Algebraic Geometry I, Fall 2021

Problem Set 2

Due Friday, September 17, 2021 at 5 pm

Definition 1. A topological space X is called *noetherian* if any decreasing sequence of closed subsets $Z_1 \supset Z_2 \supset Z_3 \supset \cdots$ stabilizes, i.e. there exists some integer n such that $Z_i = Z_n$ for all $i \geq n$.

1. Let X be a noetherian topological space.
   (a) Show that X is quasi-compact.
   (b) Show that every subset of X with the induced topology is noetherian.
   (c) Show that X has a finitely many irreducible components, say $X_1, \ldots, X_n$, and that $X = X_1 \cup X_2 \cup \cdots \cup X_n$ is the union of its irreducible components.

2. Let A be a ring.
   (a) Give a characterization in terms of ideals for when $\mathrm{Spec}(A)$ is noetherian.
   (b) Show that if A is noetherian, then $\mathrm{Spec}(A)$ is noetherian.
   (c) Given an example where $\mathrm{Spec}(A)$ is noetherian but A is not.

3. (a) Prove that every irreducible closed subset of $\mathrm{Spec}(A)$ has a unique generic point.
   (b) Give an example of a ring A and an irreducible subset $S \subset \mathrm{Spec}(A)$ which (with the induced topology) does not have a generic point.
   (c) Prove that $\mathrm{Spec}(A)$ is irreducible if and only if the nilradical $\sqrt{(0)}$ is a prime ideal, in which case $\sqrt{(0)} \in \mathrm{Spec}(A)$ is the generic point.
   (d) Consider the ring

$$A = \mathbf{C}[x, y, z]/((x+1)^4, xy + 3x + y + 3, z^4).$$

   Is $\mathrm{Spec}(A)$ irreducible?

4. Prove that a map of sheaves of sets $\phi\colon \mathcal{F} \to \mathcal{G}$ on a topological space X is an epimorphism if and only if for all $p \in X$ the induced map on stalks $\phi_p\colon \mathcal{F}_p \to \mathcal{G}_p$ is surjective.

5. Read the definition of an abelian category, e.g. in Section 6 of *Foundations of Algebraic Geometry* by Vakil. Work out why the category $\mathrm{Ab}(X)$ of sheaves of abelian groups on a topological space X is an abelian category. You only need to write up the verification of the following three ingredients in this result:
   (a) Show that every morphism in $\mathrm{Ab}(X)$ has a kernel and a cokernel. More precisely, in class for any morphism $\phi\colon \mathcal{F} \to \mathcal{G}$ in $\mathrm{Ab}(X)$, we defined sheaves $\ker(\phi)$ and $\mathrm{coker}(\phi)$; show that they satisfy the universal properties defining kernels and cokernels in category theory (see 1.6.3 of *Foundations of Algebraic Geometry* by Vakil).
   (b) Show that every monomorphism in $\mathrm{Ab}(X)$ is the kernel of its cokernel.

(c) Show that every epimorphism in $\mathrm{Ab}(X)$ is the cokernel of its kernel.

6. Let $j\colon U \to X$ be the inclusion of an open subset of a topological space X. Let $\mathcal{F}$ be a sheaf of abelian groups on U. Define $j_!^{\mathrm{pre}}\mathcal{F}$ to be the presheaf of abelian groups on X given by

$$j_!^{\mathrm{pre}}\mathcal{F}(V) = \begin{cases} \mathcal{F}(V) & \text{if } V \subset U, \\ 0 & \text{if } V \not\subset U, \end{cases}$$

for $V \subset X$ open, with the obvious restrictions maps induced by the ones for $\mathcal{F}$. Let $j_!\mathcal{F}$ denote the sheafification of $j_!^{\mathrm{pre}}\mathcal{F}$.

(a) Show that $j_!^{\mathrm{pre}}\mathcal{F}$ need not be a sheaf in general.

(b) Give a formula for the stalk of $j_!\mathcal{F}$ at any point $p \in X$ in terms of the stalk of $\mathcal{F}$.

(c) Convince yourself that this construction gives a functor $j_!\colon \mathrm{Ab}(U) \to \mathrm{Ab}(X)$ which is left adjoint to the pullback functor $j^{-1}\colon \mathrm{Ab}(X) \to \mathrm{Ab}(U)$. You don't need to write up the details.