Math 592: Algebraic Topology, Winter 2023

Problem Set 5

Due Monday, February 13, 2023 at 8pm

1. **(True/False on covering spaces).** For each of the following statements, prove if it is true or give a counterexample if it is false.
   (a) If $p\colon \tilde{X} \to X$ is a covering space and X is path-connected, then so is $\tilde{X}$.
   (b) If $p\colon \tilde{X} \to X$ is a covering space, $Y \subset X$ is a subspace, and $\tilde{Y} = p^{-1}(Y)$, then $p|_{\tilde{Y}}\colon \tilde{Y} \to Y$ is a covering space.
   (c) A covering space is an open map.
   (d) A covering space $p\colon \tilde{X} \to X$ is a local homeomorphism, i.e. for each $x \in \tilde{X}$ there is an open neighborhood such that p induces a homeomorphism $U \cong p(U)$.
   (e) A local homeomorphism is a covering space.

2. **(Subgroups of free groups are free).**
   (a) Let X be a graph, i.e. a 1-dimensional CW complex. Let $p\colon \tilde{X} \to X$ be a covering space. Show that $\tilde{X}$ is also a graph.
   (b) Using the preceding statement, show that every subgroup of a free group is a free group.

3. **(Large subgroups of free groups).** Let F_2 be the free group on 2 elements, and let F_∞ the free group on a countably infinite collection of elements. Prove that there is an injective group homomorphism $F_\infty \to F_2$.

4. **(Degree of a covering space in terms of its subgroup).** Let $p\colon \tilde{X} \to X$ be a covering space with $\tilde{X}$ and X path-connected. Let $\tilde{x}_0 \in \tilde{X}$, let $x_0 = p(\tilde{x}_0)$, and let $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \subset \pi_1(X, x_0)$. Determine the degree of $p\colon \tilde{X} \to X$ (as defined in Problem Set 3, #1) in terms of the subgroup $H \subset \pi_1(X, x_0)$.

5. **(Example of a universal cover).** Construct explicitly a covering space of $T^2 \vee S^2$ which is simply connected, where T^2 denotes the torus.

6. **(Maps to graphs).** Let G be a graph, i.e. a 1-dimensional CW complex. Prove that if X is a path-connected, locally path-connected space such that $\pi_1(X)$ is finite, then any continuous map $X \to G$ is *nullhomotopic*, i.e. homotopic to a constant map.