Extensions of Irreducible Representations of Quaternion Algebras over p-adic Fields

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Introduction

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- **Representation Theory** is the study of how groups act on vector spaces.
- Understanding how a group acts gives us an understanding of the group itself.

Group Representations

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Let $G = S_3$. One representation of G is to send each $\sigma \in S_3$ to the permutation matrix associated with σ.

$$(12) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (123) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

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We may often refer to the representation (π, V) as simply π or simply V, depending on context.

Irreducible Representations

Definition

Let (π, V) be a representation of G. $W \subset V$ is said to be a **subrepresentation** of V provided that (π, W) is a representation of G. In particular, we must have $\pi(g)W \subset W$ for all $g \in G$. If the only subrepresentations of V are 0 and V, V is said to be **irreducible**.

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When $k = \mathbb{C}$, every representation of G can be broken down into a direct sum of irreducible representations.

Note that this is not true when k is an arbitrary field!

Representations over $\overline{\mathbb{F}_p}$

Example

Let $k = \overline{\mathbb{F}_p}$. Let $G = \mathbb{Z}/p\mathbb{Z}$, and let $V = \text{span}_k(e_1, e_2)$, with representation

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If $V = V_1 \oplus V_2$ with neither 0, both must be 1-dimensional simultaneous eigenspaces for every n.

But each non-identity matrix only has eigenspace $\text{span}(e_1)$! So we cannot find a V_2. □

Quaternion Algebras

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Definition

Let F be a field. A **quaternion algebra** over F is a 4-dimensional vector space over F, with basis $\{1, i, j, k\}$, satisfying the multiplication rules

$$i^2 = a \qquad j^2 = b$$
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For the remainder of this talk, we assume $F = \mathbb{Q}_p$, the field of p-adic numbers, a and b are chosen such that D is a division algebra, and let $D^\times$ denote the units of D.

Decomposition of $D^\times$

Fact

There exists an element $\varpi \in D^\times$ with $\varpi^2 = p$, with respect to which we have the decomposition

$$D^\times \cong \varpi^{\mathbb{Z}} \ltimes \left(\mathbb{F}_{p^2}^\times \ltimes 1 + \varpi \mathcal{O} \right)$$

where $\mathcal{O}$ denotes the ring of integers in D.

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Note that $D^\times$ contains the index 2 subgroup $H = \varpi^{2\mathbb{Z}} \times \left(\mathbb{F}_{p^2}^\times \ltimes 1 + \varpi\mathcal{O}\right)$, where the product is direct since $\varpi^2 = p$ is in the center of $D^\times$.

Irreducible Representations of $D^\times$

There are two classes of irreducible representations of $D^\times$ over $\overline{\mathbb{F}_p}$:

- 1-dimensional representations given by

$$\chi_{a,b} : D^\times \to \mathrm{GL}_1(\overline{\mathbb{F}_p}) = \overline{\mathbb{F}_p}^\times$$
$$(\varpi^x, y, z) \mapsto a^x y^{(p+1)b}$$

  for $a \in \overline{\mathbb{F}_p}^\times$ and $0 \leq b \leq p-2$.

- 2-dimensional representations constructed by induction of the 1-dimensional representations of H, which are given by

$$\psi_{c,d} : H \to \mathrm{GL}_1(\overline{\mathbb{F}_p}) = \overline{\mathbb{F}_p}^\times$$
$$(\varpi^{2x}, y, z) \mapsto c^x y^d$$

  for $c \in \overline{\mathbb{F}_p}^\times$ and $0 \leq d \leq p^2-2$.

Goals

Recall that when our representation is not over $\mathbb{C}$, irreducible representations don't tell us everything!

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We want to understand how to glue together irreducible representations in more interesting ways than simply taking their direct sum.

We will do this by computing **extensions** of irreducible representations of $D^\times$.

Extensions of Irreducible Representations

Definition

Given two representations V_1 and V_2 of a group G, we say that the representation V is an **extension** of V_1 by V_2 provided that the following sequence

$$0 \longrightarrow V_2 \hookrightarrow V \twoheadrightarrow V_1 \longrightarrow 0$$

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Over $\mathbb{C}$, Maschke's Theorem tells us $\mathrm{Ext}^1_G(V, V') = 0$.

Computing Extensions

Result

We describe $\mathrm{Ext}^1_{D^\times}(V_1, V_2)$.

- When V_1 and V_2 are both 1-dimensional representations, we have that

$$\dim \mathrm{Ext}^1_{D^\times}(V_1, V_2) = \begin{cases} 2 & V_1 = V_2 \\ 0 & \text{else} \end{cases}$$

- When $V_1 = \chi_{\gamma,\delta}$ and $V_2 = \mathrm{Ind}_H^{D^\times}(\psi_{a,b})$, we have that

$$\dim \mathrm{Ext}^1_{D^\times}(V_1, V_2) = \begin{cases} 2 & a = \gamma^2, b = (p+1)\delta \\ 1 & a = \gamma^2, b - (p+1)\delta = \pm(p-1) \\ 0 & \text{else} \end{cases}$$

Computing Extensions

Result (cont.)

- When $V_1 = \operatorname{Ind}_H^{D^\times}(\psi_{a,b})$ and $V_2 = \operatorname{Ind}_H^{D^\times}(\psi_{c,d})$, we have that

$$\dim \operatorname{Ext}^1_{D^\times}(V_1, V_2) = \begin{cases} 3 & a = c, (b,d) \in S \\ 1 & a = c, b = d \text{ or } b = pd \text{ (excl. above case)} \\ 0 & \text{else} \end{cases}$$

where

$$S = \{(\lambda(p+1) \pm p, \lambda(p+1) \pm 1), (\lambda(p+1) \pm 1, \lambda(p+1) \pm 1)\}$$

Furthermore, we are able to produce explicit bases for these spaces.

Main Idea

- Recall that $D^\times$ admits the decomposition

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- Recall that $D^\times$ admits the decomposition

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- Understand $\mathrm{Hom}(1 + \varpi\mathcal{O}, \overline{\mathbb{F}_p})$.
- Extend homomorphisms to functions from $D^\times$ to $\overline{\mathbb{F}_p}$ which parameterize extensions of representations.

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- Let $F/\mathbb{Q}_p$ be a field extension of finite degree.
- Take D instead to be a quaternion algebra over F.
- Many desirable properties of $\mathbb{Q}_p$ still hold in F, and much of our work can be adapted to F with slight modifications.

Thank You!

Questions?