Math 592: Algebraic Topology, Winter 2023

Problem Set 2

Due Monday, January 23, 2023 at 8pm

1. **(Punctured surfaces).** Let Σ_g be a compact orientable genus g surface and let $p \in \Sigma_g$ be a point. Prove that $\Sigma_g \setminus \{p\}$ is homotopy equivalent to a wedge sum of copies of S^1.

2. **(Real projective space as a CW complex).** Let $\mathbf{RP}^n$ be the n-dimensional real projective space, defined as the quotient space of $\mathbf{R}^{n+1} \backslash \{0\}$ by the scaling action of $\mathbf{R}^\times = \mathbf{R} \backslash \{0\}$, i.e. the quotient by the equivalence relation $x \sim \lambda x$ for $\lambda \in \mathbf{R}^\times$. Show that $\mathbf{RP}^n$ has the structure of a CW complex with one cell of each dimension i for $0 \leq i \leq n$.

3. **(Infinite-dimensional sphere).** The infinite-dimensional sphere S^∞ is the space

$$S^\infty = \left\{ (x_1, x_2, x_3, \dots) \,\middle|\, x_i \in \mathbf{R}, \text{ all but finitely many } x_i \text{ are zero, and } \sum_i x_i^2 = 1 \right\}$$

with the topology where $U \subset S^\infty$ is open if and only if $U \cap S^n$ is open for every $n \geq 0$, where $S^n \subset S^\infty$ is embedded as the subset where $x_i = 0$ for $i \geq n+2$.

   (a) Show that S^∞ is a CW complex.

   (b) Show that S^∞ is contractible.

   Hint: First show that the identity map of S^∞ is homotopic to the shifting map $(x_1, x_2, x_3, \dots) \mapsto (0, x_1, x_2, x_3, \dots)$, and then show the latter is homotopic to a constant map.

4. **(π_1 on the homotopy category of pointed spaces).** Let Top_* denote the category of *pointed spaces*, defined as follows:

   - An object of Top_* is a pointed space (X, x_0), i.e. a topological space X together with a chosen point $x_0 \in X$.
   - A morphism $(X, x_0) \to (Y, y_0)$ is a continuous map $f \colon X \to Y$ such that $f(x_0) = y_0$.

   Let hTop_* denote the *homotopy category of pointed spaces*, defined as follows:

   - The objects of hTop_* are the same as those of Top_*.
   - Given morphisms $f, g \colon (X, x_0) \to (Y, y_0)$ in Top_*, the relation "f is homotopic to g relative to $\{x_0\} \subset X$" is an equivalence relation, and $\mathrm{Hom}_{\mathrm{hTop}_*}((X, x_0), (Y, y_0))$ is defined as the set of equivalence classes for this equivalence relation.

   (a) You do not need to write this up, but convince yourself that hTop_* as defined above is indeed a category. (You already did a very similar exercise on Problem Set 1.)

   (b) If $f \colon (X, x_0) \to (Y, y_0)$ is a morphism of pointed spaces, show that there is a well-defined group homomorphism $f_* \colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ given by $f([\alpha]) = [f \circ \alpha]$.

(c) Show that there is a functor $\pi_1 \colon \mathrm{Top}_* \to \mathrm{Grp}$ from the category of pointed spaces to the category of groups given on objects by $\pi_1(X, x_0)$ and on morphisms by f_* defined above.

(d) Show that $\pi_1 \colon \mathrm{Top}_* \to \mathrm{Grp}$ factors through the natural functor $\mathrm{Top}_* \to \mathrm{hTop}_*$.

5. **(Homotopy type of wedge sums).** Given pointed spaces (X, x_0) and (Y, y_0), we write $(X, x_0) \vee (Y, y_0)$ for their wedge sum taken at the points x_0 and y_0, i.e

$$(X, x_0) \vee (Y, y_0) = \frac{X \sqcup Y}{x_0 \sim y_0}.$$

We say that (X, x_0) and (Y, y_0) are *pointed homotopy equivalent*, denoted $(X, x_0) \sim (Y, y_0)$, if they are isomorphic in the category hTop_*.

(a) Prove that if $(X, x_0) \sim (X', x_0')$ and $(Y, y_0) \sim (Y', y_0')$, then

$$(X, x_0) \vee (Y, y_0) \sim (X', x_0') \vee (Y', y_0').$$

(b) Let X and Y be path-connected CW complexes. Show that for any points $x_0, x_1 \in X$ and $y_0, y_1 \in Y$, we have

$$(X, x_0) \vee (Y, y_0) \sim (X, x_1) \vee (Y, y_1).$$

Hint: First show that if X is a CW complex and $x_0 \in X$ is any point (not necessarily a 0-cell), then there exists a CW complex structure on X for which x_0 is a 0-cell. Then apply a theorem from class.

6. **(π_1 commutes with products).** Let (X, x_0) and (Y, y_0) be pointed spaces. Construct an isomorphism of groups $\pi_1(X, x_0) \times \pi_1(Y, y_0) \cong \pi_1(X \times Y, (x_0, y_0))$.

7. **(π_1 of a topological group).**

(a) Let A be a set equipped with two binary operations $*$ and $\bullet$, i.e. we are given two maps $A \times A \to A$ whose action on elements are denoted by $(a, b) \mapsto a * b$ and $(a, b) \mapsto a \bullet b$. Assume that:

- There exist units 1_* and $1_\bullet$ for each operation, i.e. elements such that

$$1_* * a = a = a * 1_* \quad \text{and} \quad 1_\bullet \bullet a = a = a \bullet 1_\bullet$$

for all $a \in A$.
- $(a \bullet b) * (c \bullet d) = (a * c) \bullet (b * d)$ for all $a, b, c, d \in A$.

Prove that the operations $*$ and $\bullet$ are equal, and they are commutative, i.e. $a*b = b*a$ for all $a, b \in A$.

(b) Let G be a *topological group*, i.e. G is a group with the structure of a topological space such that the multiplication map $G \times G \to G, (g, h) \mapsto gh$, and inverse map $G \to G, g \mapsto g^{-1}$, are both continuous. Let $e \in G$ be the identity element of G. Prove that $\pi_1(G, e)$ is abelian.