Math 592: Algebraic Topology, Winter 2023

Problem Set 12

Due Monday, April 17, 2023 at 11:59pm

1. Fill out the course evaluation for this class, please!

2. Let $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ be a short exact sequence of abelian groups.

   (a) Prove that the following conditions are equivalent:

   i. There exists a homomorphism $p\colon B \to A$ such that $p \circ i = \mathrm{id}_A$.

   ii. There exists a homomorphism $s\colon C \to B$ such that $j \circ s = \mathrm{id}_C$.

   iii. There exists an isomorphism $\phi\colon B \xrightarrow{\sim} A \oplus C$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{j} & C & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \mathrm{id}_A} & & \Big\downarrow{\scriptstyle \phi} & & \Big\downarrow{\scriptstyle \mathrm{id}_C} & & \\ 0 & \longrightarrow & A & \xrightarrow{\iota} & A \oplus C & \xrightarrow{\pi} & C & \longrightarrow & 0 \end{array}$$

   commutes, where $\iota(a) = (a, 0)$ and $\pi((a, c)) = c$.

   When the above equivalent conditions hold, we say the short exact sequence *splits*.

   (b) Show that if C is a free abelian group (i.e. C is isomorphic to a direct sum of copies of $\mathbf{Z}$), then $B \cong A \oplus C$.

3. Determine, with the proof, the minimal number of 2-simplices in a simplicial complex structure on S^2.

4. Let $C_\bullet$ be a chain complex of finitely generated abelian groups, with $C_n = 0$ for all but finitely many n. Let $\phi\colon C_\bullet \to C_\bullet$ be a morphism of chain complexes. Then for every n we have a map $\phi_n\colon C_n \to C_n$ as well as a map $\phi_*\colon \mathrm{H}_n(C_\bullet) \to \mathrm{H}_n(C_\bullet)$. Prove that there is an equality

$$\sum_n (-1)^n \mathrm{tr}(\phi_n\colon C_n \to C_n) = \sum_n (-1)^n \mathrm{tr}(\phi_*\colon \mathrm{H}_n(C_\bullet) \to \mathrm{H}_n(C_\bullet))$$

   Remark. We used this in our proof of the Lefschetz fixed point theorem.

5. Let X be a finite CW complex. Let k be a field. Show that the Euler characteristic of X can be computed by the formula

$$\chi(X) = \sum_n (-1)^n \dim_k \mathrm{H}_n(X; k),$$

   where $\dim_k \mathrm{H}_n(X; k)$ denotes the dimension of $\mathrm{H}_n(X; k)$ as a k-vector space.

   Remark. Above, we are using the observation that for a space Y and field k, the homology groups $\mathrm{H}_n(Y; k)$ naturally have the structure of k-vector spaces. Please convince yourself of this.

6. Let X be a topological space. Show that the following two conditions are equivalent:

(a) For all i and all primes p, $\widetilde{\mathrm{H}}_i(X; \mathbf{Z}/p) = 0$.

(b) For all i, $\widetilde{\mathrm{H}}_i(X)$ is the abelian group underlying a $\mathbf{Q}$-vector space.