Math 592: Algebraic Topology, Winter 2023

Problem Set 10

Due Monday, April 3, 2023 at 11:59pm

1. Let X be a topological space and let $A, B \subset X$ be subspaces such that $X = \mathrm{int}(A) \cup \mathrm{int}(B)$ is the union of their interiors.

   (a) If $A \cap B$ is path-connected, use Mayer-Vietoris to show that $\mathrm{H}_1(X)$ is isomorphic to the cokernel of the map

   $$(j_{A*}, -j_{B*})\colon \mathrm{H}_1(A \cap B) \to \mathrm{H}_1(A) \oplus \mathrm{H}_1(B) \tag{1}$$

   where $j_A\colon A \to A \cap B$ and $j_B\colon B \to A \cap B$ are the inclusions.

   (b) Now suppose that $A, B \subset X$ are open, and A, B, and $A \cap B$ are path-connected. Then Van Kampen together with our description of H_1 as the abelianization of π_1 gives

   $$\mathrm{H}_1(X) \cong \left(\pi_1(A) *_{\pi_1(A \cap B)} \pi_1(B)\right)^{\mathrm{ab}}.$$

   Prove directly that the right-hand side is isomorphic to the cokernel of the map (1). Thus this description is consistent with the one obtained from Mayer-Vietoris.

2. For a topological space X, its suspension SX is defined as the quotient of $X \times I$ obtained by collapsing $X \times \{0\}$ to one point and collapsing $X \times \{1\}$ to another point. Prove that for all n there are isomorphisms $\widetilde{\mathrm{H}}_n(SX) \cong \widetilde{\mathrm{H}}_{n-1}(X)$.

3. Fix an integer m. Let X be the space obtained from $S^1 \times S^1$ by attaching a Mobius strip M along the map $\partial M = S^1 \to S^1 \times S^1$ given by $z \mapsto (z^m, 1)$, where we regard $S^1 \subset \mathbf{C}$ as a subset of the complex numbers. Compute the homology of X in all degrees.

4. (a) For any map $f\colon S^{2n} \to S^{2n}$, show that there exists a point $x \in S^{2n}$ such that either $f(x) = x$ or $f(x) = -x$, and deduce that every map $\mathbf{RP}^{2n} \to \mathbf{RP}^{2n}$ has a fixed point.

   (b) Is it true that every map $\mathbf{RP}^{2n+1} \to \mathbf{RP}^{2n+1}$ has a fixed point? If yes, prove it; if no, give a counterexample.

5. For each $n \geq 1$, construct a surjective map $S^n \to S^n$ with $\deg(f) = 0$.

6. (a) Let T be a 2-dimensional torus, which we regard as the quotient of a solid square by identifying opposite edges in the usual way. Let $q\colon T \to S^2$ be the map obtained by collapsing the boundary of the square (which is a copy of $S^1 \vee S^1$ in T) to a point. Prove that q is not nullhomotopic (i.e. q is not homotopic to a constant map) by showing it induces an isomorphism on H_2.

   (b) Does there exist a map $S^2 \to T$ which induces an isomorphism on H_2? If yes, construct an example; if no, prove none exists.

7. Let G be any finitely generated abelian group and let $n \geq 1$ be an integer. Construct a topological space X such that

$$\widetilde{\mathrm{H}}_i(X) = \begin{cases} G & i = n \\ 0 & i \neq n \end{cases}$$

Hint: First do the case where $G = \mathbf{Z}/m$ for an integer m.

Remark. In fact, for an arbitrary abelian group G it is possible to construct a space with the above property.