Math 592: Algebraic Topology, Winter 2023

Problem Set 11

Due Monday, April 10, 2023 at 11:59pm

1. Let $\mathbf{CP}^n$ be complex projective n-space, i.e. the quotient of $\mathbf{C}^{n+1} \setminus \{0\}$ by the scaling action of nonzero complex numbers $\mathbf{C}^\times$.

   (a) $\mathbf{CP}^n$ has a CW complex structure with one cell e^i of each even dimension i for $0 \leq i \leq 2n$. Please convince yourself that this is true, or read about it in Hatcher Example 0.6. You do not need to submit anything for this part of the problem.

   (b) Compute the homology $\mathrm{H}_i(\mathbf{CP}^n)$ for all i.

   (c) Determine, with proof, the minimal number of cells in a CW decomposition of $\mathbf{CP}^n$.

   (d) Consider the quotient map $\pi\colon \mathbf{C}^3 \setminus \{0\} \to \mathbf{CP}^2$. Does there exist a section of π, i.e. a continuous map $s\colon \mathbf{CP}^2 \to \mathbf{C}^3 \setminus \{0\}$ such that $\pi \circ s = \mathrm{id}_{\mathbf{CP}^2}$? If yes, construct such an s; if no, prove that none exists.

2. Let X and Y be CW complexes. A continuous map $f\colon X \to Y$ is called *cellular* if it takes the n-skeleton of X to the n-skeleton of Y, i.e. $f(X^n) \subset Y^n$, for all n.

   (a) Show a cellular map f induces a natural morphism $C_\bullet^{\mathrm{CW}}(X) \to C_\bullet^{\mathrm{CW}}(Y)$ between the cellular chain complexes of X and Y, and hence a map $f_*\colon \mathrm{H}_n^{\mathrm{CW}}(X) \to \mathrm{H}_n^{\mathrm{CW}}(Y)$ on cellular homology for all n.

   (b) Show that under the isomorphism $\mathrm{H}_n^{\mathrm{CW}}(X) \cong \mathrm{H}_n(X)$ between cellular and singular homology, the map f_* constructed above corresponds to the usual pushforward map $f_*\colon \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$ on singular homology.

3. (a) For finite CW complexes X and Y, show that the Euler characteristic of the product satisfies $\chi(X \times Y) = \chi(X)\chi(Y)$. Use this to compute the Euler characteristic of an n-dimensional torus $(S^1)^n$ and of $\mathbf{RP}^n \times \Sigma_g$ where Σ_g is a compact orientable surface of genus g.

   (b) If X is a finite connected CW complex and $p\colon \tilde{X} \to X$ is a covering space of finite degree d, explain how to construct an induced finite CW complex structure on $\tilde{X}$. Using this, prove that $\chi(\tilde{X}) = d\chi(X)$.

   (c) Show that there is no free continuous action of $\mathbf{Z}/7$ on $\mathbf{CP}^5$.

   (d) Suppose that $p\colon \Sigma_g \to \Sigma_h$ is a covering space, where Σ_g and Σ_h are compact orientable surfaces of genus g and h. Prove that $g = d(h-1)+1$ where $d = \deg(p)$.

   Remark. Conversely, given g and h satisfying $g = d(h-1)+1$, it is possible to construct a covering space $p\colon \Sigma_g \to \Sigma_h$ of degree d (Hatcher Example 1.41).

4. Let X be a 2-dimensional CW complex with one 0-cell, four 1-cells a, b, c, d, and two 2-cells attached along the loops a^2bc and ab^2d. Compute the homology of X.

5. Let $n \geq 1$ and let $f\colon \mathbf{CP}^n \to \mathbf{CP}^n$ be a continuous automorphism satisfying $f^d = \mathrm{id}_{\mathbf{CP}^n}$ for some $d \geq 1$. Show that if d is odd, then f must have a fixed point.

6. Let $n \geq 1$, let $p\colon S^n \to X$ be a covering space, and assume that $f\colon S^n \to S^n$ is an automorphism of the covering space p. Determine the set of possible degrees of f.