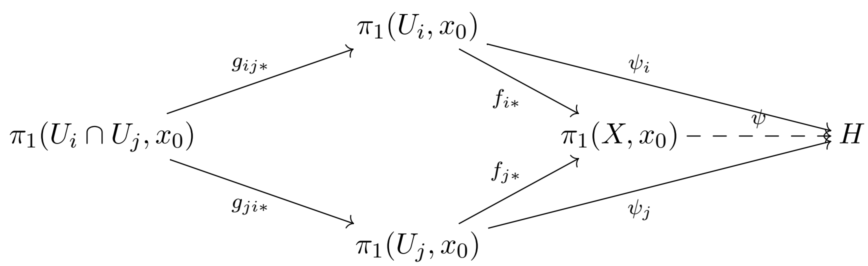
Math 592: Algebraic Topology, Winter 2023

Problem Set 4

Due Monday, February 6, 2023 at 8pm

1. **(Proof of Van Kampen's theorem).** The purpose of this problem is to complete the proof of the second part of Van Kampen's theorem from class, which states:

Theorem (Van Kampen)**.** *Let X be a topological space, $x_0 \in X$ a point, and $X = \bigcup_{i \in \mathcal{I}} U_i$ an open cover such that $x_0 \in U_i$ for every $i \in \mathcal{I}$. Assume that U_i, $U_i \cap U_j$, and $U_i \cap U_j \cap U_k$ are path-connected for every $i, j, k \in \mathcal{I}$. Let $g_{ij}\colon U_i \cap U_j \to U_i$ and $f_i\colon U_i \to X$ be the inclusions. If we are given for every $i, j \in \mathcal{I}$ a solid commutative diagram of groups*



then there exists a unique dotted arrow ψ which makes the diagram commute for all $i, j \in \mathcal{I}$.

Let us recall the setup for the proof of this result from class. Let $\alpha\colon I \to X$ be a loop at x_0. Then we may choose:

- a partition $0 = s_0 < s_1 < \cdots < s_m = 1$ of I and for $1 \le k \le m$ an element $i_k \in \mathcal{I}$ such that $\alpha([s_{k-1}, s_k]) \subset U_{i_k}$; and
- γ_k a path in $U_{i_k} \cap U_{i_{k+1}}$ from x_0 to $x_k := \alpha(s_k)$ for $1 \le k \le m-1$.

Given these choices, we let $\alpha_k\colon I \to X$ be the path $s \mapsto \alpha((1-s)s_{k-1} + s s_k))$ (i.e. $\alpha|_{[s_{k-1}, s_k]}$ reparameterized to have domain I). Then

$$[\alpha] = [\gamma_0 \cdot \alpha_1 \cdot \overline{\gamma_1}][\gamma_1 \cdot \alpha_2 \cdot \overline{\gamma_2}] \cdots [\gamma_{m-1} \cdot \alpha_m \cdot \overline{\gamma_m}]$$

is an expression of $[\alpha] \in \pi_1(X, x_0)$ as a product of loops contained in the U_{i_k}, where we set $\gamma_0 = \gamma_m$ to be the constant loop at x_0. Let $h_k = \psi_{i_k}([\gamma_{k-1} \cdot \alpha_k \cdot \overline{\gamma_k}]) \in H$, and let

$$h_\alpha = h_1 h_2 \cdots h_m \in H.$$

In class we reduced Van Kampen's theorem to two statements, which you are asked to prove below.

(a) Prove that h_α is independent of the choice of subdivision $0 = s_0 < \cdots < s_m = 1$ in the above construction. (Recall that we already proved in class that h_α is independent of the choice of $i_k \in \mathcal{I}$ and γ_k.)

(b) Prove that $h_\alpha = h_{\alpha'}$ if α is path homotopy equivalent to α'.
Hint: Let $H\colon I \times I \to X$ be a path homotopy from α to α'. First show that there exists $\delta > 0$ such that for every $(s,t) \in I \times I$, there exists $i \in \mathcal{I}$ such that

$$H(B_\delta(s,t)) \subset U_i$$

where $B_\delta(s,t) = \{(s',t') \in I \times I \mid \|(s-s', t-t')\| < \delta\}$ denotes the open ball of radius δ at (s,t) in $I \times I$. Conclude that there exists $\epsilon > 0$ such that for any $t_0, t_1 \in I$ with $0 < t_1 - t_0 < \epsilon$, we can choose a partition $0 = s_0 < s_1 < \cdots < s_m = 1$ such that for all $1 \leq k \leq m$ there exists $i_k \in \mathcal{I}$ such that $H([s_{k-1}, s_k] \times [t_0, t_1]) \subset U_{i_k}$. Note that H_{t_0} and H_{t_1} are loops in X at x_0 which are path homotopic, so we can consider the elements $h_{H_{t_0}}, h_{H_{t_1}} \in H$ constructed above; prove that they are equal, and conclude $h_\alpha = h_{\alpha'}$.

2. **(Fundamental theorem of algebra).** Read the proof of Theorem 1.8 in Hatcher. You do not need to submit anything for this problem.

3. **(Homotopy type of orientable surfaces).** Let Σ_g and Σ_h denote compact orientable surfaces of genus g and h. Prove that if $g \neq h$, then Σ_g and Σ_h are not homotopy equivalent.

4. **(CW decomposition of an orientable surface).** For each $g \geq 0$, determine with proof the minimal possible number of cells in a CW decomposition of a compact orientable surface of genus g.

5. **(π_1 of quotient of edges of a cube).** Let Y be the space consisting of all of the edges and vertices of the cube $I^3 \subset \mathbf{R}^3$. Let X be the space obtained from Y by identifying $I \times \{0\} \times \{0\}$ with $I \times \{0\} \times \{1\}$. Compute $\pi_1(X)$.

6. **(π_1 of gluing of Mobius strips).** Let M be a Mobius strip. Note that the boundary of M is a copy of S^1.

   (a) Let X be the space obtained by identifying the boundaries of two copies of M. Compute a presentation for $\pi_1(X)$.

   (b) Let Y be the space obtained by identifying the boundaries of three copies of M. Compute a presentation for $\pi_1(Y)$.