Stochastic, Size-Structured Community Assembly: Supporting Information

1 Full Derivation of the Equilibrium Partition Function

In the main body of the text, we provided an equilibrium solution of our functional Master Equation. First, we set $\frac{\partial Z}{\partial t} = 0$, so that we are looking for an equilibrium solution. Then, rewriting the Eq.[7] in terms of $\log Z$, rather than Z, we need to solve

$$\begin{aligned} 0 = &\int_{m_0}^{\infty} dm\ d(m) \frac{\delta \log Z}{\delta H(m)} \left(e^{-H(m)} - 1\right) \\ &+ \int_{m_0}^{\infty} dm\ g(m) \frac{\delta \log Z}{\delta H(m)} \frac{dH}{dm} \\ &+ \left[b \int_{m_0}^{\infty} dm \frac{\delta \log Z}{\delta H(m)} + \nu \right] \left(e^{H(m_0)} - 1\right). \end{aligned} \tag{1}$$

In the main text, we stated that this equilibrium solution took the form:

$$\log Z = -\frac{\nu}{b} \log \left[1 - \frac{b}{\nu} \int_{m_0}^{\infty} dm f(m) \left(e^{H(m)} - 1\right) \right]. \tag{2}$$

We may now confirm that (2) solves (1) by direct substitution. To do this, we must make use of the following simple rule for taking functional derivatives:

$$\frac{\delta H(m')}{\delta H(m)} = \delta(m' - m), \tag{3}$$

where $\delta(m' - m)$ is the Dirac delta function. (Informally-speaking, the delta function represents an infinitely sharp peak at $m = m'$, bounding unit area, so that the functional derivative vanishes unless m' is equal to m.)

Applying this rule in conjunction with the chain rule, we may compute the functional derivative of Eq.(2) to obtain

$$\begin{aligned} \frac{\delta \log Z}{\delta H(m)} &= \frac{\frac{\nu}{b} \frac{b}{\nu} \int_{m_0}^{\infty} d\tilde{m} f(\tilde{m}) \frac{\delta H(\tilde{m})}{\delta H(m)} e^{H(\tilde{m})}}{1 - \frac{b}{\nu} \int_{m_0}^{\infty} d\tilde{m} f(\tilde{m}) \left(e^{H(\tilde{m})} - 1\right)} \\ &= \frac{f(m) e^{H(m)}}{1 - \frac{b}{\nu} \int_{m_0}^{\infty} d\tilde{m} f(\tilde{m}) \left(e^{H(\tilde{m})} - 1\right)}. \end{aligned} \tag{4}$$

We may now check that Eq.(2) solves Eq.(1) by plugging this result into Eq.(1), and checking that we have a time-independent solution, as claimed. To see this, we show that the right-hand-side of Eq.(1) sums to zero:

$$\begin{aligned}
&\int_{m_0}^{\infty} dm \left[d(m) \frac{\delta \log Z}{\delta H(m)} \left(e^{-H(m)} - 1 \right) + g(m) \frac{\delta \log Z}{\delta H(m)} \frac{dH}{dm} \right] + \left[\nu + b \int_{m_0}^{\infty} dm \frac{\delta \log Z}{\delta H(m)} \right] \left(e^{H(m_0)} - 1 \right) \\
&= \frac{1}{1 - \frac{b}{\nu} \int_{m_0}^{\infty} d\tilde{m} f(\tilde{m}) \left(e^{H(\tilde{m})} - 1 \right)} \int_{m_0}^{\infty} \left[-d(m) f(m) \left(e^{H(m)} - 1 \right) + g(m) f(m) \frac{d}{dm} \left(e^{H(m)} - 1 \right) \right] \\
&+ \left[\nu + \frac{b}{1 - \frac{b}{\nu} \int_{m_0}^{\infty} d\tilde{m} f(\tilde{m}) \left(e^{H(\tilde{m})} - 1 \right)} \int_{m_0}^{\infty} dm f(m) e^{H(m)} \right] \left(e^{H(m_0)} - 1 \right) \\
&= \frac{e^{H(m_0)} - 1}{1 - \frac{b}{\nu} \int_{m_0}^{\infty} d\tilde{m} f(\tilde{m}) \left(e^{H(\tilde{m})} - 1 \right)} \left[-g(m_0) f(m_0) + \nu - \nu \frac{b}{\nu} \int_{m_0}^{\infty} d\tilde{m} f(\tilde{m}) \left(e^{H(\tilde{m})} - 1 \right) \right. \\
&\left. + b \int_{m_0}^{\infty} dm f(m) e^{H(m)} \right] \\
&= \frac{e^{H(m_0)} - 1}{1 - \frac{b}{\nu} \int_{m_0}^{\infty} d\tilde{m} f(\tilde{m}) \left(e^{H(\tilde{m})} - 1 \right)} \left[-b \int_{m_0}^{\infty} d\tilde{m} f(\tilde{m}) e^{H(\tilde{m})} + b \int_{m_0}^{\infty} dm f(m) e^{H(m)} \right] \\
&= 0, \qquad (5)
\end{aligned}$$

as required. We used the boundary condition for $f(m)$,

$$f(m_0) g(m_0) = \nu + b \int_{m_0}^{\infty} dm \ f(m), \qquad (6)$$

to obtain the third equality.

The central message of this derivation is that Eq.(2) does indeed solve Eq.(1), our master equation for the partition function $Z[H]$.

2 Obtaining Explicit Expressions for the Species Biomass Distribution

In the main body of the text we derived the following expression for the generating function for the species biomass distribution, $P(M)$:

$$\log z_{sbd}(h) = -\frac{\nu}{b} \log \left[1 - \frac{b}{\nu} \int_{m_0}^{\infty} dm f(m) \left(e^{hm} - 1 \right) \right]. \qquad (7)$$

The distribution $P(M)$ is defined via the following transform:

$$z_{sbd}(h) = \int dM \ P(M) e^{hM}, \qquad (8)$$

so that moments of $P(M)$ are equal to appropriate derivatives of $z_{sbd}(h)$ with respect to h, as required. Given this definition, we may compute $P(M)$ explicitly by taking the inverse Laplace transform of $z_{sbd}(-h)$:

$$P(M) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} dh \; z_{sbd}(-h) e^{hM}. \tag{9}$$

The contour is defined so that the real number γ is greater than than the real part of all singularities of $z_{sbd}(-h)$. I.e. the contour is a vertical line in the complex plane, to the right of those singularities.

2.1 The Totally Neutral Community

We gave the example in the main text of a totally neutral community, that is where all individuals have the same birth, death and growth rates, irrespective of species identity *or* size:

$$\begin{aligned} g(m) &= g \\ d(m) &= d. \end{aligned}$$

For this choice of size-structure, we have that

$$f(m) = \frac{\nu}{g\left(1 - \frac{b}{d}\right)} e^{-\frac{d}{g} m} \tag{10}$$

and the problem of finding $P(M)$ reduces to the following contour integral,

$$\begin{aligned} I &= \int_C dh \left[1 - \frac{b}{g\left(1 - \frac{b}{d}\right)} \frac{1}{h + \frac{d}{g}} + \frac{b}{d-b} \right]^{-\frac{\nu}{b}} e^{hM} \\ &= \left(1 + \frac{b}{d-b}\right)^{-\frac{\nu}{b}} \int_C dh \; \frac{\left(h + \frac{d}{g}\right)^{\frac{\nu}{b}}}{\left(h + \frac{d-b}{g}\right)^{\frac{\nu}{b}}} \; e^{hM}, \end{aligned} \tag{11}$$

where the contour C is defined as above. Changing variables to

$$\omega = \frac{gh}{b} + \frac{d}{b}, \tag{12}$$

we have

$$\begin{aligned} I &= \frac{b}{g} e^{-\frac{dM}{g}} \left(1 + \frac{b}{d-b}\right)^{-\frac{\nu}{b}} \int_C dh \; \frac{\left(\frac{b\omega}{g}\right)^{\frac{\nu}{b}}}{\left(\frac{b\omega}{g} - \frac{b}{g}\right)^{\frac{\nu}{b}}} \; e^{\frac{b\omega M}{g}} \\ &= \frac{b}{g} e^{-\frac{dM}{g}} \left(1 + \frac{b}{d-b}\right)^{-\frac{\nu}{b}} \int_C dh \; \frac{(\omega)^{\frac{\nu}{b}}}{(\omega - 1)^{\frac{\nu}{b}}} \; e^{\frac{b\omega M}{g}}. \end{aligned} \tag{13}$$

The function to be integrated has a branch cut between $\omega = 0$ and $\omega = 1$, and so we must take the contour C to be such that $\gamma > 1$.

This contour may be closed with a large semicircle to the left of C in the complex plane, and for $M > 0$ this semicircle gives a vanishing contribution. From Cauchy's integral theorem, this integral may be shrunk to an integral anticlockwise around the branch cut. This contour can be shrunk to be infinitessimally thin, so that all is left is an integral along the top of the branch cut from right to left, $\omega = 1$ to 0, and an integral along the bottom of the branch cut from left to right, $\omega = 0$ to 1. These two integrals are:

$$I_{top} = \frac{b}{g} e^{-\frac{dM}{g}} \left(1 + \frac{b}{d-b}\right)^{-\frac{\nu}{b}} \int_1^0 d\omega \; \frac{(\omega)^{\frac{\nu}{b}}}{(1-\omega)^{\frac{\nu}{b}}} \; e^{\frac{b\omega M}{g}} e^{-\frac{i\pi\nu}{b}} \tag{14}$$

and

$$I_{bottom} = \frac{b}{g} e^{-\frac{dM}{g}} \left(1 + \frac{b}{d-b}\right)^{-\frac{\nu}{b}} \int_0^1 d\omega \; \frac{(\omega)^{\frac{\nu}{b}}}{(1-\omega)^{\frac{\nu}{b}}} \; e^{\frac{b\omega M}{g}} e^{\frac{i\pi\nu}{b}}. \tag{15}$$

Adding both together we obtain

$$I = I_{top} + I_{bottom} = \frac{b}{g} e^{-\frac{dM}{g}} \left(1 + \frac{b}{d-b}\right)^{-\frac{\nu}{b}} 2i \sin(\pi\nu/b) \int_0^1 d\omega \; \frac{(\omega)^{\frac{\nu}{b}}}{(1-\omega)^{\frac{\nu}{b}}} \; e^{\frac{b\omega M}{g}} \tag{16}$$

and so for $M > 0$ the distribution is

$$P(M) = \frac{b}{g} e^{-\frac{dM}{g}} \left(1 + \frac{b}{d-b}\right)^{-\frac{\nu}{b}} \frac{\sin(\pi\nu/b)}{\pi} \int_0^1 d\omega \; \frac{(\omega)^{\frac{\nu}{b}}}{(1-\omega)^{\frac{\nu}{b}}} \; e^{\frac{b\omega M}{g}}. \tag{17}$$

Taking the limit of small ν/b we obtain the result relevant for neutral theory:

$$\begin{aligned} P(M) &= \frac{b}{g} e^{-\frac{dM}{g}} \frac{\nu}{b} \int_0^1 d\omega \; e^{\frac{b\omega M}{g}} + O\left(\frac{\nu}{b}\right)^2 \\ &= \frac{\nu}{bM} \left(e^{-\frac{d-b}{g}M} - e^{-\frac{d}{g}M}\right) \end{aligned} \tag{18}$$