Math 592: Algebraic Topology, Winter 2023

Problem Set 1

Due Friday, January 13, 2023 at 8pm

1. **(Disks and spheres).** For a positive integer n, let

$$D^n = \{x \in \mathbf{R}^n \mid |x| \leq 1\} \quad \text{and} \quad S^n = \left\{x \in \mathbf{R}^{n+1} \,\middle|\, |x| = 1\right\}$$

be the closed unit n-disk and unit n-sphere. Prove that there are homeomorphisms:

   (a) $D^n \cong [-1,1]^n$. (In particular, it follows that $D^m \times D^n \cong D^{m+n}$.)

   (b) $D^n/\partial D^n \cong S^n$, where $\partial D^n = \{x \in \mathbf{R}^n \mid |x| = 1\}$ is the boundary $(n-1)$-sphere.

Remark. Going forward in the course, you may assert "geometrically obvious" homeomorphisms without proof. The point of the above exercise is to make sure you can do it rigorously if necessary.

2. **(Homotopy category of spaces).**

   (a) Let X and Y be topological spaces. Show that the relation "f is homotopic to g" defines an equivalence relation on the set of continuous maps $X \to Y$.

   (b) Show that there is a category whose objects are topological spaces and whose morphisms are homotopy classes of continuous maps, with composition operation induced by usual composition of maps (i.e. check that this gives a well-defined composition operation on homotopy classes of maps, which satisfies the axioms of a category).

   This category is denoted hTop and called the homotopy category of spaces. By construction, there is a functor Top $\to$ hTop, and homotopy equivalences are exactly the morphisms in Top sent to isomorphisms in hTop.

   (c) Show that "homotopy equivalence" defines an equivalence relation on topological spaces.

3. **(Obstruction to a retraction).** Assume there exists a functor $F\colon \mathrm{Top} \to \mathrm{Ab}$ from the category of topological spaces to the category of abelian groups such that

$$F(S^2) = \mathbf{Z} \quad \text{and} \quad F(S^3) = 0.$$

(We will see later in the course that such a functor F does exist.) Prove that the inclusion $S^2 \hookrightarrow S^3$ of the 2-sphere as the equator of the 3-sphere does not admit a retraction $S^3 \to S^2$.

4. **(Limits and colimits of spaces).**

   (a) Familiarize yourself with the notions of limits and colimits from category theory. There are many sources for this material; see for instance §6 of May's *A Concise Course in Algebraic Topology* or Tag 002D of *The Stacks Project*.

(b) Show that the category Top of topological spaces admits all limits and all colimits, i.e. the limit and colimit of any diagram exists in Top.

(c) Is a limit of a diagram of Hausdorff spaces in Top necessarily Hausdorff? Prove if true or give a counterexample if false.

(d) Is a colimit of a diagram of Hausdorff spaces in Top necessarily Hausdorff? Prove if true or give a counterexample if false.

5. **(Coproducts of groups).** Familiarize yourself with the definition of a coproduct in a category (which is a special case of a colimit), for instance from one of the sources mentioned above.

   (a) Compute the coproduct of $\mathbf{Z}$ with $\mathbf{Z}$ in the category Ab of abelian groups.

   (b) Compute the coproduct of $\mathbf{Z}$ with $\mathbf{Z}$ in the category Grp of all groups.

6. **(Quotient space pathology).** Let $p \in S^1$ be a point. Prove that the quotient space $X = S^1/(S^1 \setminus \{p\})$ is contractible.

   Remark. This contrasts the statement from class that for a CW pair (X, A) with A contractible, the map $X \to X/A$ is a homotopy equivalence.

7. **(Homotopy type of a punctured plane).** Let $n \geq 1$ be an integer. Prove that

$$\mathbf{R}^2 \setminus \{(1,0), (2,0), \ldots, (n,0)\}$$

   is homotopy equivalent to $\bigvee_{i=1}^n S^1$, a wedge sum of n circles.