Math 592: Algebraic Topology, Winter 2023

Problem Set 6

Due Monday, February 20, 2023 at 11:59pm

1. Let $p\colon Z \to X$ and $q\colon Y \to X$ be covering spaces, where Z and Y are path-connected and X is locally path connected. Suppose that $f\colon Z \to Y$ is a morphism of covering spaces, i.e. $p = q \circ f$. Prove that $f\colon Z \to Y$ is a covering space.

2. Determine explicitly all path-connected covering spaces of $\mathbf{RP}^2 \times \mathbf{RP}^2$ up to isomorphism of covering spaces (without basepoints).

3. Let $X = S^1 \vee S^1$. Determine explicitly all path-connected covering spaces $\tilde{X} \to X$ of degree 2 and degree 3, up to isomorphism of covering spaces (without basepoints). For each of these covering spaces, determine whether or not it is Galois[1].

4. For an integer $n \geq 1$, let F_n denote the free group on a set of n elements. We call n the *rank* of F_n (which is also the rank of its abelianization and hence an intrinsic invariant of the group F_n).

   (a) Prove that F_4 does not have a finite-index subgroup isomorphic to F_8.

   (b) Construct a finite-index subgroup H of F_4 which is isomorphic to F_7, determine a free generating set of elements for H, and decide whether H is normal.

5. Let X be a path-connected, locally path-connected, semilocally simply-connected space. We say that a path-connected covering space $\tilde{X} \to X$ is *abelian* if it is Galois and the group $\mathrm{Aut}(\tilde{X}/X)$ is abelian.

   (a) Prove that there exists an abelian covering space $p\colon \tilde{X}^{\mathrm{ab}} \to X$, unique up to isomorphism, satisfying the following property: for any abelian covering space $q\colon Y \to X$, there exists a morphism $f\colon \tilde{X}^{\mathrm{ab}} \to Y$ of covering spaces of X (i.e. $p = q \circ f$). We call $\tilde{X}^{\mathrm{ab}}$ the *universal abelian cover* of X.

   (b) Describe explicitly the universal abelian cover of $S^1 \vee S^1$.

6. (a) Let X be a Hausdorff space. Let G be a finite group which acts on X freely and continuously. Prove that the quotient map $X \to X/G$ is a covering space.

   Hint: Verify that the criterion from Problem 2 on Problem Set 3 holds.

   (b) Consider the 3-sphere, regarded as a subspace $S^3 \subset \mathbf{C}^2 \cong \mathbf{R}^4$. For fixed coprime integers p and q, verify that there is a free continuous action of $\mathbf{Z}/p$ on S^3, where $1 \in \mathbf{Z}/p$ acts by $(z_1, z_2) \mapsto (e^{2\pi i/p} z_1, e^{2\pi i q/p} z_2)$ for $(z_1, z_2) \in S^3 \subset \mathbf{C}^2$. Let M be the quotient of S^3 by this action of $\mathbf{Z}/p$. Compute $\pi_1(M)$.

[1] Recall Hatcher uses the terminology "normal covering space" for what we call a "Galois covering space".