

# A New Upper Bound on the Average Error Exponent for Multiple-Access Channels

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## Abstract

A new lower bound for the average probability of error for a two-user discrete memoryless (DM) multiple-access channel (MAC) is derived. This bound has a structure very similar to the well-known sphere packing bound derived by Haroutunian. However, since it explicitly imposes independence of the users' input distributions (conditioned on the time-sharing auxiliary variable) results in a tighter sphere-packing exponent in comparison to Haroutunian's. Also, the relationship between average and maximal error probabilities is studied. Finally, by using a known sphere packing bound on the maximal probability of error, a lower bound on the average error probability is derived.

## Index Terms

Multiple-access channel, error exponents, Sphere Packing bound.

## I. INTRODUCTION

One of the most important practical questions which arises when we are designing or using an information transmission or processing system is: How much information can this system transmit or process in a given time? Information theory, developed by Claude E. Shannon during World War II, defines the notion of channel capacity and provides a mathematical model by which one can compute it. Basically, Shannon coding theorem and all newer versions of it treat the question of how much data can be reliably communicated from one point, or sets of points, to another point or sets of points.

The class of channels to be considered include multiple transmitter and a single receiver. The received signal is corrupted both by noise and by mutual interference between the transmitters. Each of transmitters is fed by an information source, and each information source generates a sequence of messages. More specifically, a two-user DM-MAC is defined by a stochastic matrix<sup>1</sup>  $W : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Z}$ , where the input alphabets,  $\mathcal{X}$ ,  $\mathcal{Y}$ , and the output

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<sup>1</sup>We use the following notation throughout this work. Script capitals  $\mathcal{U}$ ,  $\mathcal{X}$ ,  $\mathcal{Y}$ ,  $\mathcal{Z}, \dots$  denote finite, nonempty sets. To show the cardinality of a set  $\mathcal{X}$ , we use  $|\mathcal{X}|$ . We also use the letters  $P, Q, \dots$  for probability distributions on finite sets, and  $U, X, Y, \dots$  for random variables.

alphabet,  $\mathcal{Z}$ , are finite sets. The channel transition probability for sequences of length  $n$  is given by

$$W^n(\mathbf{z}|\mathbf{x}, \mathbf{y}) \triangleq \prod_{i=1}^n W(z_i|x_i, y_i) \quad (1)$$

where

$$\mathbf{x} \triangleq (x_1, \dots, x_n) \in \mathcal{X}^n, \mathbf{y} \triangleq (y_1, \dots, y_n) \in \mathcal{Y}^n$$

and

$$\mathbf{z} \triangleq (z_1, \dots, z_n) \in \mathcal{Z}^n.$$

It has been proven, by Ahlswede [1] and Liao's [6] coding theorem, that for any  $(R_X, R_Y)$  in the interior of a certain set  $\mathcal{C}$ , and for all sufficiently large  $n$ , there exists a multiuser code with an arbitrary small average probability of error. Conversely, for any  $(R_X, R_Y)$  outside of  $\mathcal{C}$ , the average probability of error is bounded away from 0. The set  $\mathcal{C}$ , called *capacity region* for  $W$ , is the closure of the set of all rate pairs  $(R_X, R_Y)$  satisfying [12]

$$0 \leq R_X \leq I(X \wedge Z|Y, U) \quad (2a)$$

$$0 \leq R_Y \leq I(Y \wedge Z|X, U) \quad (2b)$$

$$0 \leq R_X + R_Y \leq I(XY \wedge Z|U), \quad (2c)$$

for all choices of joint distributions over the random variables  $U, X, Y, Z$  of the form  $p(u)p(x|u)p(y|u)W(z|x, y)$  with  $U \in \mathcal{U}$  and  $|\mathcal{U}| \leq 4$ . As we can see, this theorem was presented in an asymptotic nature, i.e., it was proven that the error probability of the channel code can go to zero as the block length goes to infinity. It does not tell us how large the block length must be in order to achieve a specific error probability. On the other hand, in practical situations, there are limitations on the delay of the communication. Additionally, the block length of the code cannot go to infinity. Therefore, it is important to study how the probability of error drops as the block length goes to infinity. A partial answer to this question is provided by examining the error exponent of the channel.

Error exponents have been studied for discrete memoryless multiple-access channels over the past thirty years. Lower and upper bounds are known on the error exponent of these channels. The random coding bound in information theory provides a well-known lower bound for the reliability function of the best code, of a given rate and block length. This bound is constructed by upper-bounding the average error probability over an ensemble of codes. Slepian and Wolf [12], Dyachkov [3], Gallager [4], Pokorný and Wallmeier [11], and Liu and Hughes [7] have all studied the random coding bound for discrete memoryless multiple access channels. Nazari and et al. [8] investigated two different upper bounds on the average probability of error, called the typical random coding bound and the partial expurgated bound. The typical bound is basically the typical performance of the ensemble. By this, we mean that almost all random codes exhibit this performance. In addition, they have shown that the typical random code performs better than the average performance over the random coding ensemble, at least, at low rates. The random coding exponent may be improved at low rates by a process called "partial expurgation" which yields a new bound that exceeds the random coding bound at low rates.

Haroutunian [5] and Nazari [9], [10] studied upper bounds on the error exponent of multiple access channels. In Multi-user information theory, the sphere packing bound provides a well known upper bound on the reliability function for multiple access channel. The sphere packing bound that Haroutunian [5] derived on the average error exponent for DM-MAC is potentially loose, as it does not capture the separation of the encoders in the MAC. Nazari et al. [10] derived another sphere packing bound which takes into account separation of the encoders. The bound in [10] turns out to be at least as good as the bound derived in [5], however it is a valid bound only for the maximal error exponent and not the average. The sphere packing bound is a good bound in high rate regime. Nevertheless, it tends to be a loose bound in low rate regime. It can be shown that in low rate regime, the minimum distance of the code dominates the probability of error. Using the minimum distance of the code, Nazari [9] derived another upper bound for the maximal error exponent of DM-MAC. To derive the minimum distance bound, they established a connection between the minimum distance of the code and the maximum probability of error; then, by obtaining an upper bound on the minimum distance of all codes with certain rates, they derived a lower bound on the maximal error probability that can be obtained by a code with a certain rate pair.

The paper is organized as follows. Some preliminaries are introduced in section II. The main result of the paper, which is an upper bound on the reliability function of the channel, is obtained in section III. In section IV, by using a known upper bound on the maximum error exponent function, we derive an upper bound on the average error exponent function. The proofs of some of these results are given in the Appendix.

## II. PRELIMINARIES

For any alphabet  $\mathcal{X}$ ,  $\mathcal{P}(\mathcal{X})$  denotes the set of all probability distributions on  $\mathcal{X}$ . The *type* of a sequence  $\mathbf{x} = (x_1, \dots, x_n) \in \mathcal{X}^n$  is the distributions  $P_{\mathbf{x}}$ , on  $\mathcal{X}$ , defined by:

$$P_{\mathbf{x}}(x) \triangleq \frac{1}{n}N(x|\mathbf{x}), \quad x \in \mathcal{X}, \quad (3)$$

where  $N(x|\mathbf{x})$  denotes the number of occurrences of  $x$  in  $\mathbf{x}$ . Let  $\mathcal{P}_n(\mathcal{X})$  denotes the set of all types in  $\mathcal{X}^n$ , and define the set of all sequences in  $\mathcal{X}^n$  of type  $P$  as

$$T_P \triangleq \{\mathbf{x} \in \mathcal{X}^n : P_{\mathbf{x}} = P\}. \quad (4)$$

The joint type of a pair  $(\mathbf{x}, \mathbf{y}) \in \mathcal{X}^n \times \mathcal{Y}^n$  is the probability distribution  $P_{\mathbf{x}, \mathbf{y}}$  on  $\mathcal{X} \times \mathcal{Y}$  defined by:

$$P_{\mathbf{x}, \mathbf{y}}(x, y) \triangleq \frac{1}{n}N(x, y|\mathbf{x}, \mathbf{y}), \quad (x, y) \in \mathcal{X} \times \mathcal{Y}, \quad (5)$$

where  $N(x, y|\mathbf{x}, \mathbf{y})$  is the number of occurrences of  $(x, y)$  in  $(\mathbf{x}, \mathbf{y})$ . The relative entropy or *Kullback-Leibler* distance between two probability distribution  $P, Q \in \mathcal{P}(\mathcal{X})$  is defined as

$$D(P||Q) \triangleq \sum_{x \in \mathcal{X}} P(x) \log \frac{P(x)}{Q(x)}. \quad (6)$$

Let  $\mathcal{W}(\mathcal{Y}|\mathcal{X})$  denote the set of all stochastic matrices with input alphabet  $\mathcal{X}$  and output alphabet  $\mathcal{Y}$ . Then, given stochastic matrices  $V, W \in \mathcal{W}(\mathcal{Y}|\mathcal{X})$ , the conditional *I-divergence* is defined by

$$D(V||W|P) \triangleq \sum_{x \in \mathcal{X}} P(x) D(V(\cdot|x)||W(\cdot|x)). \quad (7)$$

**Definition 1.** An  $(n, M, N)$  multi-user code is a set  $\{(\mathbf{x}_i, \mathbf{y}_j, D_{ij}) : 1 \leq i \leq M, 1 \leq j \leq N\}$  with

- $\mathbf{x}_i \in \mathcal{X}^n, \mathbf{y}_j \in \mathcal{Y}^n, D_{ij} \subset \mathcal{Z}^n$
- $D_{ij} \cap D_{i'j'} = \emptyset$  for  $(i, j) \neq (i', j')$ .

The average error probability of this code for the MAC,  $W : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Z}$ , is defined as

$$e(\mathcal{C}, W) \triangleq \frac{1}{MN} \sum_{i=1}^M \sum_{j=1}^N W^n(D_{i,j}^c | \mathbf{x}_i, \mathbf{y}_j). \quad (8)$$

Similarly, the maximal error probability of this code for  $W$  is defined as

$$e_m(\mathcal{C}, W) \triangleq \max_{(i,j)} W^n(D_{i,j}^c | \mathbf{x}_i, \mathbf{y}_j). \quad (9)$$

**Definition 2.** For the MAC,  $W : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Z}$ , the average and maximal error reliability functions, at rate pair  $(R_X, R_Y)$ , are defined as:

$$E_{av}^*(R_X, R_Y) \triangleq \lim_{n \rightarrow \infty} \max_{\mathcal{C}} -\frac{1}{n} \log e(\mathcal{C}, W) \quad (10)$$

$$E_m^*(R_X, R_Y) \triangleq \lim_{n \rightarrow \infty} \max_{\mathcal{C}} -\frac{1}{n} \log e_m(\mathcal{C}, W), \quad (11)$$

where the maximum is over all codes of length  $n$  and rate pair  $(R_X, R_Y)$ .

**Definition 3.** A code  $\mathcal{C}_X = \{\mathbf{x}_i \in \mathcal{X}^n : i = 1, \dots, M_X\}$ , for some  $P_X$ , is called a bad codebook, if

$$\exists (i, j), \quad i \neq j \quad \mathbf{x}_i = \mathbf{x}_j \quad (12)$$

A codebook which is not bad, is called a good one.

**Definition 4.** A multi user code  $\mathcal{C} = \mathcal{C}_X \times \mathcal{C}_Y$  is called a good multi user code, if both individual codebooks  $\mathcal{C}_X, \mathcal{C}_Y$  are good codes.

**Definition 5.** For a good multi user code  $\mathcal{C} = \mathcal{C}_X \times \mathcal{C}_Y$ , and for a particular type  $P_{XY} \in \mathcal{P}_n(\mathcal{X} \times \mathcal{Y})$ , we define

$$R(\mathcal{C}, P_{XY}) \triangleq \frac{1}{n} \log |\mathcal{C} \cap T_{P_{XY}}| \quad (13)$$

**Definition 6.** For a sequence of joint types  $P_{XY}^n \in \mathcal{P}_n(\mathcal{X} \times \mathcal{Y})$ , with marginal types  $P_X^n$  and  $P_Y^n$ , the sequence of type graphs,  $G_n$ , is defined as follows: For every  $n$ ,  $G_n$  is a bipartite graph, with its left vertices consisting of all  $x^n \in T_{P_X^n}$  and the right vertices consisting of all  $y^n \in T_{P_Y^n}$ . A vertex on the left (say  $\tilde{x}^n$ ) is connected to a vertex on the right (say  $\tilde{y}^n$ ) if and only if  $(\tilde{x}^n, \tilde{y}^n) \in T_{P_{XY}^n}$ .

### III. MAIN RESULT

The main result of this section is a new sphere packing bound for the average error probability for a discrete memoryless multiple access channel. The idea behind the derivation of this bound is based on the property that is common among all good multi user codes with certain rate pair. In the following, we first derive a sphere packing bound for a good multiuser code. Next, we show that for any bad multiuser code, there exists a good code with the

same rate pair and smaller average probability of error. Therefore, to obtain a lower bound for the average error probability for the best code, we only need to study good codes (codes without any repeated codewords).

Now, consider a good multiuser code with blocklength  $n$ . Suppose the number of messages of the first source is  $M_X = 2^{nR_X}$  and the number of messages of the second source is  $M_Y = 2^{nR_Y}$ . Assume that all the messages of any source are equiprobable and the sources are sending data independently. Considering these assumptions, all  $M_X M_Y$  pairs are occurring with the equal probability. Thus, at the input of the channel, we can see all possible  $2^{n(R_X+R_Y)}$  (an exponentially increasing function of  $n$ ) pairs of input sequences. However, we also know that the number of possible types is a polynomial function of  $n$ . Thus, for at least one joint type, the number of pairs of sequences in the multi user code sharing that particular type, should be an exponential function of  $n$  with the rate arbitrary close to the rate of the multi user code. We will look at these pairs of sequences as a subcode, and then try to find a lower bound for the average error probability of this subcode. Following, we will show that this bound is a valid lower bound for the average probability of error for the original code.

**Lemma 1.** [9] *For any  $\delta > 0$ , for any sufficiently large  $n$ , and for any good  $(n, 2^{nR_X}, 2^{nR_Y})$  multi user code  $\mathcal{C}$ , as defined above, there exists  $P_{XY} \in \mathcal{P}_n(\mathcal{X} \times \mathcal{Y})$  such that*

$$R(\mathcal{C}, P_{XY}) \geq R_X + R_Y - \delta \quad \text{for sufficiently large } n,$$

$P_{XY}$  is called a dominant type of  $\mathcal{C}$ .

Hence, for any good code, there must exist at least a joint type which dominates the codebook. We can ask the following question: for a multiuser code, with rate  $(R_X, R_Y)$ , can any joint type potentially be its dominant type? As shown later, the answer to this question helps us characterize a tighter sphere packing bound. In response to this question, Nazari et al. [9] studied the type graphs for different joint types and proved the following result:

**Lemma 2.** [9] *For all sequences of nearly complete subgraphs of a particular type graph  $T_{P_{XY}}$ , the rates of the subgraph  $(R_X, R_Y)$  must satisfy*

$$R_X \leq H(X|U), \quad R_Y \leq H(Y|U) \tag{14}$$

for some  $P_{U|XY}$  such that  $X - U - Y$ .

Now consider a particular joint type  $P_{XY}^n$ . By the previous lemma, if there does not exist any  $P_{U|XY}$  satisfying the constraint mentioned in lemma 2, the type graph corresponding to this joint type can not contain an almost fully connected subgraph with rate  $(R_X, R_Y)$ . Consequently, it cannot be the dominant type of a good multiuser code with rate  $(R_X, R_Y)$ .

**Fact 1.** *Consider a good multiuser code  $\mathcal{C}$  with parameter  $(n, 2^{nR_X}, 2^{nR_Y})$ . A joint type  $P_{XY}^n \in \mathcal{P}_n(\mathcal{X} \times \mathcal{Y})$  can be the dominant type of  $\mathcal{C}$  if there exists a  $P_{U|XY}$ ,  $X - U - Y$ , such that*

$$R_X \leq H(X|U), \quad R_Y \leq H(Y|U), \tag{15}$$

conversely, if it does not exist such a conditional distribution, then  $P_{XY}^n$  cannot be the dominant type of any good multiuser code with parameter  $(n, 2^{nR_X}, 2^{nR_Y})$ .

**Theorem 1.** Fix any  $R_X \geq 0$ ,  $R_Y \geq 0$ ,  $\delta > 0$  and a sufficiently large  $n$ . Consider a good multiuser code  $\mathcal{C}$  with parameter  $(n, 2^{nR_X}, 2^{nR_Y})$  which has a dominant type  $P_{XY}^* \in \mathcal{P}_n(\mathcal{X} \times \mathcal{Y})$ . The average error exponent of such a code is bounded above by

$$E_{sp}(R_X, R_Y, W) \triangleq \min_{V_{Z|XY}} D(V_{Z|XY} || W | P_{XY}^*). \quad (16)$$

Here, the minimization is over all possible conditional distributions  $V_{Z|XY} : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Z}$ , which satisfy at least one of the following conditions

$$I_V(X \wedge Z|Y) \leq R_X \quad (17)$$

$$I_V(Y \wedge Z|X) \leq R_Y \quad (18)$$

$$I_V(XY \wedge Z) \leq R_X + R_Y. \quad (19)$$

*Proof:* The proof is provided in Appendix A.1. ■

In theorem 1, we obtained a sphere packing bound on the average error exponent for a good multiuser code with a certain dominant type. For a more general code, we do not know the dominant type of the code. However, we do have the condition for a joint type to be the potential dominant type of a code with certain parameter. By combining the result of theorem 1 and fact 1, we can obtain the following sphere packing bound for any good multiuser code:

**Theorem 2.** For any given multiple access channel  $W$  and any good multi user code with rate pair  $(R_X, R_Y)$ , the reliability function,  $E(R_X, R_Y, W)$ , is bounded above by

$$E_{sp}(R_X, R_Y, W) \triangleq \max_{P_{U,XY}} \min_{V_{Z|XY}} D(V_{Z|XY} || W | P_{XY}). \quad (20)$$

Here, the maximum is taken over all possible joint distributions satisfying  $X - U - Y$  and

$$R_X \leq H(X|U), R_Y \leq H(Y|U), \quad (21)$$

and the minimum over all channels  $V_{Z|XY}$  that satisfy at least one of the following conditions

$$I_V(X \wedge Z|Y) \leq R_X \quad (22)$$

$$I_V(Y \wedge Z|X) \leq R_Y \quad (23)$$

$$I_V(XY \wedge Z) \leq R_X + R_Y. \quad (24)$$

Thus far, we have obtained a lower bound on the average error probability for all good multiuser codes with certain rate pairs. Here, we show that the result of the previous theorem is indeed a valid bound for any multiuser code regardless of whether it is good or bad. This approach shows that for any bad code there exists a good code with the same number of codewords and a better performance. Therefore, to obtain a lower bound on the error

probability of the best code, we only need to consider codes without any repeated codewords. In lemma 3, we prove this result for a single-user code and later, by using the result of lemma 3 several times, we prove the same result for the multiuser scenario.

**Lemma 3.** *Suppose  $C_X$  is a codebook of size  $M_X$  for which all codewords are selected from  $T_{P_X}$ . Moreover, suppose  $\mathbf{x}_i$  is repeated  $N_i$  times in the codebook and  $M_X = N_1 + N_2 + \dots + N_M$ , where  $M$  is the number of distinct sequences in  $C_X$ . If  $M_X \leq |T_{P_X}| - 1$ , there exists another code  $C'_X$  with better probability of error, such that*

$$\begin{aligned} |C_X| &= |C'_X| \\ N'_i &= N_i \quad i = 1, \dots, M-1 \\ N'_M &= N_M - 1 \\ N'_{M+1} &= 1 \end{aligned} \tag{25}$$

Here,  $N'_{M+1} = 1$  is the number of occurrences of the new sequence  $\mathbf{x} \in T_{P_X}$  which does not belong to  $C_X$ .

*Proof:* The proof is provided in Appendix A.2. ■

**Lemma 4.** *For any bad multi user code with codewords that belong to  $T_{P_X}$ , and  $T_{P_Y}$ , with rate pair  $(R_X, R_Y)$ , there exists a good multi user code with the same rate pair and a better probability of error.*

*Proof:* For a bad multi user code, we know that at least one of the individual codebooks is bad. If we apply lemma 3 several times to any of the bad single user codes, with the appropriate cardinality, we will end up with a good multiuser code and a better probability of decoding error. ■

Finally, by combining the result of lemma 4 and the result of theorem 2, we deduce an upper bound on the reliability function for all multiuser codes.

**Theorem 3.** *For any given multiple access channel  $W$ , and any good multi user code with rate pair  $(R_X, R_Y)$ , the reliability function,  $E(R_X, R_Y, W)$ , is bounded above by*

$$E_{sp}(R_X, R_Y, W) \triangleq \max_{P_{XY}} \min_{V_{Z|XY}} D(V_{Z|XY} || W | P_{XY}). \tag{26}$$

Here, the maximum is taken over all possible joint distributions, and the minimum over all channels  $V_{Z|XY}$  which satisfy at least one of the following conditions

$$\begin{aligned} I_V(X \wedge Z|Y) &\leq R_X \\ I_V(Y \wedge Z|X) &\leq R_Y \\ I_V(XY \wedge Z) &\leq R_X + R_Y \end{aligned} \tag{27}$$

#### IV. ANOTHER SPHERE PACKING BOUND

In point to point communications systems, one can show that a lower bound for the maximal error probability of the best code is also a lower bound on the average probability of error for such a code. However, in multiuser communications, this is not the case. It has been shown that for multiuser channels, in general, the maximal error capacity region is smaller than the average error capacity region [2]. Therefore, we cannot hope a sphere packing bound for maximal error probability to be equal to the one for the average probability of error. In the following, we show an approach to derive an upper bound on the average error exponent by using a known upper bound for the maximal error exponent.

**Lemma 5.** *Fix any DM-MAC  $W : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Z}$ ,  $R_X \geq 0$ ,  $R_Y \geq 0$ . Assume that, the maximal reliability function is bounded as follows:*

$$E_m^L(R_X, R_Y) \leq E_m^*(R_X, R_Y) \leq E_m^U(R_X, R_Y), \quad (28)$$

therefore, the average reliability function can be bounded by

$$E_m^L(R_X, R_Y) \leq E_{av}^*(R_X, R_Y) \leq E_m^U(R_X, R_Y) + R, \quad (29)$$

where  $R = \min\{R_X, R_Y\}$ . Similarly, if the average reliability function is bounded as follows:

$$E_{av}^L(R_X, R_Y) \leq E_{av}^*(R_X, R_Y) \leq E_{av}^U(R_X, R_Y), \quad (30)$$

it can be concluded that the maximal reliability function satisfies the following constraint

$$E_{av}^L(R_X, R_Y) - R \leq E_m^*(R_X, R_Y) \leq E_{av}^U(R_X, R_Y). \quad (31)$$

*Proof:* The proof is provided in Appendix A.3. ■

In [10], the authors derived a sphere packing bound on the maximal reliability function for DM-MAC. This results is only a valid upper bound for the maximal error reliability function and not the average one. We can simply use the previous lemma to derive a new upper bound on the average error reliability function for DM-MAC.

**Theorem 4.** *For any  $R_X, R_Y > 0$ ,  $\delta > 0$  and any DM-MAC,  $W : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Z}$ , every  $(n, M_X, M_Y)$  code,  $\mathcal{C}$  with*

$$\frac{1}{n} \log M_X \geq R_X + \delta \quad (32a)$$

$$\frac{1}{n} \log M_Y \geq R_Y + \delta, \quad (32b)$$

has average probability of error

$$e(\mathcal{C}, W) \geq \frac{1}{2} \exp(-n(E_{sp}^m(R_X, R_Y, W) + R)(1 + \delta)), \quad (33)$$

where  $E_{sp}^m$  is the sphere packing bound derived in [10], and  $R = \min\{R_X, R_Y\}$ .



## V. APPENDIX

## A. Appendix A.1

For a given MAC  $W : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Z}$  and a good multi user code  $C = C_X \times C_Y$ , where  $C_X = \{ \mathbf{x}_i \in \mathcal{X}^n : i = 1, \dots, M_X \}$  and  $C_Y = \{ \mathbf{y}_j \in \mathcal{Y}^n : j = 1, \dots, M_Y \}$ , with decoding sets  $D_{i,j} \subset \mathcal{Z}^n$ , we have

$$e(C, W) = \frac{1}{M_X M_Y} \sum_{i=1}^{M_X} \sum_{j=1}^{M_Y} W(D_{i,j}^c | \mathbf{x}_i, \mathbf{y}_j) \quad (34)$$

$$= \frac{1}{M_X M_Y} \sum_{P_{XY}} \frac{M_{XY}}{M_{XY}} \sum_{(i,j) \in C_{XY}} W(D_{i,j}^c | \mathbf{x}_i, \mathbf{y}_j) \quad (35)$$

where  $C_{XY}$  is the set that includes all pairs in  $C_X \times C_Y$  which have the same type  $P_{XY}$ ,  $M_{XY}$  denotes the cardinality of this set, and  $R_{XY} = \frac{1}{n} \log M_{XY}$ . For a fixed  $(i, j)$ ,  $T_V(\mathbf{x}_i, \mathbf{y}_j)$ s are disjoint subsets of  $\mathcal{Z}^n$  for different conditional types  $V : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Z}$ . Therefore,

$$e(C, W) = \frac{1}{M_X M_Y} \sum_{P_{XY}} \frac{M_{XY}}{M_{XY}} \sum_{(i,j) \in C_{XY}} \sum_V W(D_{i,j}^c \cap T_V(\mathbf{x}_i, \mathbf{y}_j) | \mathbf{x}_i, \mathbf{y}_j) \quad (36)$$

$$= \frac{1}{M_X M_Y} \sum_{P_{XY}} M_{XY} \sum_V \frac{1}{M_{XY}} \sum_{(i,j) \in C_{XY}} W(T_V(\mathbf{x}_i, \mathbf{y}_j) | i, j) \frac{|D_{i,j}^c \cap T_V(\mathbf{x}_i, \mathbf{y}_j)|}{|T_V(\mathbf{x}_i, \mathbf{y}_j)|} \quad (37)$$

$$= \frac{1}{M_X M_Y} \sum_{P_{XY}} M_{XY} \sum_V 2^{-nD(V||W|P_{XY})} \left[ 1 - \frac{1}{M_{XY}} \sum_{(i,j) \in C_{XY}} \frac{|D_{i,j} \cap T_V(\mathbf{x}_i, \mathbf{y}_j)|}{|T_V(\mathbf{x}_i, \mathbf{y}_j)|} \right] \quad (38)$$

$$\geq \frac{1}{M_X M_Y} \sum_{P_{XY}} M_{XY} \sum_V 2^{-nD(V||W|P_{XY})} \left[ 1 - \frac{1}{M_{XY}} \sum_{(i,j) \in C_{XY}} \frac{|D_{i,j} \cap T_V(\mathbf{x}_i, \mathbf{y}_j)|}{2^{nH(Z|X,Y)}} \right] \quad (39)$$

$$= \frac{1}{M_X M_Y} \sum_{P_{XY}} M_{XY} \sum_V 2^{-nD(V||W|P_{XY})} \left[ 1 - \frac{1}{M_{XY}} \frac{|\bigcup_{(i,j) \in C_{XY}} D_{i,j} \cap T_V(\mathbf{x}_i, \mathbf{y}_j)|}{2^{nH(Z|X,Y)}} \right] \quad (40)$$

$$\geq \frac{1}{M_X M_Y} \sum_{P_{XY}} M_{XY} \sum_V 2^{-nD(V||W|P_{XY})} \left[ 1 - \frac{1}{M_{XY}} \frac{|T_Z|}{2^{nH(Z|X,Y)}} \right] \quad (41)$$

$$\geq \frac{1}{M_X M_Y} \sum_{P_{XY}} M_{XY} \sum_V 2^{-nD(V||W|P_{XY})} \left[ 1 - \frac{1}{M_{XY}} \frac{2^{nH(Z)}}{2^{nH(Z|X,Y)}} \right] \quad (42)$$

$$= \frac{1}{M_X M_Y} \sum_{P_{XY}} M_{XY} \sum_V 2^{-nD(V||W|P_{XY})} \left[ 1 - 2^{-n[R_{XY} - I_V(XY \wedge Z)]} \right] \quad (43)$$

We define

$$V_{bad}^{XY} = \{V : R_{XY} \geq I_V(XY \wedge Z)\} \quad (44)$$

So, form the last inequality,

$$e(C, W) \geq \frac{1}{M_X M_Y} \sum_{P_{XY}} M_{XY} \sum_{V \in V_{bad}^{XY}} 2^{-nD(V||W|P_{XY})} \quad (45)$$

$$\geq \frac{1}{M_X M_Y} \sum_{P_{XY}} M_{XY} 2^{-n[\min_{V \in V_{bad}^{XY}} D(V||W|P_{XY})]} \quad (46)$$

$$= \frac{1}{M_X M_Y} \sum_{P_{XY}} 2^{-n[\min_{V \in V_{bad}^{XY}} D(V||W|P_{XY}) - R_{XY}]} \quad (47)$$

$$\geq \frac{1}{M_X M_Y} 2^{-n[\min_{P_{XY}} \min_{V \in V_{bad}^{XY}} D(V||W|P_{XY}) - R_{XY}]} \quad (48)$$

Thus,

$$e(C, W) \geq 2^{-n[\min_{P_{XY}} \min_{V \in V_{bad}^{XY}} D(V||W|P_{XY}) + R_X + R_Y - R_{XY}]} \quad (49)$$

On the other hand, if we use the fact that  $D_{i,j}^c \subseteq \bigcup_{j'} \bigcup_{i' \neq i} D_{i',j'}$ , we can conclude

$$e(C, W) = \frac{1}{M_X M_Y} \sum_{i=1}^{M_X} \sum_{j=1}^{M_Y} W(D_{i,j}^c | \mathbf{x}_i, \mathbf{y}_j) \quad (50)$$

$$\geq \frac{1}{M_Y} \sum_{j=1}^{M_Y} \frac{1}{M_X} \sum_{i=1}^{M_X} W\left(\bigcup_{j' \neq i} D_{i',j'} | \mathbf{x}_i, \mathbf{y}_j\right) \quad (51)$$

$$\text{Define } D_i^c \triangleq \bigcup_{j'} \bigcup_{i' \neq i} D_{i',j'} \quad (52)$$

$$= \frac{1}{M_X M_Y} \sum_{P_{XY}} \sum_i \sum_{j:(i,j) \in C_{XY}} \sum_V W(D_i^c \cap T_V(\mathbf{x}_i, \mathbf{y}_j) | \mathbf{x}_i, \mathbf{y}_j) \quad (53)$$

$$= \frac{1}{M_X M_Y} \sum_{P_{XY}} \sum_V 2^{-nD(V||W|P_{XY})} \sum_i \sum_{j:(i,j) \in C_{XY}} \frac{|D_i^c \cap T_V(\mathbf{x}_i, \mathbf{y}_j)|}{|T_V(\mathbf{x}_i, \mathbf{y}_j)|} \quad (54)$$

$$= \sum_{P_{XY}} \sum_V 2^{-nD(V||W|P_{XY})} \frac{1}{M_X M_Y} \sum_i \sum_{j:(i,j) \in C_{XY}} \left[1 - \frac{|D_i \cap T_V(\mathbf{x}_i, \mathbf{y}_j)|}{|T_V(\mathbf{x}_i, \mathbf{y}_j)|}\right] \quad (55)$$

$$= \sum_{P_{XY}} \sum_V 2^{-nD(V||W|P_{XY})} \frac{M_{XY}}{M_X M_Y} \left[1 - \frac{1}{M_{XY}} \sum_i \sum_{j:(i,j) \in C_{XY}} \frac{|D_i \cap T_V(\mathbf{x}_i, \mathbf{y}_j)|}{|T_V(\mathbf{x}_i, \mathbf{y}_j)|}\right] \quad (56)$$

$$\geq \sum_{P_{XY}} \sum_V 2^{-nD(V||W|P_{XY})} \frac{M_{XY}}{M_X M_Y} \left[1 - \frac{1}{M_{XY}} \sum_{i=1}^{M_X} \sum_{j=1}^{M_Y} \frac{|D_i \cap T_V(\mathbf{x}_i, \mathbf{y}_j)|}{|T_V(\mathbf{x}_i, \mathbf{y}_j)|}\right] \quad (57)$$

$$\geq \sum_{P_{XY}} \sum_V 2^{-nD(V||W|P_{XY})} \frac{M_{XY}}{M_X M_Y} \left[1 - \frac{1}{M_{XY}} \sum_{j=1}^{M_Y} \sum_{i=1}^{M_X} \frac{|D_i \cap T_V(\mathbf{x}_i, \mathbf{y}_j)|}{2^{nH(Z|X,Y)}}\right] \quad (58)$$

$$\geq \sum_{P_{XY}} \sum_V 2^{-nD(V||W|P_{XY})} \frac{M_{XY}}{M_X M_Y} \left[1 - \frac{1}{M_{XY}} \sum_{j=1}^{M_Y} \frac{2^{nH(Z,X|Y)}}{2^{nH(Z|X,Y)}}\right] \quad (59)$$

$$\geq \sum_{P_{XY}} \sum_V 2^{-nD(V||W|P_{XY})} \frac{M_{XY}}{M_X M_Y} \left[1 - 2^{-n[R_{XY} - R_Y - I_V(Z \wedge X|Y)]}\right] \quad (60)$$

and now, let us define

$$V_{bad}^X \triangleq \{V : R_{XY} - R_Y \geq I_V(Z \wedge X|Y)\} \quad (61)$$

Hence, it easily can be seen

$$e(C, W) \geq \frac{1}{M_X M_Y} \sum_{P_{XY}} M_{XY} \sum_{V \in V_{bad}^X} 2^{-nD(V||W|P_{XY})} \quad (62)$$

$$\geq \frac{1}{M_X M_Y} \sum_{P_{XY}} M_{XY} 2^{-n[\min_{V \in V_{bad}^X} D(V||W|P_{XY})]} \quad (63)$$

$$= \frac{1}{M_X M_Y} \sum_{P_{XY}} 2^{-n[\min_{V \in V_{bad}^X} D(V||W|P_{XY}) - R_{XY}]} \quad (64)$$

$$\geq \frac{1}{M_X M_Y} 2^{-n[\min_{P_{XY}} \min_{V \in V_{bad}^X} D(V||W|P_{XY}) - R_{XY}]} \quad (65)$$

So,

$$e(C, W) \geq 2^{-n[\min_{P_{XY}} \min_{V \in V_{bad}^X} D(V||W|P_{XY}) + R_X + R_Y - R_{XY}]} \quad (66)$$

Using the same idea for  $Y$  and defining  $D_j^c \triangleq \bigcup_{i'} \bigcup_{j' \neq j} D_{i', j'}$ , we can easily see

$$e(C, W) \geq \frac{1}{M_X M_Y} \sum_{P_{XY}} M_{XY} \sum_{V \in V_{bad}^Y} 2^{-nD(V||W|P_{XY})} \quad (67)$$

$$\geq \frac{1}{M_X M_Y} \sum_{P_{XY}} M_{XY} 2^{-n[\min_{V \in V_{bad}^Y} D(V||W|P_{XY})]} \quad (68)$$

$$= \frac{1}{M_X M_Y} \sum_{P_{XY}} 2^{-n[\min_{V \in V_{bad}^Y} D(V||W|P_{XY}) - R_{XY}]} \quad (69)$$

$$\geq \frac{1}{M_X M_Y} 2^{-n[\min_{P_{XY}} \min_{V \in V_{bad}^Y} D(V||W|P_{XY}) - R_{XY}]} \quad (70)$$

So,

$$e(C, W) \geq 2^{-n[\min_{P_{XY}} \min_{V \in V_{bad}^Y} D(V||W|P_{XY}) + R_X + R_Y - R_{XY}]} \quad (71)$$

where

$$V_{bad}^Y = \{V : R_{XY} - R_X \geq I_V(Z \wedge Y|X)\} \quad (72)$$

From (64),(81),(86),

$$e(C, W) \geq 2^{-n[\min_{P_{XY}} \min_{V \in V_{bad}^X \cup V_{bad}^Y \cup V_{bad}^{XY}} D(V||W|P_{XY}) + R_X + R_Y - R_{XY}]} \quad (73)$$

Equivalently, for the exponent of  $e(C, W)$

$$E(C, W) \leq \min_{P_{XY}} \min_{V \in V_{bad}^X \cup V_{bad}^Y \cup V_{bad}^{XY}} D(V||W|P_{XY}) + R_X + R_Y - R_{XY} \quad (74)$$

If we define  $V_{bad} = V_{bad}^X \cup V_{bad}^Y \cup V_{bad}^{XY}$ , for every code  $C$ , we have

$$E(C, W) \leq \max_C \min_{P_{XY}} \min_{V \in V_{bad}} D(V||W|P_{XY}) + R_X + R_Y - R_{XY} \quad (75)$$

$$= \max_{\underline{R} \in \mathcal{R}} \min_{P_{XY}} \min_{V \in V_{bad}} D(V||W|P_{XY}) + R_X + R_Y - R_{XY} \quad (76)$$

Where  $\underline{R}$  is a vector with elements  $R(C, P_{XY})$  and  $\mathcal{R}$  is the set of all possible vectors  $\underline{R}$ . The last inequality follows from the fact that  $E(C, W)$  is only a function of  $R_{XY}$ s. Since  $P_{XY}^*$  is the dominant type of the code, we

conclude that

$$E(C, W) \leq \max_{R \in \mathcal{R}} \min_{V \in V_{bad}} D(V \| W | P_{XY}^*) + R_X + R_Y - R_{XY}^* \quad (77)$$

$$= \max_{R \in \mathcal{R}} \min_{V \in V_{bad}} D(V \| W | P_{XY}^*). \quad (78)$$

However, this expression does not depend on  $R$ . Therefore

$$E(C, W) \leq \min_{V \in V_{bad}} D(V \| W | P_{XY}^*), \quad (79)$$

where  $V_{bad} = \{V : I_V(XY \wedge Z) \leq R_X + R_Y \text{ or } I_V(Y \wedge Z|X) \leq R_Y \text{ or } I_V(X \wedge Z|Y) \leq R_X\}$

### B. Appendix A.2

Suppose the decoding regions for  $C_X$  are  $D_1, D_2, \dots, D_M$ . Hence,

$$\begin{aligned} e(C_X, W) &= \frac{1}{M_X} \sum_{i=1}^{M_X} W(D_i^c | i) \\ &= \frac{1}{M_X} \left[ \sum_{i=1}^M (N_i W(D_i^c | \mathbf{x}_i) + (N_i - 1) W(D_i | \mathbf{x}_i)) \right] \\ &= \frac{1}{M_X} \left( M_X - M + \sum_{i=1}^M W(D_i^c | \mathbf{x}_i) \right). \end{aligned} \quad (80)$$

Let us randomly choose  $\mathbf{x} \in T_{P_X}$  that does not belong to  $C_X$ . Define

$$V_0 \triangleq \arg \min_V \{D(V \| W | P_X) + H(V | P_X)\} \quad (81)$$

it is proved that if  $\mathbf{y} \in T_{V_0}(\mathbf{x})$

$$\begin{aligned} W^n(\mathbf{y} | \mathbf{x}) &= 2^{-n[\min_V \{D(V \| W | P_X) + H(V | P_X)\}]} \\ &\geq 2^{-n\{D(V \| W | P_X) + H(V | P_X)\}} \quad \text{any } V \\ &= W^n(\mathbf{y} | \mathbf{x}') \end{aligned} \quad (82)$$

for some  $\mathbf{x}'$  such that  $\mathbf{y} \in T_V(\mathbf{x}')$ . Thus,

$$W^n(\mathbf{y} | \mathbf{x}) \geq W^n(\mathbf{y} | \mathbf{x}_i) \quad \text{any } i = 1, \dots, M \quad (83)$$

Choose  $\mathbf{y} \in T_{V_0}(\mathbf{x}) \cap D_k$  for some  $k$  with  $|D_k| \geq 2$ . Now, let us look at  $C'_X$  which contains all codewords in  $C_X$  except one of the repeated ones, i.e one of the  $\mathbf{x}_M$  which is replaced with  $\mathbf{x}$ , and define the decoding sets

$$D'_i = D_i \quad i \neq k \quad (84)$$

$$D'_k = D_k - \{\mathbf{y}\} \quad (85)$$

$$D'_{M+1} = \{\mathbf{y}\} \quad \text{where } \mathbf{x}'_{M+1} \triangleq \mathbf{x}. \quad (86)$$

By following a similar approach, we conclude that

$$\begin{aligned}
e'(C_X, W) &= \frac{1}{M_X} \left( M_X - M + \sum_{i=1, i \neq k}^M W(D_i^c | \mathbf{x}_i) + W(D_k'^c | \mathbf{x}_k) - W(\mathbf{y} | \mathbf{x}) \right) \\
&= \frac{1}{M_X} \left( M_X - M + \sum_{i=1}^M W(D_i^c | \mathbf{x}_i) W(D_k'^c | \mathbf{x}_k) - W(\mathbf{y} | \mathbf{x}) - W(D_k^c | \mathbf{x}_k) \right) \\
&= e(C_X, W) + \frac{1}{M_X} \left( W(\mathbf{y} | \mathbf{x}_k) - W(\mathbf{y} | \mathbf{x}) \right) \\
&\leq e(C_X, W),
\end{aligned} \tag{87}$$

where the last inequality follows from the fact that  $W(\mathbf{y} | \mathbf{x}_k) \leq W(\mathbf{y} | \mathbf{x})$ .

### C. Appendix A.3

The left hand side of (29) is straightforward, since for all multiuser codes,  $\mathcal{C}$ ,  $e_m(\mathcal{C}, W) \geq e(\mathcal{C}, W)$ . By (28), for all multiuser codes with rate pair  $(R_X, R_Y)$ , we can conclude that

$$e_m(\mathcal{C}, W) \geq 2^{-n E_m^U(R_X, R_Y)}. \tag{88}$$

Let us assume that there exists a code  $\mathcal{C}$  with rate pair  $(R_X, R_Y)$  for which the right hand side of (29) does not hold. Without loss of generality, we assume  $R_X \leq R_Y$ . For Therefore,

$$e(\mathcal{C}, W) < \frac{1}{2} 2^{-n(E_m^U(R_X, R_Y) + R_X)},$$

which is equivalent to

$$\frac{1}{M_X M_Y} \sum_{i=1}^{M_X} \sum_{j=1}^{M_Y} W(D_{i,j}^c | \mathbf{x}_i, \mathbf{x}_j) < \frac{1}{2} 2^{-n(E_m^U(R_X, R_Y) + R_X)}, \tag{89}$$

which can be written as

$$\frac{1}{M_Y} \sum_{j=1}^{M_Y} \frac{1}{M_X} \sum_{i=1}^{M_X} W(D_{i,j}^c | \mathbf{x}_i, \mathbf{x}_j) < \frac{1}{2} 2^{-n(E_m^U(R_X, R_Y) + R_X)}, \tag{90}$$

therefore, there exist  $M_Y^1 \geq \frac{M_Y}{2}$  codewords in  $\mathcal{C}_Y$  that satisfy

$$\frac{1}{M_X} \sum_{i=1}^{M_X} W(D_{i,j}^c | \mathbf{x}_i, \mathbf{x}_j) < 2^{-n(E_m^U(R_X, R_Y) + R_X)}. \tag{91}$$

Let us call this set of codewords as  $\mathcal{C}_Y^1$ . By multiplying both sides of (91) with  $M_X$ , and considering the fact that all terms in summation are non-negative, it can be concluded that for every  $\mathbf{x}_i \in \mathcal{C}_X$ ,  $\mathbf{y}_j \in \mathcal{C}_Y^1$ ,

$$W(D_{i,j}^c | \mathbf{x}_i, \mathbf{x}_j) < 2^{-n(E_m^U(R_X, R_Y))}. \tag{92}$$

Therefore, the new multiuser code  $\mathcal{C}^1 = \mathcal{C}_X \times \mathcal{C}_Y^1$ , has a rate pair very close to the original code, and its maximal probability of error satisfies

$$e_m(\mathcal{C}^1, W) < 2^{-n(E_m^U(R_X, R_Y))}. \tag{93}$$

(93) contradicts our assumption in (88), therefore it can be concluded that the assumption must be false and that its opposite must be true. Similarly, we can show the bounds in (31) by assumption in (30).

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