Appendix C from R. D'Andrea et al., "Revising the Tolerance-Fecundity Trade-Off; or, On the Consequences of Discontinuous Resource Use for Limiting Similarity, Species Diversity, and Trait Dispersion"

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The Competition Kernel

Here we show that the competition kernel in Muller-Landau's model, $a(f_1, f_2)$, possesses a discontinuous first derivative at $f_1 = f_2$ if and only if the tolerance function is discontinuous.

It follows from the definition of derivatives that the first derivative of a function is discontinuous at a point if and only if the second derivative diverges to infinity at that point. We will use that to show that the competition kernel has a divergent second derivative if and only if the tolerance function is discontinuous. We will use the transition function g defined in appendix A, rather than the tolerance function T. Recall that $g(x, f)$ describes the probability that the species with fecundity f can tolerate stress level at the threshold of the species with fecundity x.

From equation (A2), if $g(x, f) = \Theta(x - f)$, the per capita growth rate is

$$R(f) = \frac{1}{p(f)}\frac{dp}{dt} = -m\left(f\int_{f_0}^{f_m}\frac{h'(x)\Theta(x-f)}{\int_{f_0}^{f_m} yp(y)\Theta(x-y)dy}dx - 1\right).$$

From this we calculate the competition kernel:

$$a(f_1, f_2) = \frac{\delta R(f_1)}{\delta p(f_2)} = mf_1 f_2\int_{f_0}^{f_m}\frac{h'(x)\Theta(x-f_1)\Theta(x-f_2)}{(I(x))^2}dx,$$

where $I(x) \equiv \int_{f_0}^{f_m} yp(y)\Theta(x-y)dy$, and take the first derivative with respect to f_2 (we could equally well pick f_1 instead):

$$\frac{\partial}{\partial f_2}a(f_1, f_2) = mf_1\left(\int_{f_0}^{f_m}\frac{h'(x)\Theta(x-f_1)\Theta(x-f_2)}{(I(x))^2}dx - f_2\int_{f_0}^{f_m}\frac{h'(x)\Theta(x-f_1)\delta(x-f_2)}{(I(x))^2}dx\right).$$

The second derivative then is

$$\begin{aligned}\frac{\partial^2}{\partial f_2^2}a(f_1, f_2) &= mf_1\left\{-2\int_{f_0}^{f_m}\frac{h'(x)\Theta(x-f_1)\delta(x-f_2)}{(I(x))^2}dx - f_2\frac{\partial}{\partial f_2}\left[\int_{f_0}^{f_m}\frac{h'(x)\Theta(x-f_1)\delta(x-f_2)}{(I(x))^2}dx\right]\right\}\\ &= mf_1\left\{-2\frac{h'(f_2)}{(I(f_2))^2}\Theta(f_2-f_1) - f_2\frac{\partial}{\partial f_2}\left[\frac{h'(f_2)}{(I(f_2))^2}\Theta(f_2-f_1)\right]\right\}. \qquad \text{(C1)}\end{aligned}$$

Since we are only interested in whether or not the second derivative diverges, we can stop here and look at the behavior of the terms. Since $h(f)$ is well behaved and $I(f) \neq 0$ for all $f > f_0$, the first term never diverges and is therefore of no consequence. The second term, however, contains a derivative of $\Theta(f_2 - f_1)$. The step function is discontinuous at $f_1 = f_2$, and the derivative of a discontinuous function diverges at the point of discontinuity (more specifically, the derivative of a function at a point of discontinuity is proportional to the Dirac delta function, which is uniformly zero at all points except for where its argument vanishes, at which point it is infinite). Precisely at $f_1 = f_2$, then, we have a singularity in equation

(C1). This proves that the competition kernel has a discontinuous first derivative at the point where two species coincide if the transition function $g(x,f)$ is discontinuous when $x = f$.

In the same vein, the competition kernel when the transition function g is continuous can be shown to be smooth, that is, it does not contain any point where the first derivative is discontinuous—no kinks. To see that, we can follow the same steps as above and replace the discontinuous unit function with any continuous function g:

$$a(f_1,f_2) = mf_1f_2\int_{f_0}^{f_m}\frac{h'(x)g(x-f_1)g(x-f_2)}{(J(x))^2}dx,$$

$$\frac{\partial}{\partial f_2}a(f_1,f_2) = mf_1\left[\int_{f_0}^{f_m}\frac{h'(x)g(x-f_1)g(x-f_2)}{(J(x))^2}dx - f_2\int_{f_0}^{f_m}\frac{h'(x)g(x-f_1)g'(x-f_2)}{(J(x))^2}dx\right],$$

where $J(x) \equiv \int_{f_0}^{f_m} yp(y)g(x-y)dy$. Proceeding to the second derivative, we have

$$\frac{\partial^2}{\partial f_2^2}a(f_1,f_2) = mf_1\left\{-2\int_{f_0}^{f_m}\frac{h'(x)g(x-f_1)g'(x-f_2)}{(I(x))^2}dx - f_2\frac{\partial}{\partial f_2}\left[\int_{f_0}^{f_m}\frac{h'(x)g(x-f_1)g'(x-f_2)}{(I(x))^2}dx\right]\right\}.$$

Now it should be easy to see that none of the terms above diverge at any point, since they contain only smooth, well-behaved functions and their derivatives. This proves that the kernel is smooth.