Math 592: Algebraic Topology, Winter 2023

Problem Set 3

Due Monday, January 30, 2023 at 8pm

1. **(Degree of covering space).** Let $p\colon \tilde{X} \to X$ be a covering space. Prove that the function $X \to \mathbf{Z}_{\geq 0} \cup \{\infty\}$ given by $x \mapsto |p^{-1}(x)|$ (where $|p^{-1}(x)|$ denotes the cardinality of $p^{-1}(x)$) is constant on connected components of X. In particular, if X is connected then $|p^{-1}(x)|$ does not depend on $x \in X$, and is known as the *degree* of the covering space $p\colon \tilde{X} \to X$.

2. **(Covering spaces from group quotients).** Let G be a group equipped with the discrete topology. Let X be a topological space. We say that G *acts continuously* on X if we are given an action of the group G on the set X (in the usual sense) such that the action map $G \times X \to X, (g, x) \mapsto g(x)$ is continuous.

   (a) In the above situation, assume that for every $x \in X$, there exists an open neighborhood $x \in U \subset X$ such that $g(U) \cap U = \varnothing$ for any non-identity element $1 \neq g \in G$. Prove that the quotient map $X \to X/G$ is a covering space of degree $|G|$.

   (b) Construct a covering space $S^n \to \mathbf{RP}^n$ of degree 2.

3. **(Abelianization).** The purpose of this problem is to review a basic construction from algebra that turns any group into an abelian group.

   (a) Let G be a group. The commutator subgroup $[G, G]$ is defined as the subgroup generated by elements of the form $[g, h] := ghg^{-1}h^{-1}$ for $g, h \in G$ (i.e. $[G, G]$ is the smallest subgroup containing all of these elements). Show that $[G, G] \subset G$ is normal, and the quotient group
   $$G^{\mathrm{ab}} := G/[G, G]$$
   is abelian. The group G^{ab} is called the *abelianization* of G.

   (b) Show that the quotient map $\pi\colon G \to G^{\mathrm{ab}}$ satisfies the following universal property: For any abelian group H and group homomorphism $f\colon G \to H$, there exists a unique group homomorphism $\overline{f}\colon G^{\mathrm{ab}} \to H$ such that $f = \bar{f} \circ \pi$.

   (c) Write F_n for the free group on n elements, i.e. F_n is the free product of n copies of $\mathbf{Z}$. Compute F_n^{ab} and use it to prove that $F_n^{\mathrm{ab}} \ncong F_m^{\mathrm{ab}}$ for $n \neq m$.

4. **(Group presentations).** The purpose of this problem is to review a method for representing groups in terms of generators and relations.

   For a set S we write $F_S = *_{s \in S}\mathbf{Z}$ for the free product of S copies of $\mathbf{Z}$; we typically regard the copy of $\mathbf{Z}$ indexed by $s \in S$ as an infinite cyclic group written multiplicatively, with generator denoted s, so that elements of F_S can be expressed as "words" in the symbols s^n for $s \in S, n \in \mathbf{Z}$. Let $R \subset F_S$ be a subset of elements. We write $\langle S \,|\, R\rangle$ for the quotient of F_S by the normal subgroup generated by R. In this construction, we call S the *generators* and R the *relations*. If G is a group then an isomorphism $\langle S \,|\, R\rangle \cong G$ is called a *presentation* of G. For example, there is a presentation $\langle a \,|\, a^n\rangle \cong \mathbf{Z}/n$ which sends a to $1 \in \mathbf{Z}/n$.

(a) Show that there exists a presentation of any group G.

(b) Let $\phi_1 \colon H \to G_1$ and $\phi_2 \colon H \to G_2$ be group homomorphisms. Let $G_1 = \langle S_1 \,|\, R_1 \rangle$, $G_2 = \langle S_2 \,|\, R_2 \rangle$, and $H = \langle S_3 \,|\, R_3 \rangle$ be group presentations. Construct a presentation for the pushout $G_1 *_H G_2$ in terms of the given presentations.

(c) Describe the abelianization of $\langle S \,|\, R \rangle$ as the cokernel of a map between free abelian groups (in terms of S and R). Using this, compute explicitly the abelianization of $\langle a, b \,|\, a^2, b^2, (ab)^3 \rangle$.

5. **(π_1 of punctured manifolds).** Let M be a path-connected topological manifold of dimension n.[1] Let $p \in M$ be a point.

   (a) Show that $M \setminus \{p\}$ is path-connected if $n \geq 2$.

   (b) Assume $n \geq 2$, let $q \neq p$, and consider the homomorphism $\pi_1(M \setminus \{p\}, q) \to \pi_1(M, q)$ induced by the inclusion $M \setminus \{p\} \to M$. Prove that it is an isomorphism if $n \geq 3$, and that it is surjective but not necessarily injective when $n = 2$.

6. **(π_1 of complement of lines in $\mathbf{R}^n$).** Let $n \geq 3$. Let X be the complement of $m \geq 1$ distinct lines $L_1, \ldots, L_m \subset \mathbf{R}^n$ which pass through the origin. Show that X is path-connected and compute $\pi_1(X)$.

7. **($\pi_1(\mathbf{RP}^n)$).** Compute $\pi_1(\mathbf{RP}^n)$ for $n \geq 2$ in two different ways:

   (a) By mimicking our computation of $\pi_1(S^1)$ using the covering space $p \colon S^n \to \mathbf{RP}^n$ from Problem 1.

   (b) By using Van Kampen's theorem and induction on n.

8. **(π_1 of gluing of two tori).** Let $x_0 \in S^1$ be a point, and let X be the space obtained from $(S^1 \times S^1) \sqcup (S^1 \times S^1)$ by identifying the circle $S^1 \times \{x_0\} \subset S^1 \times S^1$ in the first torus with the corresponding circle $S^1 \times \{x_0\} \subset S^1 \times S^1$ in the second. Compute $\pi_1(X)$.

[1] For the purpose of this problem, all you need to know about a topological manifold of dimension n is that if M is one, then for any point $p \in M$, there exists a neighborhood $p \in U \subset M$ such that U is homeomorphic to $\mathbf{R}^n$.