Algebraic Topology Notes

University of Michigan

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These are a set of notes in algebraic topology, following a reading group done at the University of Michigan in Winter 2020. The reading group webpage can be found at

`http://www.umich.edu/~alxwang/math/algtop/`

Our reading group loosely follows *Algebraic Topology* by Allen Hatcher, as well as *A Concise Course in Algebraic Topology* by J. P. May, both of which are available for free online. Thanks to Joe Carter, Vikram Mathew, Ethan Mook, Raviv Sarch, Steven Schaefer, and Alex Wang for writing, editing, and updating these notes.

Contents

1 Introduction

In this set of notes, we present a write-up of material covered in our reading group in algebraic topology.

2 The Fundamental Group (January 27, 2020)

Speaking broadly, the Fundamental Group is what makes Algebraic Topology simultaneously "Algebraic" and "Topological." We'll see that this group is indeed fundamental insofar as it can be used to solve group theoretic problems topologically and topology problems group-theoretically. It is the building block of much of the theory of Algebraic Topology.

2.1 Paths and Homotopy

In Hatcher, this section develops specifically the homotopy of paths. I'll begin with the more general definition because I personally find it more intuitive, and then move on to the specific case needed to build a fundamental group.

Definition: HOMOTOPIC MAPS. Letting X and Y being topological spaces, continuous maps $f, g : X \to Y$ are *homotopic* (written $f \simeq g$) provided that there exists a continuous map $F : X \times I \to Y$ so that

$$\forall\ x \in X,\ F(x, 0) = f(x)\ \&\ F(x, 1) = g(x)$$

We put the natural product topology on $X \times I$. Intuitively, we can think of homotopic f and g having the property that one can be continuously deformed to another in a space of *maps* $X \to Y$, a space we parametrize with I. The map F as defined above is referred to as a *homotopy* between f and g This concept lends some structure to the set of continuous maps $X \to Y$. Observe, for instance, that homotopy defines an equivalence relation on this set.
We wish to focus on homotopies of paths, which are defined as follows:

Definition: PATH. Given topological space X and letting I be the unit interval in $\mathbb{R}$, a path is simply a continuous map $I \to X$

In this, we can begin to talk about homotopy classes of paths. That is, homotopies between continuous maps $I \to X$. A homotopy in this case is a family of functions $f_t : I \to X$ where t runs from 0 to 1. Hatcher also enforces the requirement that all the paths have the same endpoints, which I can't get to follow from the more general definition. Either I'm missing something or, I expect, this is just to make the definition more meaningful and make some of the machinery work. For instance:

Example: LINEAR HOMOTOPIES:. We can construct a homotopy between any two paths $f_1, f_2 : I \to \mathbb{R}^n$ that share the same endpoints via $f_0(x) + t(f_1(x) - f_0(x))$. The "linear" in "linear homotopy" comes from the fact that for a fixed $x \in [0, 1]$, this parametrizes the line segment between $f_0(x)$ and $f_1(x)$.

We can now begin to build our algebraic structure on path homotopies. Let's start with the following binary operation on paths f and g with the property that $f(1) = g(0)$:

$$f * g(x) = \begin{cases} f(2x) & \text{if } x \in [0, 1/2] \\ g(2s - 1) & \text{if } x \in [1/2, 1] \end{cases}$$

Note that if we have $f_0 \simeq f_1$ and $g_0 \simeq g_1$ with homotopies f_t, g_t, then $f_t \simeq g_t$ is defined for all t so long as $f_0 * g_0$ and $f_1 * g_1$ are, and hence $f_0 * g_0 \simeq f_1 * g_1$.
We now have two tools to use over paths f, g on a topological space: we can construct a path homotopy provided that $f(0) = g(0)$ and $f(1) = g(1)$ and we can construct a product path provided that $f(1) = g(0)$. It follows that if we want to use both, we have to require both paths start and end on the same point. These paths are called *loops*, and the point in question is called a *basepoint*.

Definition: FUNDAMENTAL GROUP. The *fundamental group* of a topological space X about base-point x_0, written $\pi_1(X, x_0)$ is the set of homotopy classes of loops about x_0 under the composition law defined above.

Proposition . The fundamental group works.

Proof. We have already observed that the product path respects homotopy classes. As an identity element, we take the homotopy class of the "constant path," c. Looking at a specific path f, $c * f$ constitutes a reparametrization of f, meaning f and $c * f$ have the same image in X, so they are trivially homotopic. Meanwhile, the inverse path for f, defined $\overline{f} = f(1-x)$ defines a homotopy class that is inverse to $[f]$. For further justification, see Hatcher Prop. 1.3.

□

Example: . What is $\pi_1(S, x_0)$ when S is starlike about x_0? Then, any loop is homotopic to the constant loop via the linear homotopy. Thus, the funamental group is trivial. What if we pick a different base point, x_1? Well, since the set is starlike, we can fix a path h from x_1 to x_0, and consider its homotopy class. Then, any loop about x_1 is equivalent to a loop about x_0 via conjugation by h. This conjugation defines the *change of basepoint map*.

Theorem . If X is path-connected, then $\forall\ x_0, x_1 \in X$, $\pi_1(X, x_0) \cong \pi_1(X, x_1)$ via the change of basepoint map.

2.2 The Fundamental Group of the Circle

Definition: COVERING SPACE. Given topological space X, topological space $\tilde{X}$ is considered an X covering space when there is a map $p : \tilde{X} \to X$ with the property that $\forall\ x \in X\ \ \exists$ open U_x such that

$$p^{-1}(U_x) = \bigsqcup_{i \in I} O_i$$

with each $O_i \subset \tilde{X}$ open and homeomorphic to U_x

Some covering space terminology: X is sometimes called the *base space*, p the *covering map* and for $x \in X$, $p^{-1}(x)$ is the *fiber over x*.

Definition: LIFT. A *lift* of a map $f : Y \to X$ is a map $\tilde{f} : Y \to \tilde{X}$ such that $p \circ \tilde{f} = f$.

Lemma . Given a map $F : Y \times I \to X$ and a map lifting $F|_{X \times \{0\}}$, there is a unique lift $\tilde{F} : Y \times I \to \tilde{X}$ that restricts over $X \times$

Proof. Let $\omega_n(t) = e^{2\pi int}$ for $0 \leqslant t \leqslant 1$. Let $p : \mathbb{R} \to S^1$ defined by $x \mapsto e^{2\pi ix}$. Note that p gives $\mathbb{R}$ as a covering space for S^1. Note further that the ω_n's define paths on S^1. In particular, note that $[\omega_1]^n = [\omega_n] \in \pi_1(S^1, (1, 0))$. We mean to show that any loop in S^1 with that basepoint is homotopic to some ω_n. Fix f a loop in S_1. □

3 Category Theory (February 4, 2020)

This section aims to provide an introduction to category theory, assuming no prior knowledge of categories. We hope to cover all necessary categorical background for the remainder of the algebraic topology topics covered in these notes.

3.1 What is a Category?

We begin by defining a category. Informally speaking, a category generalizes the notion of mathematical objects, and the maps between these objects. It is important for the reader to have many examples in mind when thinking about categories.

Definition: . A **category** C is a collection[1] of objects, $\mathrm{Obj}(C)$, together with a collection of morphisms, $\mathrm{Mor}(C)$, satisfying the following properties:

1. Each morphism $f \in \mathrm{Mor}(C)$ has a source object A and target object B, with $A, B \in \mathrm{Obj}(C)$, which we denote $f : A \to B$.

2. For $X, Y, Z \in \mathrm{Obj}(C)$, and morphisms $f : X \to Y$ and $g : Y \to Z$, there exists a unique morphism $g \circ f : X \to Z$, the **composition** of f and g.

Notationally, for $A, B \in \mathrm{Obj}(C)$, we denote

$$\mathrm{Hom}(A, B) = \{f \in \mathrm{Mor}(C) : f : A \to B\}$$

We can rephrase the condition above by stating that for all $X, Y, Z \in \mathrm{Obj}(C)$, there exists a binary operation

$$\begin{aligned} \circ : \mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) &\to \mathrm{Hom}(A, C) \\ (f, g) &\mapsto g \circ f \end{aligned}$$

In addition, we require that the category C satisfies the following properties.

1. (Identity) For every object $A \in \mathrm{Obj}(C)$, there exists a morphism, denoted $\mathrm{Id}_A : A \to A$, such that for any morphism $f : A \to B$, we have that . This is often called the **identity morphism**.

2. (Associativity) For $W, X, Y, Z \in \mathrm{Obj}(C)$, and for $f : W \to X$, $g : X \to Y$, and $h : Y \to Z$, we have that
$$h \circ (g \circ f) = (h \circ g) \circ f$$

The (perhaps not so) observant reader may notice that this may remind them of sets and functions, which is in fact the canonical example.

Example: . We list many examples of categories. Generally, we denote categories by boldfaced text. If it is not obvious why these are categories, please verify the axioms!

1. **Set**, the category of sets, where the morphisms are functions.

2. **Grp**, the category of groups, where the morphisms are group homomorphisms.

3. **Ab**, the category of abelian groups, where the morphisms are group homomorphisms.

4. (k-)**Vect**, the category of vector spaces (over a field k), where the morphisms are linear maps.

5. **Ring**, the category of rings, where the morphisms are ring homomorphisms.

6. **R-Mod**, the category of R-modules, where the morphisms are R-module homomorphisms.

7. **Top**, the category of topological spaces, where the morphisms are continuous maps.

Note that in these categories, the morphisms must respect the structure of the algebraic objects. However, we emphasize that while categories are often algebraic objects with morphisms given by the appropriate functions between them, there are plenty of categories which do not look like this.

1. We can consider the category of logical expressions, where morphisms are implications.

[1] We intentionally avoid a discussion of classes versus sets, because this is not something we are interested in distinguishing at the moment.

2. Let X be a topological space. Then, $\text{Open}(X)$ is the category whose objects are open sets in X, and for U, V open in X, there exists a morphism $U \to V$ if and only if $U \subset V$.

3. Given any category C, we can consider C^{op}, the opposite category, which has the same objects as C but with all morphisms reversed.

4. A group can be thought of as a category with one object, and each element of the group corresponds to a morphism.

We emphasize, especially in the final example, that sometimes, the most important part of a category is not the collection of objects, but the collection of morphisms.

3.2 Special Morphisms

We now define some special objects and morphisms in a category.

Definition: . Let C be a category, and let $A, B \in \text{Obj}(C)$. Suppose we have $f \in \text{Hom}(A, B)$ and $g \in \text{Hom}(B, A)$, satisfying

1. $g \circ f = \text{Id}_A$

2. $f \circ g = \text{Id}_B$

Then, f and g are said to be **isomorphisms**, and the objects A and B are said to be **isomorphic**.

We note that this certainly agrees with our notion of isomorphism in groups, rings, vector spaces, etc, but this definition is completely independent of any notion of injection or surjection. We can, however, generalize these notions as well.

Definition: . Let $f : A \to B$ be a morphism. Then, f is said to be

- a **monomorphism** if for all $g_1, g_2 : B \to C$, we have that

$$f \circ g_1 = f \circ g_2 \implies g_1 = g_2$$

- an **epimorphism** if for all $g_1, g_2 : C \to A$, we have that

$$g_1 \circ f = g_2 \circ f \implies g_1 = g_2$$

We verify that monomorphisms correspond to injective functions and epimorphisms correspond to surjective functions in **Set**. In fact, we can make a stronger statement.

Definition: . We say that a category C is **concrete** provided that objects in C have an underlying set. For example, **Grp**, **Ab**, **Vect**, and **Ring** are all concrete categories.

3.3 Special Objects

We now define a special kind of object.

Definition: . Let C be a category, and let A be an object of C. We say that A is

- **initial** provided that for all $B \in \text{Obj}(C)$, there exists a unique morphism $f : A \to B$.
- **final** provided that for all $B \in \text{Obj}(C)$, there exists a unique morphism $g : B \to A$.
- **terminal** provided that it is either initial or final.
- a **zero object** provided that it is both initial and final.

Of course, we must give plenty of examples. Note that not all categories have terminal objects, but some categories do.

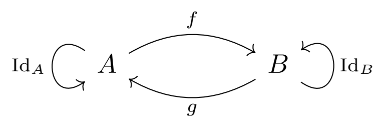
Example: . We give some examples of terminal objects.

1. In **Set**, we have that $\varnothing$ is initial, and $\{*\}$, any singleton set, is final.
2. In **Grp** (and similarly **Ab**), we have that the trivial group is a zero object.
3. In **Ring**, we have that $\mathbb{Z}$ is the ~~only~~ initial object, and the zero ring is final.
4. In the category of fields, there is no initial object (Why?).

Notice that in **Set**, any singleton set is final. This motivates us to prove the following proposition.

Proposition . Let C be a category. All initial objects, and dually, terminal objects, in C are unique up to unique isomorphism.

Proof. We prove this for initial objects. Let A, B be initial objects in C. Then, there exists a unique $f \in \text{Hom}(A, B)$, and unique $g \in \text{Hom}(B, A)$.



Note additionally that there is exactly one map in $\text{Hom}(A, A)$, namely, Id_A, and similarly, there is exactly one map in $\text{Hom}(B, B)$, namely, Id_B. Thus, we must have that

$$g \circ f = \text{Id}_A \qquad f \circ g = \text{Id}_B$$

and thus we have that A and B are isomorphic as desired. □

3.4 Functors

Throughout our discussion, we have been emphasizing that in a category, the relationship between objects is sometimes more important than the objects themselves. Can we apply this to categories? (Hint: yes!)

Definition: . Let C, D be categories. We say that $F : C \to D$ is a **(covariant) functor** provided that

- For all $A \in \text{Obj}(C)$, we have that $F(A) \in \text{Obj}(D)$.
- For all $f : X \to Y$ in $\text{Mor}(C)$, we have that $F(f) : F(X) \to F(Y)$ in $\text{Mor}(D)$.

Additionally, we require that F satisfies

- For every $X \in \text{Obj}(C)$, we have that $F(\text{Id}_X) = \text{Id}_{F(X)}$.
- For every $f : X \to Y$ and $g : Y \to Z$ in $\text{Mor}(C)$, we have that $F(g \circ f) = F(g) \circ F(f)$.

A **contravariant functor** $F : C \to D$ is a covariant functor $C^{\text{op}} \to D$.

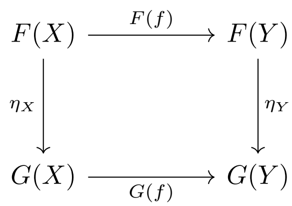
Let's give some examples. Verify that these are indeed functors (where do they send the morphisms?).

1. The forgetful functor on any concrete category C, which takes every object to the underlying set.
2. The free functor, which associates any set to the "free" object associated with that set (free group, free vector space, etc).
3. **The fundamental group** is a functor from **Top** to **Grp**.
4. The dual vector space functor sends vector spaces to the associated dual space.

5. The tangent space functor sends a manifold to the tangent space at a point.
6. A sheaf of rings on a topological space X is a contravariant functor from $\text{Open}(X) \to \textbf{Ring}$.

We can even take this a step further.

Definition: . Let C, D be two categories, and let $F, G : C \to D$ be two functors between them. A **natural transformation** between F and G is a collection of morphisms η_X for every $X \in \text{Obj}(C)$ such that the following diagram commutes.



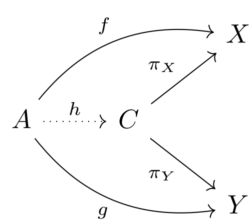
We can think of this as a map between functors, which are related in a natural way. Some examples include the identity functor and the opposite functor in **Grp**, which sends a group to the opposite group, as well as the identity functor and the double dual functor in **Vect**.

3.5 Universal Properties

Category theory allows us to abstract away from many of the concrete definitions we make, and the way that we do this is through universal properties.

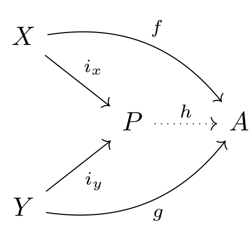
Definition: . (Informal) A **universal property** is a property which uniquely determines an object.

This is best demonstrated through example. For example, we present the universal property of **products**



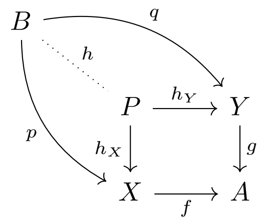
An object C satisfies the universal property of products provided that for any object A and maps $f : A \to X$ and $g : A \to Y$, there exists a unique map $h : A \to C$ making the above diagram commute. When this is the case, we say that P is the product of X and Y in this category. Note that when we are working in **Set**, we can take P to be $X \times Y$. The important point to note, however, is that we have defined the product completely independently of sets. However, not every category has products, and the existence of products is something that needs to be proved about categories.

We can similarly define **coproducts** using a universal property.

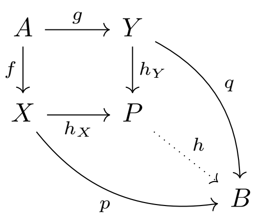


Let X and Y be objects. We say that an object P satisfies the universal property of coproducts provided that for any object A and morphisms $f : X \to A$ and $g : Y \to A$, there exists a unique map $h : P \to A$ which makes the above diagram commute. What are the coproducts in **Set**?

Another important construction is the pullback and pushout, defined by the following universal properties.



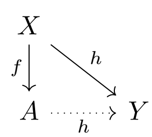
Let X, Y be objects with maps $f : X \to A$ and $g : Y \to A$. The pullback is the object P satisfying the universal property that given any object B with maps $p : B \to X$ and $q : B \to Y$, there exists a unique map $h : B \to P$ making the above diagram commute.



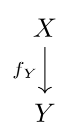
We can define the pushout dually by the diagram above. Many other objects can be defined using universal properties, which illustrates an important point about category theory: objects can sometimes be defined by the properties they satisfy.

Proposition . Universal objects are unique up to unique isomorphism.

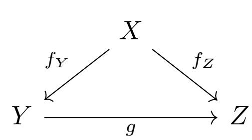
Proof. We prove this using a slick argument. Consider this example: Let C be a category, and suppose that an object A of C satisfies the universal property that for any map $f : X \to A$, and any object Y with a map $g : X \to Y$, there exists a unique map $h : A \to Y$.



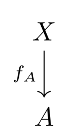
Then, let D be the category whose objects are diagrams of the form



for every $Y \in \text{Obj}(C)$ and whose morphisms are arrows g of the form



Then, we note that the diagram



is initial in D, and thus unique up to unique isomorphism by previous result. We can do this process for (most) universal properties. □

This illustrates the notion of a comma category, which is beyond the scope of these notes.

3.6 Adjunction

An important relationship between two functors is the concept of adjunction.

Definition: . Let C, D be categories, and let F, G be functors $F : C \to D$ and $G : D \to C$. We say that F and G are **adjoint** provided that for all objects $X \in C$ and $Y \in D$, we have that

$$\mathrm{Hom}_D(F(X), Y) \cong \mathrm{Hom}_C(X, G(Y))$$

When this is true, we say that F is **left adjoint** and G is **right adjoint**, and we denote this by $F \dashv G$.

For our purposes, there are two important examples of functor adjunctions.

1. A free functor is generally adjoint to a forgetful functor. For example, the free functor on sets is adjoint to the forgetful functor on groups, vector spaces, etc.

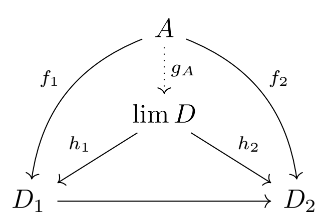
2. Let A be an R-module. The $- \otimes_R A$ functor is adjoint to the $\mathrm{Hom}(A, -)$ functor. This can be seen by writing
$$\mathrm{Hom}(X \otimes_R A, Z) \cong \mathrm{Hom}(X, \mathrm{Hom}(A, Z))$$

In particular, one neat example is the forgetful functor from **Top** to **Set**. We notice that in fact, this functor is left adjoint to the "free" functor which assigns the discrete topology, and right adjoint to the "free" functor which assigns the trivial topology.

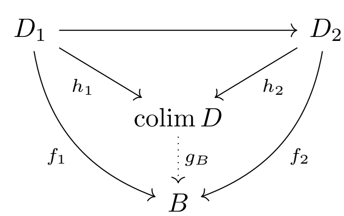
3.7 Limits

We further generalize the notion of a universal property to that of a limit.

Definition: . Let D be a diagram in a category C, and let A be an object such that for all objects $X \in D$, we have a map $f_X : A \to X$, such that each f_X and all maps in D commute. Then, we define the **limit** of D, denoted $\lim D$, to be the unique object such that for any objects A as above, there exists a unique map $g_A : A \to \lim D$ making everything commute.



Dually, we can define a **colimit**, denoted $\mathrm{colim}\, D$. Let B be an object such that for all objects X in D, we have a map $f_X : X \to B$, such that all maps in sight commute. Then, we define the colimit of D to be the unique object such that for any object B as above, there is a unique map $g_B : \mathrm{colim}\, D \to B$ making everything commute.



These generalize the previous universal property definitions. How can we express the universal property of products and coproducts as a limit or colimit?

It is a fact that right adjoints preserve limits, and left adjoints preserve colimits.

3.8 Homological Algebra

We have already seen that category theory can help us redefine many ideas in terms of categorical notions. However, we need some additional structure to define concepts like homology. In this section, we assume our category has a notion of adding and subtracting morphisms, kernels and cokernels, images, zero objects, and quotients.

Definition: . A **chain complex** is a sequence of objects

$$C_\bullet = \cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots$$

together with morphisms between each object such that $d_{i-1} \circ d_i = 0$. Note that this implies that $\operatorname{im}(d_i) \subset \ker(d_{i-1})$. The d_i are said to be the **boundary maps** or **differentials**. We typically take the category to be **Ab** or **R-Mod**. A **cochain complex** has the indices reversed.

We say that a chain complex is **exact** at C_i provided that $\operatorname{im}(d_{i+1}) = \ker(d_i)$. We say that a chain complex is an **exact sequence** if it is exact everywhere.

Example: . Some examples of chain complexes include

1. Differential forms on a manifold, with the exterior derivative.
2. Manifolds with the boundary operator.

Definition: . We can measure how much a chain complex fails to be exact by computing the **homology** of the chain complex, which is defined to be

$$H_n(C_\bullet) = \ker(d_n)/\operatorname{im}(d_{n+1})$$

This produces its own chain complex, whose differentials are 0.

A sequence of the form

$$0 \longrightarrow A \longrightarrow B \longrightarrow 0$$

forces A and B to be isomorphic (why?).

A sequence of the form

$$0 \longrightarrow A \xhookrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

is called a **short exact sequence**. Note that α is injective and β is surjective (why?). An exact sequence which is longer is called a **long exact sequence**.

4 Van Kampen's Theorem (February 11, 2020)

This section explores the first big theorem in algebraic topology. We will see how compute fundamental groups by decomposing spaces into component parts.

4.1 A Motivating Example

Consider the following image,
TODO: Put image here. pg 40 Hatcher is a good picture.
Fun Fact: colloquially this is known as a "figure eight" or "infinity", however in algebraic geometry this is also known as a *lemniscate*. I will choose this terminology simply because it is cool.
The lemniscate is made up of two circles that share a basepoint, so intuitively we may expect that its fundamental group somehow decomposes into two copies of $\pi_1(S_1)$. To check this hypothesis let us compute the fundamental group of the lemniscate, then we will later try to make this hypothesis more general and more

rigorous.

Let this space be called X such that X is formed by two circles A and B intersecting at a single point x_0. Because A and B are both circles, we know that $\pi_1(A) \cong \pi_1(B) \cong \pi_1(S_1) \cong \mathbb{Z}$. Define $a \in \pi_1(A)$ as a loop around the circle A. Define powers of a in the obvious way, and similarly define b in the obvious manner.

Then loops in $\pi_1(X)$ are formed by going around each circle independently as many times as we want. This reduces to the words formed with characters from $\pi_1(A)$ and $\pi_1(B)$. Write an a if you navigate around circle A, and similarly write b. For instance, one element of $\pi_1(X)$ may look like $a^{-2}b^3a$. Read the word right-to-left, so that $a^{-2}b^3a$ consists of first going around the circle A clockwise, then going around B three times clockwise, and finally going around A two times counterclockwise. It also suffices to consider these words in reduced form by combining adjacent characters that are from the same group – that is, a^2ab^3 may be simplified to a^3b^3.

Because loops in $\pi_1(X)$ are formed in this manner, we find that $\pi_1(X)$ is the free product of $\pi_1(A)$ and $\pi_1(B)$. Symbolically,

$$\pi_1(X) \cong \pi_1(A) * \pi_1(B) \cong \mathbb{Z} * \mathbb{Z}$$

Before fully defining $*$, the free product of groups, take note that just as X is decomposed into A and B, the fundamental group $\pi_1(X)$ is decomposed into $\pi_1(A)$ and $\pi_1(B)$. This parallel notion of decomposition is the key intuition behind Van Kampen's theorem.

4.2 Free Product of Groups

For some collection of groups $\{G_\alpha\}$, the free product $*_\alpha G_\alpha$ contains all words $g_1 g_2 \cdots g_m$ of arbitrary finite length $m \geqslant 0$. Again, reduce words if adjacent elements are in the same group. For convenience and without loss of generality, most work is done with words in reduced form. The group operation in $*_\alpha G_\alpha$ is "juxtaposition" or "concatenation". Simply join the words as follows,

$$(g_1 g_2) \cdot (h_1 h_2) = g_1 g_2 h_1 h_2$$

One can check that $(*_\alpha G_\alpha, \cdot)$ satisfies all the group axioms and they will find that this is indeed a group.

We found one example of a free product above with the lemniscate. Here is another such example.

Example: $\mathbb{Z}_2 * \mathbb{Z}_2$. Consider two copies of $\mathbb{Z}_2$, one presented as $\{e, a\}$ and the other as $\{e, b\}$. Then the elements of $\mathbb{Z}_2 * \mathbb{Z}_2$ are given as,

$$\{e, a, b, ab, ba, aba, bab, abab, baba, \dots\}$$

Note that although this is a free product of groups, it is not a free group as it is subject to the relations $a^2 = b^2 = e$.

Side note: This is also called the **infinite dihedral group**.

4.3 Van Kampen's Theorem

TODO define the i's in hatcher p. 43. These serve to mod out the loops in the intersection

Theorem Van Kampen. Let X be a topological space.

1. If X is the union of path connected A_α each containing basepoint x_0, and if each $A_\alpha \cap A_\beta$ is path connected, then $\varphi : *_\alpha \pi_1(A_\alpha) \to \pi_1(X)$ is surjective.

2. If in addition each $A_\alpha \cap A_\beta \cap A_\gamma$ is path connected, then $N := \ker \varphi$ is the normal subgroup generated by all elements of the form $i_{\alpha\beta}(\omega)i_{\beta\alpha}(\omega)^{-1}$ for $\omega \in \pi_1(A_\alpha \cap A_\beta)$, and hence φ induces an isomorphism $\pi_1(X) \cong *_\alpha \pi_1(A_\alpha)/N$.

TODO pf vk1, statement on how vk2 is extraneous, pf vk2, sidebar on the categorical approach.

5 Covering Spaces and Deck Transformations (February 18, 2020)

This section discusses a characterization of covering spaces and describes how we can view deck transformations as group actions

5.1 Background Information and Motivating Examples

Recall: . A covering space of topological space X is a topological space $\tilde{X}$ equipped with a map $p : \tilde{X} \to X$ satisfying the following condition

> $\forall\ x \in X,\ \exists$ open neighborhood $U \ni x$ such that $p^{-1}[U] = \bigsqcup_i O_i$ with O_i open and disjoint in $\tilde{X}$, each of which is map homeomorphically onto U via p

Saying $p : \tilde{X} \to X$ is a covering space is typical abuse of notation. Additionally, such a U is called **evenly covered** and the O_i are called **sheets** of $\tilde{X}$

Fact: The # of sheets over U is the cardinality of $p^{-1}(x)$ for $x \in U$. As x varies over X, this number is locally constant.

$$\implies |p^{-1}(x)| = c \text{ if } X \text{ is connected}$$

Example: . Let us look at some covering spaces of S^1 which we have considered before. We have seen

$$\begin{aligned} \varphi : \mathbb{R} &\to S^1 \\ x &\mapsto (\cos(2\pi x), \sin(2\pi x)) \end{aligned}$$

is a covering space and for $n \in \mathbb{N}$

$$\begin{aligned} \varphi_n : S^1 &\to S^1 \\ z &\mapsto z^n \end{aligned}$$

is a covering space if we consider $S^1 \subseteq \mathbb{C}$. Since both covering spaces are connected, let us answer how many sheets each has. Due to the periodic nature of sin and cos we see that $|\varphi^{-1}((x, y))| = \infty$ and over the complex plane, we know each number as n complex nth roots. Thus we conclude that $|\varphi_n^{-1}(z)| = n$. One might be curious if there are other connected covering spaces besides the ones listed above. Using the theory we will develop below, we shall see that there are in fact none. This is related to the idea that $\pi_1(S^1) = \mathbb{Z}$ and all of the subgroups of $\mathbb{Z}$ are $n\mathbb{Z}$ for some positive n

From the first talk, we know that continuous maps between topological spaces induces a group homomorphism between the spaces' fundamental groups. Additionally, since the image of a group homomorphism is a group, we might be curious about what subgroup $p_*[\pi_1(\tilde{X}, \tilde{x})] \subseteq \pi_1(X, x_0)$ is? In the examples above, we see that

$$\varphi_*[\pi_1(\mathbb{R})] = \varphi_*[\{0\}] = \{0\}$$

and

$$\varphi_{n*}[\pi_1(S^1)] = \varphi_{n*}[\mathbb{Z}] = n\mathbb{Z}$$

This is the relationship we'd like to explore in general as we classify covering spaces. We would also like to know how our choice of basepoint in $\tilde{X}$ affects which subgroup of $\pi_1(X, x_0)$ is realized as this image. We will come to discover that changing $\tilde{x}$ will amount to conjugating $p_*[\pi_1(\tilde{X}, \tilde{x})]$. The conjugating element if $\pi_1(X, x_0)$ is represented by any loop that is the projection of a path in $\tilde{X}$ joining the basepoints together.

5.2 Lifiting Properties