Recursions and Colored Hilbert Schemes

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Outline

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The Problem

Object of Study: The Hilbert scheme of type (m_0, m_1).

Long Term Goal: Find the Poincaré polynomial of the punctual Hilbert scheme of type (m_0, m_1).

Why? The Poincarè polynomial is a *topological invariant*, meaning it doesn't change with stretching and bending.

Short Term Goal: Count the points of the punctual Hilbert scheme of type (m_0, m_1).

Why? We can use this to find a generating function, which we can then use in the Weil conjectures, to find the Poincaré polynomial.

What is the punctual Hilbert scheme of type (m_0, m_1)?

We'll first introduce some background:

1. Monomial ideals
2. Young diagrams
3. Group actions
4. Colored Young diagrams

Ideals

Definition

An **ideal** $I \subset k[x, y]$ for a field k is a set of polynomials, but with a few rules attached:

- $\alpha, \beta \in I$ implies $\alpha + \beta \in I$
- $\alpha \in I$ and $m \in k[x, y]$ implies $\alpha \cdot m \in I$

Theorem

Every polynomial ideal can be written as

$$\langle g_1, \dots, g_n \rangle = \{f_1 g_1 + \cdots + f_n g_n \mid f_i \in k[x, y]\},$$

which is the set of all $k[x, y]$ linear combinations of the g_i. These g_i are called the generators.

Monomial ideals

Definition

A monomial ideal is an ideal generated by monomials.

Example

- $\langle x\rangle = \{Ax \mid A \in k[x,y]\}$
- $\left\langle x^3, y^3\right\rangle = \{Ax^3 + By^3 \mid A, B \in k[x,y]\}$

are monomial ideals within the polynomial ring $k[x,y]$.

- $\langle x+y\rangle = \{A(x+y) \mid A \in k[x,y]\}$

is not a monomial ideal.

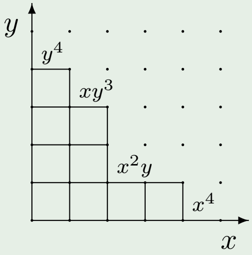
Young diagrams

A **Young diagram** is a visual representation of a monomial ideal.

Example

We'll construct a Young diagram for the monomial ideal

$$\langle x^4, x^2y, xy^3, y^4 \rangle \subset k[x, y]$$



Group actions

Our $\mathbb{Z}_2$ **group action** is defined by $x \mapsto -x$, $y \mapsto -y$

Example

Under the given transformation:

$$\begin{aligned} 1 = x^0y^0 &\mapsto (-x)^0(-y)^0 = 1, & \therefore 1 \mapsto 1 \\ x^3y^5 &\mapsto (-x)^3(-y)^5 = x^3y^5, & \therefore x^3y^5 \mapsto x^3y^5 \\ y^5 = x^0y^5 &\mapsto (-x)^0(-y)^5 = -y^5, & \therefore y^5 \mapsto -y^5 \end{aligned}$$

and in general, $x^ay^b \mapsto (-1)^{a+b}x^ay^b$

Colored Young diagrams

We can combine the **Young diagram** and the **group action** to "color the Young diagram."

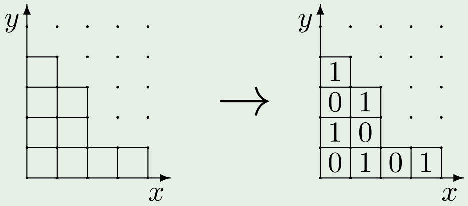
Procedure:

1. Draw the Young diagram as before
2. If the monomial for a box maps to itself, we "color" the box with a 0
3. If the monomial for a box maps to the negative of itself, we "color" the box with a 1

Example: coloring a Young diagram

Example

Recall our previous ideal, $\langle x^4, x^2y, xy^3, y^4\rangle \subset k[x,y]$.
We can see that $1 \mapsto 1$, $x \mapsto -x$, $y \mapsto -y$, $xy \mapsto xy$, etc.



And finally... the punctual Hilbert scheme

The **punctual Hilbert scheme of type** (m_0, m_1) is defined as

$$\mathrm{Hilb}_0^{(m_0,m_1)} k^2 = \left\{ I \subseteq k[x,y] \,\middle|\, \frac{k[x,y]}{I} \simeq m_0\rho_0 + m_1\rho_1, V(I) = 0 \right\}$$

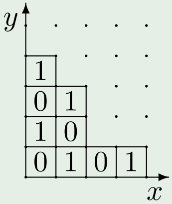
where $k = \mathbb{F}_q$ is a finite field of order q. But the subset of $\mathrm{Hilb}_0^{(m_0,m_1)} k^2$ made of monomial ideals looks like

$$\{\text{Young diagrams with } m_0 \text{ 0's and } m_1 \text{ 1's}\}$$

Example: punctual Hilbert scheme of type $(4, 5)$

Example

Recall the monomial ideal $\langle x^4, x^2y, xy^3, y^4 \rangle \subset k[x, y]$.



By definition, the ideal $\langle x^4, x^2y, xy^3, y^4 \rangle$ is in $\mathrm{Hilb}_0^{(4,5)} k^2$, since the ideal has 4 0's and 5 1's when represented as a colored Young diagram under our group action.

Outline

Stratified Hilbert schemes

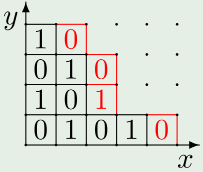
The *stratified punctual Hilbert scheme of type* $(m_0, m_1), (d_0, d_1)$, written $\mathrm{Hilb}_0^{(m_0,m_1),(d_0,d_1)} k^2$, is a subset of $\mathrm{Hilb}_0^{(m_0,m_1)} k^2$. The strata is determined by the first box outside the Young diagram in each row.

- $d_0 =$ number of those boxes containing zero
- $d_1 =$ number of those boxes containing one.

Stratified Hilbert scheme example

Example

$$\langle y^4, xy^3, x^2y, x^4\rangle \in \mathrm{Hilb}_0^{(4,5),(3,1)} k^2$$

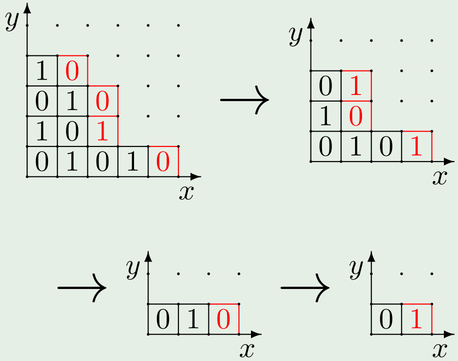


$$\mathrm{Hilb}_0^{(m_0,m_1)} k^2 = \bigcup_{d_0,d_1\geq 0} \mathrm{Hilb}_0^{(m_0,m_1),(d_0,d_1)} k^2$$

$$\Rightarrow \#\mathrm{Hilb}_0^{(m_0,m_1)} k^2 = \sum_{d_0,d_1\geq 0} \#\mathrm{Hilb}_0^{(m_0,m_1),(d_0,d_1)} k^2$$

Example

Example



Recursion

$$\#\mathrm{Hilb}_0^{(m_0,m_1),(d_0,d_1)}k^2 = \sum_{\substack{0\le d_0'\le d_1 \\ 0\le d_1'\le d_0 \\ d_1'-d_0'=d_0-d_1+(-1)^{(d_0'+d_1')}((d_0'+d_1'+d_0+d_1)\%2)}} q^r \cdot \#\mathrm{Hilb}_0^{(m_0-d_1,m_1-d_0),(d_0',d_1')}k^2$$

where

$$r = \begin{cases} d_0' & \text{if } d_0+d_1 \equiv 0 \mod 2 \\ d_1' & \text{if } d_0+d_1 \equiv 1 \mod 2 \end{cases}$$

Let $a, b, c, d \in \mathbb{Z}_{\ge 0}$. The base cases are

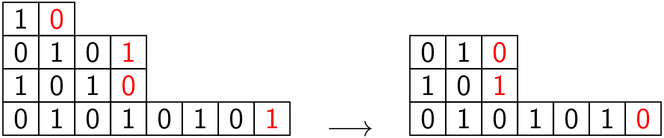
$$\#\mathrm{Hilb}_0^{(0,b>0),(c,d)}k^2 = 0 \qquad \#\mathrm{Hilb}_0^{(a,b),(c>b,d)}k^2 = 0$$

$$\#\mathrm{Hilb}_0^{(a,b),(c,d>a)}k^2 = 0 \qquad \#\mathrm{Hilb}_0^{(a\neq 0,b),(0,0)}k^2 = 0$$

$$\#\mathrm{Hilb}_0^{(0,0),(0,0)}k^2 = 1$$

Why $(m_0 - d_1, m_1 - d_0)$?

- Recurse by “chopping off” first column and sliding diagram over, then counting which ideals give same diagram
- Same as removing last block in each row
- 0 outside diagram $\Rightarrow$ 1 in last box
- 1 outside diagram $\Rightarrow$ 0 in last box
- Also requires $d_0' \leq d_1$ and $d_1' \leq d_0$ in smaller scheme



Recursion

$$\#\mathrm{Hilb}_0^{(m_0,m_1),(d_0,d_1)} k^2 = \sum_{\substack{0\le d_0'\le d_1 \\ 0\le d_1'\le d_0 \\ d_1'-d_0'=d_0-d_1+(-1)^{(d_0'+d_1')}((d_0'+d_1'+d_0+d_1)\%2)}} q^r \cdot \#\mathrm{Hilb}_0^{(m_0-d_1,m_1-d_0),(d_0',d_1')} k^2$$

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$$\#\mathrm{Hilb}_0^{(0,0),(0,0)} k^2 = 1$$

Some Special Cases

- $\#\mathrm{Hilb}_0^{(k,k+1),(k+1,k-1)} k^2 = 1$
- $\#\mathrm{Hilb}_0^{(k,k+1),(k+1,k-1)} k^2 = 1$
- $\#\mathrm{Hilb}_0^{(k,k),(1,0)} k^2 = q^k$
- $\#\mathrm{Hilb}_0^{(k+1,k),(0,1)} k^2 = q^k$
- $\#\mathrm{Hilb}_0^{(k,k+1),(2,0)} k^2 = \left\lceil \frac{k}{2} \right\rceil q^{k-1}$
- $\#\mathrm{Hilb}_0^{(k+1,k),(0,2)} k^2 = \left\lfloor \frac{k}{2} \right\rfloor q^k$

Outline

Generating Function

Prove Dr. Gholampour's conjectured generating function for the number of points in the punctual Hilbert scheme of n points

$$\sum_{n_0,n_1\geq 0} (\#\mathrm{Hilb}^{n_0,n_1})\, t_0^{n_0} t_1^{n_1} = \prod_{j\geq 1} \frac{1}{(1-q^{j-1}(t_0t_1)^j)(1-q^j(t_0t_1)^j)} \cdot \sum_{m\in\mathbb{Z}} t_0^{m^2} t_1^{m^2+m}$$

Future Work

Current future tasks and questions include

- Complete long term task
  - Prove generating function for $\#\mathrm{Hilb}_0^{(m_0,m_1)}k^2$
  - Use this to prove generating function for the non-punctual version of the Hilbert scheme
  - Apply Weil conjectures to generating function; get Poincaré polynomial
- Study different group actions

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