic Topology, Winter 2023

Problem Set 8

Due Monday, March 20, 2023 at 11:59pm

1. Verify the details of the construction of the long exact sequence on homology induced by a short exact sequence of chain complexes. Try to do this yourself, but if you get stuck you can also read this in Theorem 2.16 of Hatcher. You do not need to turn anything in for this problem.

2. Look up the statement of the 5-lemma (see for instance page 129 of Hatcher), and write out the proof. Try to do this yourself, without reading the proof from the book.

3. Let $0 \to A \to B \to C \to 0$ be a short exact sequence of abelian groups.

   (a) Show that if C is a free abelian group (i.e. isomorphic to a direct sum of copies of $\mathbf{Z}$), then there exists an isomorphism $B \cong A \oplus C$.

   (b) Give an example showing that the previous statement can fail if C is not a free abelian group.

   (c) Determine up to isomorphism all groups B that fit into an exact sequence as above with $A = \mathbf{Z}/6$ and $C = \mathbf{Z}/2$.

4. Prove the following basic results about the reduced homology of a space X:

   (a) If $X = \varnothing$ is empty, then

   $$\mathrm{H}_n(X) = 0 \text{ for all } n, \quad \text{and} \quad \widetilde{\mathrm{H}}_n(X) \cong \begin{cases} 0 & n \neq -1 \\ \mathbf{Z} & n = -1. \end{cases}$$

   (b) If $X \neq \varnothing$, then for $n \neq 0$ there is an isomorphism

   $$\widetilde{\mathrm{H}}_n(X) \cong \mathrm{H}_n(X).$$

   For $n = 0$, there is a short exact sequence

   $$0 \to \widetilde{\mathrm{H}}_0(X) \to \mathrm{H}_0(X) \to \mathbf{Z} \to 0,$$

   and an isomorphism $\mathrm{H}_0(X) \cong \widetilde{\mathrm{H}}_0(X) \oplus \mathbf{Z}$.

   (c) For any n, show that there is a functor $\widetilde{\mathrm{H}}_n \colon \mathrm{hTop} \to \mathrm{Ab}$ which on objects is given by $X \mapsto \widetilde{\mathrm{H}}_n(X)$ and on morphisms is defined similarly to the functor $\mathrm{H}_n \colon \mathrm{hTop} \to \mathrm{Ab}$ for (unreduced) homology.

   (d) If X is path-connected, then $\widetilde{\mathrm{H}}_0(X) = 0$.

   (e) If X is contractible, then $\widetilde{\mathrm{H}}_n(X) = 0$ for all n.

   (f) Assume X is nonempty and let $x_0 \in X$. Express the relative homology $\mathrm{H}_n(X, \{x_0\})$ in terms of the (reduced) homology of X.

5. Let $f\colon X \to Y$ be a map of path-connected spaces, and let $x_0 \in X$ and $y_0 = f(x_0) \in Y$. Then we have an induced homomorphism $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$.

   (a) Explain why the map f_* induces a homomorphism $f_*^{\mathrm{ab}}\colon \pi_1(X, x_0)^{\mathrm{ab}} \to \pi_1(Y, y_0)^{\mathrm{ab}}$ on the abelianizations.

   (b) Show that there is a commutative diagram

$$\begin{tikzcd} \pi_1(X, x_0)^{\mathrm{ab}} \arrow[r, "f_*^{\mathrm{ab}}"] \arrow[d, "\sim"'] & \pi_1(Y, y_0)^{\mathrm{ab}} \arrow[d, "\sim"] \\ \mathrm{H}_1(X) \arrow[r, "f_*"] & \mathrm{H}_1(Y) \end{tikzcd}$$

   where the vertical arrows are the isomorphisms we constructed in class, and the bottom arrow is the map induced by f on H_1.

6. Let $n \geq 1$, let $A \subset S^n$ be a finite set of points, and let $X = S^n/A$ be the quotient space collapsing A to a point. Compute the singular homology groups $\mathrm{H}_i(X)$ for all i.

7. Let $X = (S^1 \times S^1)/(S^1 \times \{1\})$ be the quotient space obtained from a torus by collapsing $S^1 \times \{1\}$ to a point. Compute the singular homology groups $\mathrm{H}_n(X)$ for all n.