Poincaré Polynomial of Moduli Space via Weil Conjectures

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Outline

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1. The main algebro-geometric object we have been working with is the Hilbert scheme of points, a moduli scheme
2. Moduli spaces commonly occur in classification problems, like ours
3. Topological spaces can be classified by discovering topological properties inherent to the space

Definition

Topological invariant: A property of a topological space that does not change under stretching and bending of the object.

The topological invariant we will be working with is known as the Betti numbers.

- Informally, the k^{th} Betti number is the number of k dimensional holes in a topological space
- The informal definition only makes sense up to dimension 3, but Betti numbers have a formal definition that generalizes for any dimension

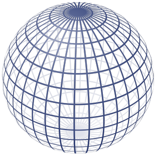
How to apply Betti numbers

The Euler number

For a topological space X, the Euler number of X is defined as

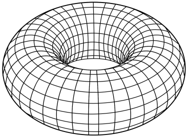
$$\chi(X) = \sum_i (-1)^i b_i(X).$$

Euler numbers are a homotopy invariant.



$$\chi(S^2) = 1-0+1 = 2$$

$$\chi(\mathbf{T}^2) = 1-2+1 = 0$$



How to apply Betti numbers

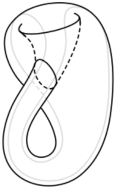


Figure: $\chi(\mathsf{Klein}) = 1 - 1 + 0 = 0$

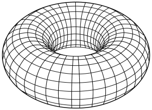


Figure: $\chi(\mathbf{T}^2) = 1 - 2 + 1 = 0$

How to apply Betti numbers

Poincaré polynomial

The generating function of the Betti numbers for a space X,

$$p(X, z) = \sum_i b_i(X) z^i.$$

- $p(\text{Klein}, z) = 1z^0 + 1z^1$
- $p(\mathbf{T}^2, z) = 1z^0 + 2z^1 + 1z^2$

How to acquire Betti Numbers

- There are different methods for computing Betti numbers, but the route that our research employed was the Weil conjectures
- The Weil conjectures apply to particular algebraic varieties, solution sets of polynomials
- The crucial aspect for our work is that if you can determine a zeta function that counts the number of points of the variety over a finite field you can essentially use that to determine the Betti numbers of the variety

Outline

The Problem

Find the Poincaré polynomial of the
punctual Hilbert scheme of type (m_0, m_1)

"the punctual Hilbert scheme of type (m_0, m_1)" $=$???

What is the punctual Hilbert scheme of type (m_0, m_1)?

We'll first introduce some background:

1. Monomial ideals
2. Young diagrams
3. Group actions
4. Colored Young diagrams

Monomial ideals

Definition

A **monomial ideal** $I \subset k[x, y]$ is a set of polynomials, but with a few rules attached:

- $\alpha, \beta \in I$ implies $\alpha + \beta \in I$
- $\alpha \in I$ and $m \in k[x, y]$ implies $\alpha \cdot m \in I$

Monomial ideals

We define $\langle x^2, y\rangle \subset k[x,y]$ to be the set of all polynomials of the form $Ax^2 + By$ for some polynomials $A, B \in k[x,y]$.

We'll accept without proof that $\langle x^2, y\rangle \subset k[x,y]$ is an ideal.

Examples of Monomial ideals

Example

- $\langle x^3, y^3 \rangle$
- $\langle xy, y^2, y^5 \rangle$
- $\langle x, y, x^2y^2 \rangle$

are all monomial ideals within the polynomial ring $k[x, y]$.

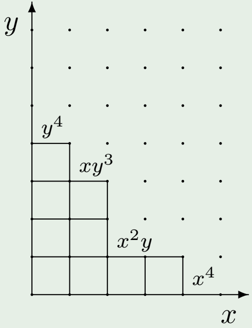
Young diagrams

A **Young diagram** is a visual representation of a monomial ideal.

Example

We'll construct a Young diagram for the monomial ideal

$$\langle x^4, x^2y, xy^3, y^4 \rangle \subset k[x,y]$$



Group actions

- Our **group action** is defined by $x \mapsto -x$, $y \mapsto -y$

Example

Under the given transformation:

$$\begin{aligned} 1 = x^0y^0 &\mapsto (-x)^0(-y)^0 = 1, && \therefore 1 \mapsto 1 \\ x^3y^5 &\mapsto (-x)^3(-y)^5 = x^3y^5, && \therefore x^3y^5 \mapsto x^3y^5 \\ y^5 = x^0y^5 &\mapsto (-x)^0(-y)^5 = -y^5, && \therefore y^5 \mapsto -y^5 \end{aligned}$$

and in general, $x^ay^b \mapsto (-1)^{a+b}x^ay^b$

Colored Young diagrams

We can combine the **Young diagram** and the **group action** to "color the Young diagram."

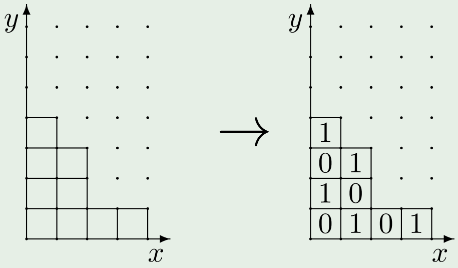
Procedure:

1. Draw the Young diagram as before
2. If the monomial for a box maps to itself, we "color" the box with a 0
3. If the monomial for a box maps to the negative of itself, we "color" the box with a 1

Example: coloring a Young diagram

Example

Recall our previous ideal, $\langle x^4, x^2y, xy^3, y^4 \rangle \subset k[x,y]$.
We can see that $1 \mapsto 1$, $x \mapsto -x$, $y \mapsto -y$, $xy \mapsto xy$, etc.



And finally... the punctual Hilbert scheme of type (m_0, m_1)

The **punctual Hilbert scheme of type** (m_0, m_1) is defined in two *semi*-synonymous ways:

$$\mathrm{Hilb}_0^{(m_0,m_1)} k^2 = \{\text{Young diagrams with } m_0 \text{ } 0'\text{s and } m_1 \text{ } 1'\text{s}\}$$

$$OR$$

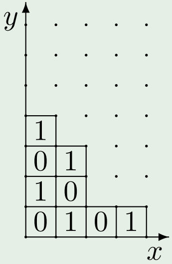
$$\mathrm{Hilb}_0^{(m_0,m_1)} k^2 = \left\{ I \subseteq k[x,y] \mid \frac{k[x,y]}{I} \simeq m_0\rho_0 + m_1\rho_1, V(I) = 0 \right\}$$

where $k = \mathbb{F}_q$ is a finite field of order q.

Example: punctual Hilbert scheme of type $(4, 5)$

Example

Recall the monomial ideal $\langle x^4, x^2y, xy^3, y^4\rangle \subset k[x, y]$.



By definition, the ideal $\langle x^4, x^2y, xy^3, y^4\rangle$ is in $\mathrm{Hilb}_0^{(4,5)}k^2$, since the ideal has 4 0's and 5 1's when represented as a colored Young diagram under our group action.

Outline

The problem: restated

The ultimate goal of our research has been to **find the Poincaré polynomial of the punctual Hilbert scheme of type (m_0, m_1).**

(Recall that the Poincaré polynomial is the generating function for the Betti numbers, which are important topological invariants.)

Plan

1. Prove generating function for $\#\mathrm{Hilb}_0^{(m_0,m_1)} k^2$
2. Apply Weil conjectures to generating function, get Poincaré polynomial
3. Use Poincaré polynomial to read off Betti numbers

Outline

Generating Function

Prove Dr. Gholampour's conjectured generating function for the number of points in the punctual Hilbert scheme of n points

$$\sum_{n_0,n_1\geq 0} \left(\#\mathrm{Hilb}^{n_0,n_1}\right) t_0^{n_0} t_1^{n_1} = \prod_{j\geq 1} \frac{1}{(1-q^{j-1}(t_0t_1)^j)(1-q^j(t_0t_1)^j)} \cdot \sum_{m\in\mathbb{Z}} t_0^{m^2} t_1^{m^2+m}$$

Outline

Cores

We define a p-**core** as a staircase-shaped colored Young diagram with base length p.

Example

$$\bullet \quad \text{is a } 0-\text{core}$$

$$\begin{array}{|c|}\hline 0 \\ \hline \end{array} \quad \text{is a } 1-\text{core}$$

$$\begin{array}{|c|c} \cline{1-1} 1 & \\ \hline 0 & \multicolumn{1}{c|}{1} \\ \hline \end{array} \quad \text{is a } 2-\text{core}$$

$$\begin{array}{|c|c|c} \cline{1-1} 0 & \multicolumn{1}{c}{} & \\ \cline{1-2} 1 & 0 & \\ \hline 0 & 1 & \multicolumn{1}{c|}{0} \\ \hline \end{array} \quad \text{is a } 3-\text{core}$$

$$\ldots$$

But. . . why should we care about cores?

Answer: **We care about cores because each colored Young diagram corresponds to a unique core.**[3]

Finding the core of a colored Young diagram

Method:

1. Suppose the boxes are actually physical boxes affected by gravity pointing towards the lower left corner
2. Remove “border strips” (horizontal or vertical pairs of boxes) until you can’t remove any more without displacing other boxes

Example

Start with the Young diagram $\begin{array}{|c|c|c|}\cline{1-2} 1 & 0 & \multicolumn{1}{c}{} \\ \hline 0 & 1 & 0 \\ \hline \end{array}$.
We’ll remove border strips one at a time, until we get to a core:

$$\begin{array}{|c|c|c|}\cline{1-2} \color{red}{1} & \color{red}{0} & \multicolumn{1}{c}{} \\ \hline 0 & 1 & 0 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|}\hline 0 & \color{red}{1} & \color{red}{0} \\ \hline \end{array} \longrightarrow \begin{array}{|c|}\hline 0 \\ \hline \end{array}$$

Big example: finding the core

Example

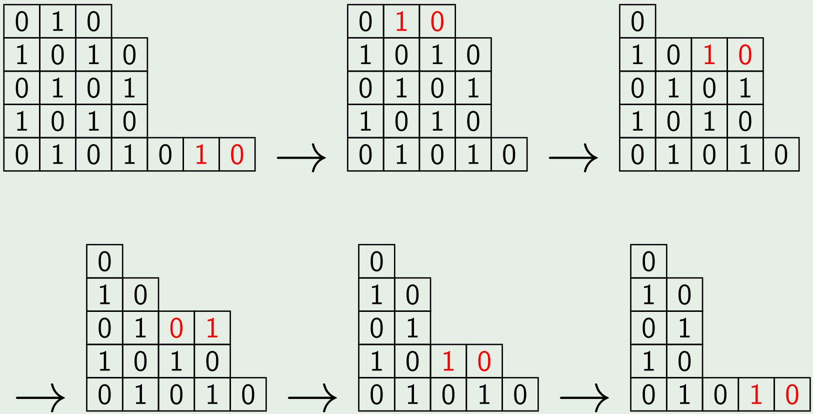
Start with the Young diagram

0	1	0				
1	0	1	0			
0	1	0	1			
1	0	1	0			
0	1	0	1	0	1	0

Again, we'll remove border strips one at a time, until we get to a core.

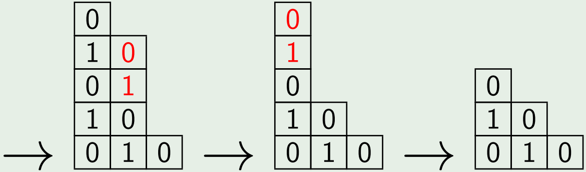
Big example: finding the core, cont'd

Example



Big example: finding the core, cont'd

Example



We can thus see that a 3-core was hiding inside of the given Young diagram.

Recall proposed generating function

$$\sum_{n_0,n_1\geq 0}(\#\mathrm{Hilb}^{n_0,n_1})t_0^{n_0}t_1^{n_1}=\prod_{j\geq 1}\frac{1}{(1-q^{j-1}(t_0t_1)^j)(1-q^j(t_0t_1)^j)}\cdot\sum_{m\in\mathbb{Z}}t_0^{m^2}t_1^{m^2+m}$$

Part of this generating function:

$$\sum_{m\in\mathbb{Z}}t_0^{m^2}t_1^{m^2+m}$$

is the generating function for cores.

$\therefore$ Understanding how cores fit into the overall picture is imperative.

Outline

Using the Weil conjectures

An important result regarding the Weil conjectures is that if $\#X(\mathbb{F}_q) = f(q)$, with $f(x) \in \mathbb{Z}[x]$ then the Poincaré polynomial is achieved by the substitution $q \mapsto z^2$.

For us this means if $\#\mathrm{Hilb}_0^{(n_0,n_1)} k^2$ is given by an integral polynomial $f(q)$, then $f(z^2)$ gives the Poincaré polynomial.

Example

$\#\mathbb{P}$ over $\mathbb{F}_q$ is given by $q+1$, thus the Poincaré polynomial is given by z^2+1 which gives the Betti numbers $b_0 = 1, b_1 = 0, b_2 = 1$.

Using the Weil conjectures

Fortunately for our work, we were able to show that the proposed generating function for $\#\mathrm{Hilb}_0^{(n_0,n_1)}k^2$ met the requirements for the substitution $q \mapsto z^2$ to give the Poincaré polynomial.

Unfortunately, our mentor brought it to our attention that the punctual Hilbert scheme we are working with is not smooth, thus the Weil conjectures do not apply and we will have to work with a resolution.

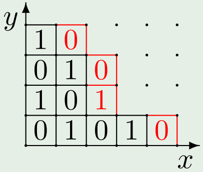
Outline

Stratified Hilbert schemes

The *stratified punctual Hilbert scheme of type* $(m_0, m_1), (d_0, d_1)$, written $\mathrm{Hilb}_0^{(m_0,m_1),(d_0,d_1)} k^2$, is a subset of $\mathrm{Hilb}_0^{(m_0,m_1)} k^2$.

Example

$$\langle y^4, xy^3, x^2y, x^4 \rangle \in \mathrm{Hilb}_0^{(4,5),(3,1)} k^2$$



$$\mathrm{Hilb}_0^{(m_0,m_1)} k^2 = \bigcup_{d_0,d_1 \geq 0} \mathrm{Hilb}_0^{(m_0,m_1),(d_0,d_1)} k^2$$

$$\Rightarrow \#\mathrm{Hilb}_0^{(m_0,m_1)} k^2 = \sum_{d_0,d_1 \geq 0} \#\mathrm{Hilb}_0^{(m_0,m_1),(d_0,d_1)} k^2$$

Recursion

$$\#\mathrm{Hilb}_0^{(m_0,m_1),(d_0,d_1)}k^2 = \sum_{\substack{0\le d_0'\le d_1\\ 0\le d_1'\le d_0\\ d_1'-d_0'=d_0-d_1+(-1)^{(d_0'+d_1')}((d_0'+d_1'+d_0+d_1)\%2)}} q^r\cdot\#\mathrm{Hilb}_0^{(m_0-d_1,m_1-d_0),(d_0',d_1')}k^2$$

where

$$r = \begin{cases} d_0' & \text{if } d_0+d_1\equiv 0 \mod 2\\ d_1' & \text{if } d_0+d_1\equiv 1 \mod 2\end{cases}$$

Let $a,b,c,d\in\mathbb{Z}_{\ge 0}$. The base cases are

$$\#\mathrm{Hilb}_0^{(0,b>0),(c,d)}k^2 = 0 \qquad \#\mathrm{Hilb}_0^{(a,b),(c>b,d)}k^2 = 0$$

$$\#\mathrm{Hilb}_0^{(a,b),(c,d>a)}k^2 = 0 \qquad \#\mathrm{Hilb}_0^{(a\neq 0,b),(0,0)}k^2 = 0$$

$$\#\mathrm{Hilb}_0^{(0,0),(0,0)}k^2 = 1$$

Why $(m_0 - d_1, m_1 - d_0)$?

- Recurse by "chopping off" first column and sliding diagram over, then counting which ideals give same diagram
- Same as removing last block in each row
- 0 outside diagram $\Rightarrow$ 1 in last box
- 1 outside diagram $\Rightarrow$ 0 in last box
- Also requires $d_0' \leq d_1$ and $d_1' \leq d_0$ in smaller scheme

$$\begin{array}{|c|c|c|c|c|c|c|c|} \cline{1-2} 1 & 0 \\ \cline{1-4} 0 & 1 & 0 & 1 \\ \cline{1-4} 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|c|c|c|c|} \cline{1-3} 0 & 1 & 0 \\ \cline{1-3} 1 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \hline \end{array}$$

Some Special Cases

- $\#\mathrm{Hilb}_0^{(k,k+1),(k+1,k-1)}k^2 = 1$
- $\#\mathrm{Hilb}_0^{(k,k+1),(k+1,k-1)}k^2 = 1$
- $\#\mathrm{Hilb}_0^{(k,k),(1,0)}k^2 = q^k$
- $\#\mathrm{Hilb}_0^{(k+1,k),(0,1)}k^2 = q^k$
- $\#\mathrm{Hilb}_0^{(k,k+1),(2,0)}k^2 = \left\lceil \frac{k}{2} \right\rceil q^{k-1}$
- $\#\mathrm{Hilb}_0^{(k+1,k),(0,2)}k^2 = \left\lfloor \frac{k}{2} \right\rfloor q^k$

Outline

Current future tasks and questions include

- Work with the resolution for the Poincaré polynomial
- Work on other group actions
- Consider more refined topological invariants

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