Large parameter asymptotic of rational solutions of Painlevé III (D6) equation near zero.

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Painlevé III (D6) equation

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- Painlevé III equation

$$u'' = \frac{(u')^2}{u} - \frac{u'}{x} + \alpha \frac{u^2}{x} + \frac{\beta}{x} + \gamma u^3 + \frac{\delta}{u}.$$

- Painlevé III (D6)

$$\gamma \neq 0, \quad \delta \neq 0.$$

- After scaling transformation $u(x) \to c_1 u(c_2 x)$ we can assume that

$$\alpha = 4\lambda + 4m, \quad \beta = 4\lambda - 4m, \quad \gamma = 4, \quad \delta = -4,$$

- Denote the solution of such equation $u_\lambda(x, m)$

Bäcklund transformation

- Given $u_n(x, m)$ one can consider the following expressions

$$u_{\lambda+1}(x) = \frac{1}{u_\lambda(x)} - \frac{2(2\lambda - 2m + 1)}{xu'_\lambda(x) + 2xu^2_\lambda(x) + 2x + (2\lambda - 2m + 1)u_\lambda(x)}$$

$$u_{\lambda-1}(x) = u_\lambda(x) - \frac{2(2\lambda + 2m - 1)u^2_\lambda(x)}{xu'_\lambda(x) - 2xu^2_\lambda(x) - 2x + (2\lambda + 2m - 1)u_\lambda(x)}$$

- They satisfy the Painlevé III (D6) equation with shifted parameter $\lambda \to \lambda \pm 1$.

Discrete Painlevé equations

- Eliminating the derivative one can obtain the discrete equation associated to Painlevé-III (D6)

$$\frac{2\lambda+1}{u_{\lambda+1}u_\lambda-1}+\frac{2\lambda-1}{u_\lambda u_{\lambda-1}-1}+2m+2\lambda+2x\left(u_\lambda+\frac{1}{u_\lambda}\right)=0$$

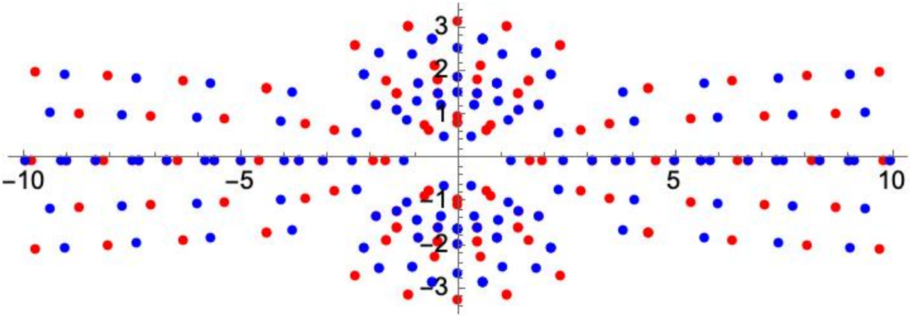
- It has symmetry type $2A_1^{(1)}$ and the surface type $D_6^{(1)}$ in the Sakai's classification.

Solutions of Painlevé III (D6) equation

Painlevé III (D6) solutions

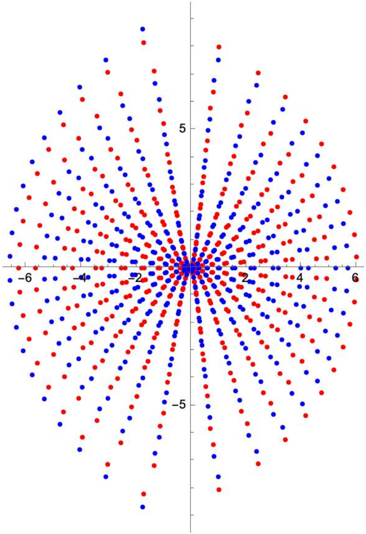
- For $\lambda = 0$ there is a solution $u_0(x, m) = 1$. Iterating it one obtains rational solutions $u_n(x, m)$ of discrete and continuous Painlevé equation.
- For $\lambda = \frac{1}{2}$ there is a Bessel solution $u_{\frac{1}{2}}(x, m) = -\frac{J_{m-\frac{1}{2}}(2z)}{J_{m+\frac{1}{2}}(2z)}$. Iterating it one obtains special functions solutions $u_{n+\frac{1}{2}}(x, m)$ of discrete and continuous Painlevé equation.
- For general λ solutions can't be expressed in terms of elementary functions.

Poles and zeroes of $u_{\frac{5}{2}}(x, -5)$



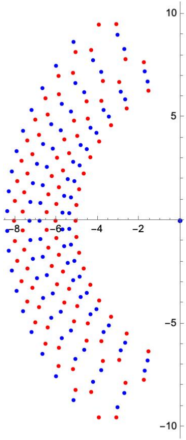
Red circles indicate zeroes, blue circles indicate poles.

Plot of poles and zeroes for $u_{20}(x, 0.25)$



Red circles indicate zeroes, blue circles indicate poles.

Plot of poles and zeroes for $u_{20}(x, 2.5)$



Red circles indicate zeroes, blue circles indicate poles.

Works on rational solutions of Painlevé III (D6) equation

- Clarkson, 2003
- Bothner, Miller, Sheung, 2018
- Bothner, Miller, 2020

Rational solutions near zero

Behaviour near zero

- Denote $v_n(x,m) = u_n\left(\frac{x}{n}, m\right)$. It satisfies the equation

$$v_n'' = \frac{(v_n')^2}{v_n} - \frac{v_n'}{z} + \frac{\alpha_n v_n^2}{z} + \frac{\beta_n}{z} + \gamma_n v_n^3 + \frac{\delta_n}{v_n}, \tag{1}$$

where

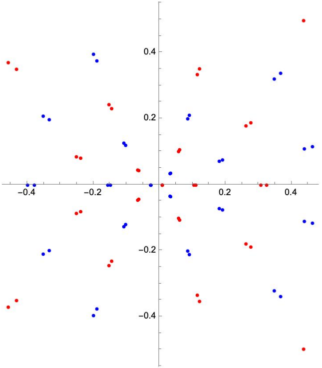
$$\alpha_n := 4 + \frac{4m}{n}, \quad \beta_n := 4 - \frac{4m}{n}, \quad \gamma_n := \frac{4}{n^2}, \quad \delta_n := -\frac{4}{n^2}. \tag{2}$$

- Painlevé III (D8) equation

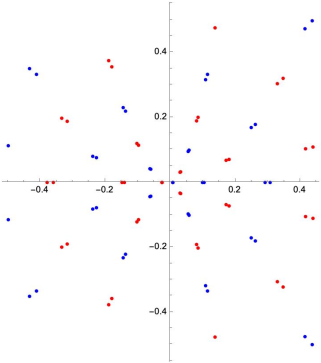
$$w'' = \frac{(w')^2}{w} - \frac{w'}{x} + \frac{4w^2}{x} + \frac{4}{x}.$$

- Reasonable expectation: $v_n(x,m) \underset{n\to\infty}{\to} w(x,m)$,

Behaviour near zero



$u_{19}(x, 0.25)$



$u_{20}(x, 0.25)$

Conjecture(Bothner, Miller, Sheng, 2018)

$$v_{2j}(x,m) \underset{j\to\infty}{\to} w_1(x,m), \quad v_{2j+1}(x,m) \underset{j\to\infty}{\to} w_2(x,m),$$

Main result

Main result

- Consider solution $w(x, m)$ of Painlevé III (D8) holomorphic at zero with initial condition

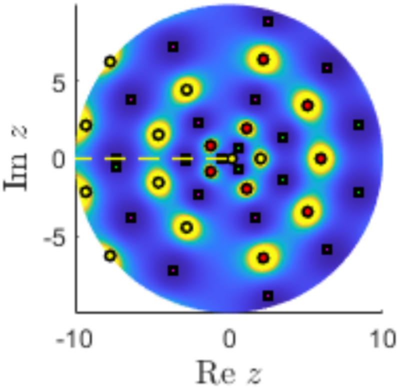
$$w(0, m) = \tan\left(\frac{\pi(2m+1)}{4}\right)$$

Theorem (Barhoumi, Lisovyy, Miller, Prokhorov)

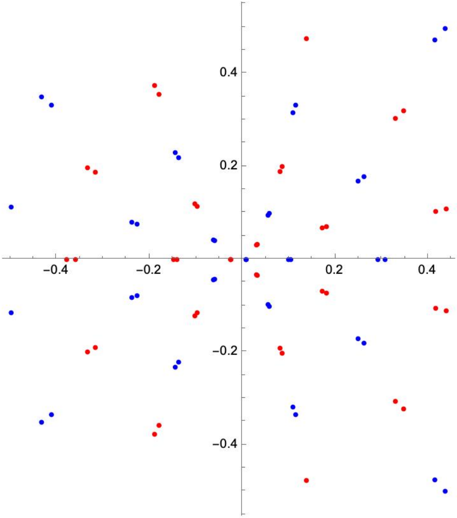
$$\lim_{j\to\infty} u_{2j}\left(\frac{x}{2j}, m\right) = w(x, m), \quad m \neq \frac{1}{2} + \mathbb{Z}$$

$$\lim_{j\to\infty} u_{2j+1}\left(\frac{x}{2j+1}, m\right) = -\frac{1}{w(x, m)}, \quad m \neq \frac{1}{2} + \mathbb{Z}$$

Numerics

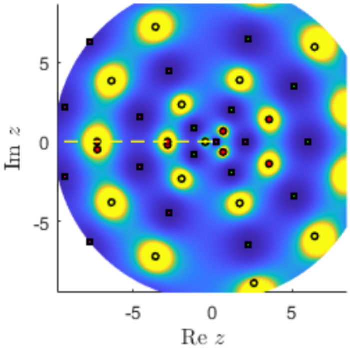


$w(x, 0.25)$, blue highlight=zeroes, yellow highlight=poles
Code by Fasondini, Fornberg, Weidemann (2018)

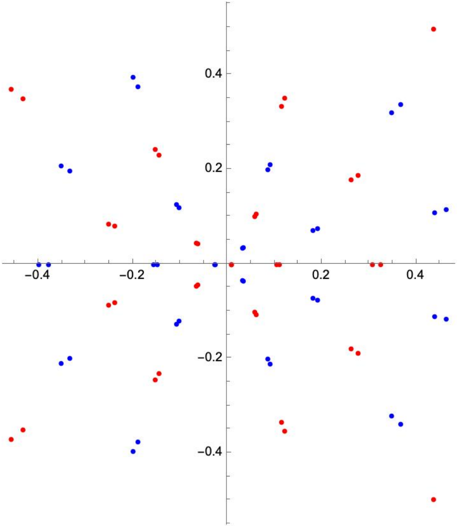


$u_{20}(x, 0.25)$, red=zeroes, blue=poles

Numerics



$-(w(x, 0.25))^{-1}$, blue highlight=zeroes, yellow highlight=poles
Code by Fasondini, Fornberg, Weidemann (2018)



$u_{19}(x, 0.25)$, red=zeroes, blue=poles

Proof

Proof, Step 1: compute $w(0, m)$.

Lemma (Clarkson, Law, Lin, 2018 (Corollary 4.2))

$$u_{2j}(0, m) = \frac{\prod_{k=1}^{j}((m - \frac{1}{2})^2 - (2k-1)^2)}{\prod_{k=1}^{j}((m + \frac{1}{2})^2 - (2k-1)^2)}.$$

$$u_{2j+1}(0, m) = \frac{(m - \frac{1}{2}) \cdot \prod_{k=1}^{j}((m - \frac{1}{2})^2 - (2k)^2)}{(m + \frac{1}{2}) \cdot \prod_{k=1}^{j}((m + \frac{1}{2})^2 - (2k)^2)}.$$

Proof, Step 1: compute $w(0, m)$.

Using the classical product formulas

$$\sin(x) = x \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{\pi^2 k^2}\right), \quad \cos(x) = \prod_{k=1}^{\infty} \left(1 - \frac{4x^2}{\pi^2 (2k-1)^2}\right)$$

we get

Corollary

$$\lim_{j\to\infty} u_{2j}(0, m) = \tan\left(\frac{\pi(2m+1)}{4}\right),$$

$$\lim_{j\to\infty} u_{2j+1}(0, m) = -\cot\left(\frac{\pi(2m+1)}{4}\right).$$

Proof, Step 2: derive the recurrence

- Because of the differential equation the coefficients of the Taylor series $v_n(x,m) = \sum_{k=0}^{\infty} c_k x^k$ satisfy the recurrence

$$c_1 = \beta_n + \alpha_n c_0^2,$$

$$c_2 = \frac{1}{4c_0}\left(3\alpha_n c_0^2 c_1 + \beta_n c_1 + \gamma_n c_0^4 + \delta_n\right)$$

$$c_{k+1} = \frac{1}{c_0(k+1)^2}\left[\sum_{j=0}^{k+1} j(k+1-2j)c_j c_{k+1-j} + \alpha_n \sum_{j_1=0}^{k}\sum_{j_2=0}^{k-j_1} c_{j_1}c_{j_2}c_{k-j_1-j_2}\right.$$

$$\left. +\beta_n c_k + \gamma_n \sum_{j_1=0}^{k-1}\sum_{j_2=0}^{k-1-j_1}\sum_{j_3=0}^{k-1-j_1-j_2} c_{j_1}c_{j_2}c_{j_3}c_{k-1-j_1-j_2-j_3}\right] = 0,\, k \geq 2.$$

Proof, Step 3: take the limit

- Taking the limit formally we obtain that the limiting coefficients satisfy the recurrence for Painlevé III (D_8).
- To justify it we need uniform convergence with respect to n of the Taylor series for $v_n(x, m)$. It can be done following the book "From Gauss to Painlevé", Proposition 1.1.1, page 261 by Iwasaki, Kimura, Shimomura, Yoshida .

Remarks

- Monodromy data of the solution $w(x, m)$ can be identified using the PhD thesis of David Niles, 2009.
- We have alternative proof of our main result using Riemann-Hilbert approach.

Further questions

- Other interesting regimes allowing simultaneous grows of m and n.
- Large parameter asymptotic of $u_{\frac{1}{2}+n}(x, m)$.

Thank you!