Math 592: Algebraic Topology, Winter 2023

Problem Set 7

Due Monday, March 13, 2023 at 11:59pm

1. For each of the following spaces, describe a Δ-complex structure and use it to compute the simplicial homology:
   (a) $S^1 \times S^1$
   (b) $\mathbf{RP}^2$
   (c) D^2
   (d) The Klein bottle
   (e) The space obtained from $\Delta^2 = [v_0, v_1, v_2]$ by identifying all of the vertices v_0, v_1, v_2 to one point.

2. Recall the following theorem from algebra.

   Theorem 1 (Smith normal form)**.** *Let A be an $m \times n$ matrix with integer*[1] *entries. Then there exist an invertible $m \times m$ and $n \times n$ integer matrices S and T such that*

   $$SAT = \operatorname{diag}(a_1, a_2, \ldots, a_r, 0, \ldots, 0)$$

   is a diagonal matrix with diagonal entries $a_1, a_2, \ldots, a_r, 0, \ldots, 0$ such that a_i divides a_{i+1} for $1 \leq i \leq r-1$. The integer r is called the rank *of A, and the elements $a_1, \ldots, a_r$ are unique up to multiplication by ± 1 and called the* invariant factors *of A.*

   Suppose that B is an $\ell \times m$ integer matrix such that $BA = 0$. Note that we may think of A and B as maps of abelian groups $A \colon \mathbf{Z}^n \to \mathbf{Z}^m$ and $B \colon \mathbf{Z}^m \to \mathbf{Z}^\ell$, so the condition $BA = 0$ says that the sequence

   $$\mathbf{Z}^n \xrightarrow{A} \mathbf{Z}^m \xrightarrow{B} \mathbf{Z}^\ell$$

   can be thought of as a complex $C_\bullet$ of abelian groups, with $C_1 = \mathbf{Z}^n$, $C_0 = \mathbf{Z}^m$, $C_{-1} = \mathbf{Z}^\ell$, and all other $C_i = 0$.

   Compute the homology $\mathrm{H}_0(C_\bullet)$ in terms of m, the ranks of A and B, and the invariant factors of A and B.

   Remark 2. Since Smith normal form can be computed algorithmically, this gives an algorithm for computing the homology of complexes whose terms are finitely generated free abelian groups, like the kind that show up in computing the simplicial homology of a Δ-complex with finitely many simplices in each degree.

[1] There is an analogous version of the theorem for matrices over any principal ideal domain.

3. For a space X with basepoint x_0, in class we constructed a homomorphism

$$\psi \colon \pi_1(X, x_0)^{\mathrm{ab}} \to \mathrm{H}_1(X).$$

   If X is path-connected, then ψ is an isomorphism. We proved the surjectivity in class; read the proof of injectivity from Section 2.A of Hatcher. You do not need to submit anything for this problem.

4. Let X be a connected graph with finitely many vertices and edges, say V vertices and E edges. State and prove a formula for the singular homology of X in terms of V and E.

   You may use the fact (which we will eventually prove in class) that singular and simplicial homology are isomorphic.

5. Let D^3 be the closed 3-disk, with boundary $\partial D^3 = S^2$. Prove that there does not exist a retraction $r \colon D^3 \to S^2$, and use this to prove that any continuous map $D^3 \to D^3$ has a fixed point.

   You may use the fact (which we will eventually prove in class) that singular and simplicial homology are isomorphic.