CLASS FIELD THEORY NOTES

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Abstract. This is the note for the Class Field Theory seminar. In this talk, I want to first prove some properties about the zeta functions and the L functions, and then use those properties to prove the Universal Norm Inequality, and maybe the Chebotarev density theorem.

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1. Zeta function

Firstly, in the case of both zeta function and the L function, we actually see that they are a certain kind of series. This is the general object that we want to study.

Definition 1.0.1. Dirichlet Series is any series of the form $\sum_{n=1}^{\infty} a_n n^{-s}$ where a_n are complex numbers and that s is the complex variable with the common notation $\sigma = Re(s)$.

Example 1.0.2. (1) In the case where $a_n = 1$, we recover the Riemann zeta function.
(2) If we have any characters $\chi(n) : G \to \mathbb{C}$, we can define $a_n = \chi(n)$, in this case, we recover the L-function.

1.1. Partial Summation Formula.

There is a common technique we use a lot in the analytic number theory which we will use several times in this note, which is called the Abel summation formula or partial summation. We will call it partial summation for the rest of the note, since the name Abel summation might be mixed with Abel summation method.

Lemma 1.1.1. *Let (a) and (b) be sequences in an arbitrary ring R.*
Let $A_n = \sum_{i=0}^{n} a_i$ be the partial sum of (a) from m to n.
Then:

$$\sum_{i=m}^{n} a_i b_i = \sum_{i=m}^{n-1} A_i (b_i - b_{i+1}) + A_n b_n - A_{m-1} b_m$$

In particular

$$\sum_{i=0}^{n} a_i b_i = \sum_{i=0}^{n-1} A_i (b_i - b_{i+1}) + A_n b_n$$

Proof. Take $A_n = \sum_{i=0}^{n} a_i$, we have

$$\sum_{i=m}^{n} a_i b_i = \sum_{i=m}^{n} (A_i - A_{i-1}) b_i$$
$$= \sum_{i=m}^{n-1} A_i (b_i - b_{i+1}) + A_n b_n - A_{m-1} b_m$$

If we take $m = 0$, then $A_{-1} = \sum_{i=0}^{-1} a_i = 0$. Thus we have

$$\sum_{i=0}^{n} a_i b_i = \sum_{i=0}^{n-1} A_i (b_i - b_{i+1}) + A_n b_n$$

□

Note that if we think about this, noting that $b_{i+1} - b_i = \Delta b_i$, then this is similar to the summation by parts in the Calculus, which is

$$\int f g' dx = fg - \int f' g dx$$

Example 1.1.2. (Baby example of how one usually uses the partial summation.)
Given $\sum b_n$ converges, a_n is monotonely bounded, then $S_N = \sum a_n b_n$ is converging uniformly.

Proof. We consider the Cauchy's criterion.

$$S_M - S_N = a_M B_M - a_N B_N + \sum_{i=N}^{M-1} B_i(a_i - a_{i+1})$$
$$= (a_M - a)B_M - (a_N - a)B_N + a(B_M - B_N) + \sum_{i=N}^{M-1} B_i(a_i - a_{i+1})$$

Note that as well

$$|\sum_{i=N}^{M-1} B_i(a_i - a_{i+1})| \leq \sum_{i=N}^{M-1} |B_i||(a_i - a_{i+1})| \leq B \sum |a_{n+1} - a_n|$$

.

Thus as $N, M \to \infty$, there exist N_0, such that we have that $|S_M - S_N| \leq \epsilon$ for $M, N > N_0$. □

1.2. **Dirichlet Series.**
Now we can proceed with our study of the Dirichlet series.

Proposition 1.2.1. *If* $\sum a_n n^{-s}$ *converge for* $s = s_0$*, then it converges for every* s*, such that* $\mathrm{Re}(s) > \mathrm{Re}(s_0)$ *uniformly on compact subset (to an analytic function).*

Proof. Note that to consider the uniform convergence, we just need to see the Cauchy's criterion. Note that

$$\sum_{n=M}^{N} a_n n^{-s} = \sum_{n=M}^{N} \frac{a_n}{n^{s_0}} \frac{1}{n^{s-s_0}}$$

, therefore we have

$$\sum_{n=M}^{N} a_n n^{-s} = \sum_{n=M}^{N} \frac{a_n}{n^{s_0}} \frac{1}{n^{s-s_0}}$$
$$= P_N(s_0)\frac{1}{N^{s-s_0}} + \sum_{n=M}^{N-1} P_N(s_0)(\frac{1}{n^{s-s_0}} - \frac{1}{(n+1)^{s-s_0}}) - P_{M-1}(s_0)\frac{1}{M^{s-s_0}}$$

Moreover we have that $\mathrm{Re}(s) > \mathrm{Re}(s_0) + \delta$ then using the identity

$$\frac{1}{n^{s-s_0}} - \frac{1}{(n+1)^{s-s_0}} = (s - s_0)\int_n^{n+1} \frac{dz}{z^{s-s_0+1}}$$

We have

$$\left|\frac{1}{n^{s-s_0}} - \frac{1}{(n+1)^{s-s_0}}\right| = \left|(s - s_0)\int_n^{n+1} \frac{dz}{z^{s-s_0+1}}\right|$$
$$\leq |s - s_0|\frac{1}{n^{\delta+1}}$$

Therefore we have

$$\left|\sum_{n=M}^{N-1} P_N(s_0)(\frac{1}{n^{s-s_0}} - \frac{1}{(n+1)^{s-s_0}})\right| \leq C\sum_{n=M}^{N-1} \frac{1}{n^{\delta+1}}(s - s_0)$$

Therefore, given any $\epsilon > 0$, exists $N_0 > 0$ such that for $N, M > N_0$,

$$\left|\sum_{n=M}^{N} a_n n^{-s}\right| = \left|P_N(s_0)\frac{1}{N^{s-s_0}} + \sum_{n=M}^{N-1} P_N(s_0)(\frac{1}{n^{s-s_0}} - \frac{1}{(n+1)^{s-s_0}}) - P_{M-1}(s_0)\frac{1}{M^{s-s_0}}\right|$$
$$< \epsilon_1 + \epsilon_2 + \epsilon_3 = \epsilon$$

□

Therefore we see in the previous proposition that the definition about "the line" of convergence will make sense, actually we have the following definition.

Definition 1.2.2. If σ_0 is the smallest real number such that $\sum a_n n^{-s}$ converges uniformly on compact subset for $\mathrm{Re}(s) > \sigma_0$, then we call that σ_0 is the abscissa of convergence.

Remark 1.2.3. Note that $s_1 = \sigma_1 + it_1$ is such that $\sum a_n n^{-s}$ converge at s_1 then $a_n = O(n^{\sigma_1})$ since $\frac{a_n}{n^{s_1}} \to 0$. Therefore, we have that the convergence is absolute on $\mathrm{Re}(s) \geq \sigma_1 + 1 + \delta$ for $\delta > 0$ using the comparison test with $\sum \frac{1}{n^{1+s}}$.

1.3. **Estimate for abscissa.**

Proposition 1.3.1. *$A_n = a_1 + \ldots + a_n$, then $|A_n| \leq Cn^{\sigma_1}$ for some absolute constant C, $\sigma_1 \geq 0$, then the abscissa of convergence is $\leq \sigma_1$.*
Note that A_n is bounded, then the abscissa of convergence is ≤ 1.

Proof. by partial summation we have

$$\sum_{n=M}^{N} a_n n^{-s} = A_N N^{-s} + \sum_{n=M}^{N-1} A_n(\frac{1}{n^s} - \frac{1}{(n+1)^s}) - \frac{A_{M-1}}{M^s}$$

Suppose we have $\sigma > \sigma_1 + \delta$ and $|\frac{A_N}{N^s} \leq \frac{C}{N^\delta}|$, then we immediately will have

$$|\frac{A_{M-1}}{M^s}| \leq C/M^\delta, \text{ and } |\frac{A_{N-1}}{N^s}| \leq C/N^\delta$$

Moreover, we have that

$$\sum_{n=M}^{N-1} A_n(\frac{1}{n^s} - \frac{1}{(n+1)^s}) = s\sum_{n=M}^{N-1} A_n \int_n^{n+1} \frac{1}{z^s} dz$$

Notice that using the condition in question, we have $\sigma > \sigma_1 + \delta$ with some $\delta > 0$, then

$$|A_n \int_n^{n+1} \frac{1}{z^s} dz| \leq c\int_n^{n+1} \frac{1}{z^{\sigma-\sigma_1} dz}$$

and thus

$$\begin{aligned}\sum_{n=M}^{N-1} A_n \int_n^{n+1} \frac{1}{z^s} dz &\leq c\int_M^N \frac{1}{z^{\sigma-\sigma_1}} dz \\ &\leq c(\frac{1}{M^{\sigma-\sigma_1+1}} - \frac{1}{N^{\sigma-\sigma_1+1}})\frac{1}{\sigma-\sigma_1} \\ &\leq c(\frac{1}{M^{\delta+1}} - \frac{1}{N^{\delta+1}})\frac{1}{\delta}\end{aligned}$$

Thus we have that the partial sum is convergent uniformly on compact subset if $\sigma > \sigma_1$, and the abscissa of convergence is thus $\leq \sigma_1$ □

Now we are back to zeta function. With the tool above, we can deduce the desired corollary as below.

Corollary 1.3.2. *$\zeta(s)$ is a analytic for $\mathrm{Re}(s) > 0$ except simple pole at $s = 1$ with residue 1.*

Proof. Apply the proposition we just proved to $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ with $A_n = n$, thus we have that $\sigma_1 = 1$ and thus $\zeta(s)$ converges for $\mathrm{Re}(s) > 1$.

now we claim that $\zeta(s)$ has only one simple pole at $s = 1$. Consider the alternating zeta functions for all $r \in \mathbb{Z}^+$ with a property $|A_n| \le r$,

$$\zeta_2(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \dots$$

$$\zeta_3(s) = 1 + \frac{1}{2^s} - \frac{2}{3^s} - \frac{1}{4^s} + \dots$$

$$\zeta_r(s) = 1 + \frac{1}{2^s} + \dots + \frac{1}{(r-1)^s} - \frac{1}{r^s} + \dots$$

Then we can see that there are identities between the alternating zeta function and the normal one, i.e.

$$(1 - \frac{1}{r^{s-1}})\zeta(s) = \zeta_r(s)$$

This analytic extends $\zeta(s)$ to the complex right half plane with possible pole $s = \frac{2\pi i}{\log r} + 1$ but this is not possible, since if there is a pole on the zeta function, then we need that $s = \frac{2\pi i}{\log r_1} + 1 = \frac{2\pi i}{\log r_2} + 1$, take $r_1 = 2$, $r_2 = 3$, we will thus have $2^n = 3^m$ for some number, which suggest that $n = m = 0$ and thus $s = 1$ will be the only simple pole if there is a pole there.

Notice as well on $\mathrm{Re}(s) > 1$, we have

$$\frac{1}{s-1} = \int_1^{\infty} \frac{1}{x^s} dx \le \zeta(s) \le 1 + \int_2^{\infty} \frac{1}{(x-1)^s} dx = \frac{1}{s-1}$$

Thus we have

$$1 \le (s-1)\zeta(s) \le s$$

with $s > 1$. Thus we have that $\zeta(s)$ is necessary to have a pole at $s = 1$ with residue 1.

Therefore, we have that $\zeta(s)$ is a analytic for $\mathrm{Re}(s) > 0$ except simple pole at $s = 1$ with residue 1. □

Combining this result with Proposition 1.3.1 again, we have a very useful variation of the proposition 1.3.1 as a corollary.

Corollary 1.3.3. *Consider $f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$, if we have $|A_n - np| \le Cn^{\sigma_1}$, then we have $f(s)$ is analytic on $\mathrm{Re}(s) > \sigma_1$, except for simple pole ρ at $s = 1$.*

Proof. Take $g(s) = f(s) - \rho\zeta(s)$, then we can just apply the proposition 1.3.1 on $g(s)$ and get the desired result. □

1.4. **Euler product.** We will prove that $\zeta(s)$ has an Euler product expansion as

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}$$

Heuristically, if we do Taylor expansion on each $\frac{1}{1-p^{-s}}$, we will get that

$$\prod_p \frac{1}{1-p^{-s}} = \prod_p (1 + p^{-s} + p^{-2s} + \ldots + p^{-ns} + \ldots) = \sum_n \frac{1}{n^s} = \zeta(s)$$

However, there is a serious problem that we need to show here, i.e., the product we defined above, might not be an analytic function at all and is thus not well defined.

However, this is not a hard work to show it.

Note that if z is complex with $|z| < 1$, then we have $-\log(1+z)$ well defined, and have corresponding Taylor series expansion and we have $|-\log(1+z)| \leq |z|$. Therefore, we have that $|-\sum_p \log\left(1 - \frac{1}{p^s}\right)| \leq \sum_p |\frac{1}{p^s}|$ converges absolutely. Therefore, the exponential of this function is well defined for $\mathrm{Re}(s) > 1$ and equals to $\prod(1 - \frac{1}{p^s})^{-1}$, converges for $\mathrm{Re}(s) > 1$.

Note that as well, $\sum_{p,m\geq 2} \frac{1}{mp^{ms}}$ converges uniformly and absolutely for $\mathrm{Re}(s) \geq 1/2 + \delta$, so we see that only $\sum_p \frac{1}{p^s}$ contributes to the pole. Therefore, we have

$$\zeta(s) \sim \frac{1}{s-1}$$

and

$$\log \zeta(s) \sim \log\left(\frac{1}{s-1}\right) \sim \sum_p \frac{1}{p^s}$$

where $f \sim g$ if two function have singular point at 1 and differ by an analytic function at 1.

Remark 1.4.1. Note that aside from the Euler product we have other product expansion. For example, using the Weierstrauss factorization theorem/Hadamard product theorem, we can expand zeta function into a hadamard product

$$\zeta(s) = \frac{e^{\log(2\pi)} - 1 - \gamma/2}{2(s-1)\Gamma(1+s/2)} \prod_\rho (1 - s/\rho) e^{s/\rho}$$

where in this expansion, the simple pole at $s = 1$, the trivial zeros at $-2, -4, \ldots$ and nontrivial zeros at $s = \rho$ are all well displayed in this expansion.

1.5. **General Zeta function.**

Definition 1.5.1. Denote K as a number field. We thus define the Dedekind Zeta function

$$\zeta_K(s) = \prod_{\mathfrak{p}} (\frac{1}{1 - N\mathfrak{p}^{-s}}) = \sum_Q \frac{1}{NQ^{-s}}$$

Therefore similar to the ζ function case, we have

$$\log \zeta_K(s) \sim \sum_{\mathfrak{p}} \frac{1}{N\mathfrak{p}^s} \sim \log \frac{1}{s-1} \sim \sum_{deg \mathfrak{p}=1} \frac{1}{N\mathfrak{p}^s}$$

Also, we have

$$\zeta_K(s) = \sum_{X \in I/P} \zeta(s, X)$$

where

$$\zeta(s, X) = \sum_{Q \in X} \frac{1}{NQ^s} = \sum_m \frac{a(X, m)}{m^s}$$

with $a(X, m) = \#\{Q \in X | NQ = m\}$.

Note that with the above definition we have that

$$\sum_{m=1}^{n} a(X.m) = \#\{Q \in X | NQ \leq n\} = \rho m + O(m^{1-1/d})$$

by Lang Chapter VI, Theorem 3. Therefore, using Corollary 1.3.3 we have that $\zeta(s, X)$ is analytic on $\mathrm{Re}(s) > 1 - 1/d$ with a simple pole at $s = 1$, and the residue there is ρ.

Furthermore, using the relation that $\zeta_K(s) = \sum_{X \in I/P} \zeta(s, X)$, we have that $\zeta_K(s)$ is analytic on $\mathrm{Re}(s) > 1 - 1/d$ except for a simple pole at $s = 1$ with the residue $h\rho$, where h is the class number.

Remark 1.5.2. Note that in the original theorem, there is an explicit formula for ρ, combining with the result we just get we can deduce class number formula within several lines.

Moreover, note that we can replace I/P by I_c/P_c, we can define another zeta function similarly.

Definition 1.5.3. Denote K as a number field, with $X \in I_c/P_c$. We thus define the Zeta function

$$\zeta_K(s, c) = \prod_{(\mathfrak{p}, c)=1} (\frac{1}{1 - N\mathfrak{p}^{-s}}) = \sum_{(Q, c)=1} \frac{1}{NQ^{-s}} = \sum_{X \in I_c/P_c} \zeta_c(s, X)$$

$$\zeta_c(s, X) = \sum_{Q \in X} \frac{1}{NQ^s}$$

Moreover, we have that $\zeta_K(s, c)$ is analytic on $\mathrm{Re}(s) > 1 - 1/d$ with simple pole at $s = 1$ with residue $h_c p_c$.

2. L-function and the Universal Norm Index Inequality

2.1. **L-function.** Given K a number field, and c a cycle, we define a character χ of I_c/P_c to be

$$\chi : I_c/P_c \to \mathbb{C}^\times$$

With the definition of the character, we can thus define L-function with this character.

Definition 2.1.1. Given K a number field, and c a cycle, and a character χ of I_c/P_c, we define

$$L(\chi, s) = \prod_{(\mathfrak{p},c)=1} \left(\frac{\chi(\mathfrak{p})}{1 - N\mathfrak{p}^{-s}}\right) = \sum_{(Q,c)=1} \frac{\chi(Q)}{NQ^{-s}}$$

where we can think of χ as a character on I_c which is trivial on P_c.

Moreover, similar to the arguments in zeta function case. we have

$$\log(L(s,\chi)) \sim \sum_{m,\mathfrak{p}\nmid c} \frac{\chi(\mathfrak{p})^m}{mNp^{ms}} \sim \sum_{\mathfrak{p}} \frac{\chi(\mathfrak{p})}{N\mathfrak{p}^s}$$

Also, due to the fact that

$$L(\chi, s) = \sum_{X \in I_c/P_c} \chi(X)\zeta(s, X)$$

We have that $L(\chi, s)$ is analytic on $\mathrm{Re}(s) > 1 - 1/d$, with possible pole at $s = 1$ of order 1, and the residue there is

$$Res(L(\chi, s), 1) = \sum_{X \in I_c/P_c} \chi(X)$$

, which is nonzero if and only if χ is trivial character.

2.2. **Universal norm index inequality.**

Theorem 2.2.1. *Given K/k Galois, c cycle in k divisible by all primes ramified in K/k then*

$$(I_c : P_c n(c)) \leq [K : k]$$

where $n(c)$ is the subgroup of $I(c)$ consisting of all norms $N_k^K A$, where A is a fractional ideal prime to c.

Proof. Let χ be any character of $I_c/P_c n(c) =: G$, we can thus view it as a charcter of I_c/P_c.

Since $L(\chi, s)$ is a analytic at $s = 1$, thus we know that $L(\chi, 1)$ has order of zero $m(\chi) \geq 0$. By the result in complex analysis, we know that we can write

$$L(\chi, s) = (s-1)^{m(\chi)} g(s, \chi)$$

where $g(1, \chi) \neq 0$ and $m(\chi) \in \mathbb{Z} \cap [0, \infty)$. Therefore, we have that

$$\log(L(s,\chi)) \sim m(\chi) \log(s-1) \sim -m(\chi) \log\left(\frac{1}{s-1}\right)$$

Also we know that

$$\log(L(s,\chi)) \sim \sum_{X \in I_c/P_c} \chi(X) \sum_{\mathfrak{p} \in X} \frac{1}{N\mathfrak{p}^s}$$

Therefore, after summing all the χ, we have that

$$\log(\zeta_K(s)) + \sum_{\chi \neq Id} \sim \sum_{\chi} \sum_{X \in G} \chi(X) \sum_{\mathfrak{p} \in X} \frac{1}{N\mathfrak{p}^s}$$

$$\sim |G| \sum_{X \in P_c n(c)/P_c} \sum_{\mathfrak{p} \in X} \frac{1}{n\mathfrak{p}^s}$$

Therefore, as $s \to 1$, we have

$$(1 - \sum_{\chi} m(\chi)) \log\left(\frac{1}{s-1}\right) \sim |G| \sum_{\mathfrak{p} \in P_c n(c), \deg \mathfrak{p}=1} \frac{1}{N\mathfrak{p}^s}$$

$$\geq |G| \sum_{\mathfrak{p} \in Spl(K/k)} \frac{1}{N\mathfrak{p}^s}$$

$$= \frac{|G|}{[K:k]} \sum_{\mathfrak{p} \in K, \mathfrak{p} \text{ splits over } k} \frac{1}{N\mathfrak{p}^s}$$

$$\geq \frac{|G|}{d} \sum_{\mathfrak{p}, \deg \mathfrak{p}=1} \frac{1}{N\mathfrak{p}^s}$$

$$\sim \frac{|G|}{[K:k] \log\left(\frac{1}{s-1}\right)}$$

As $s \to 1$, we have that $\frac{1}{s-1}$ is large, thus we have all $m(\chi) = 0$ since $m(chi)$ can only be non negative integers. thus we have that

$$1 \geq \frac{|G|}{[K:k]}$$

and thus

$$[K:k] \geq |G| = (I_c : P_c n(c))$$

□

2.3. **Density theorem.** Moreover, using the notation as above, class field theory gives that given any group $I_c \supset H \supset P_c$, there exist an abelian extension K/k such that $H = P_c n(c, K/k)$. Therefore, the conclusion that $m(\chi) = 0$ hold for all nontrivial χ of I_c/P_c.

Theorem 2.3.1. *Given a cycle c in k, χ be a nontrivial character of I_c/P_c. Then $L_c(1, \chi) \neq 0$*

Corollary 2.3.2. *given $h_c = [I_c : P_c]$, and an ideal class $X \in I_c/P_c$, we have for s real, $s \to 1^+$,*

$$\log\left(\frac{1}{s-1}\right) \sim h_c \sum_{\mathfrak{p} \in X} \frac{1}{N\mathfrak{p}^s}$$

Proof. The proof is a similar computation to the proof we did in UNI.

Note that we have the relation

$$\log L_c(s, \chi) \sim \sum_{Y \in I_c/P_c} \chi(Y) \sum_{\mathfrak{p} \in X}$$

If we multiple the relation by $\chi(X^{-1})$ and sum over all χ, we have

$$\log \zeta_k(s) \sim \sum_{Y \in I_c/P_c} \sum_{\chi} \chi(Y) \sum_{\mathfrak{p} \in X}$$

But then this sum is 0 unless the case where $YX^{-1} = 1$, which occurs only when $X = Y$. Thus

$$\log \zeta_k(s) \sim \log\left(\frac{1}{s-1}\right) \sim h_c \sum_{\mathfrak{p}\in X} \frac{1}{N\mathfrak{p}^s}$$

□

Definition 2.3.3. If M is a set of primes in k, then we consider the limit

$$\lim_{s\to 1^+} \frac{\sum_{\mathfrak{p}\in M} \frac{1}{N\mathfrak{p}^s}}{\log\left(\frac{1}{s-1}\right)}$$

This is what we call the Dirichlet Density of M, if it exists.

Corollary 2.3.4. *Given an ideal class of I_c/P_c, the Dirichlet density of it is $1/h_c$*

Proof. This is essentially a rewritting of the result of Cor 2.3.2. □