Balanced Triangulations of Sphere Products with Minimal Vertices

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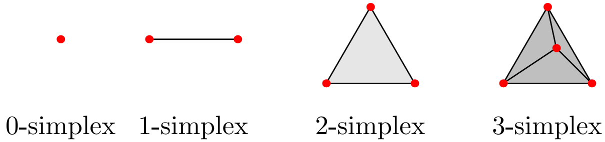
Definitions

An *n-simplex* is the convex hull of $n+1$ vertices in $\mathbb{R}^n$.



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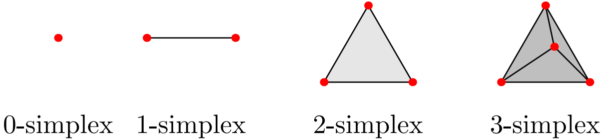
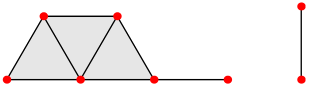


A *simplicial complex* Δ is a collection of simplicies closed under inclusion and intersection.



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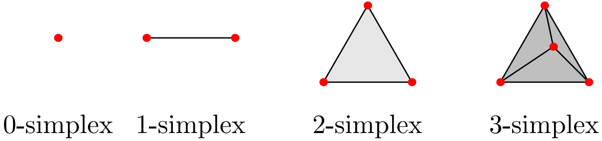

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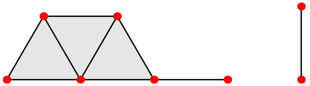
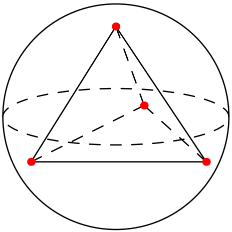
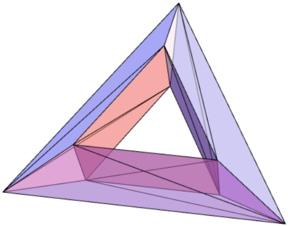


A *face* of Δ is a simplex contained in Δ, and a *facet* is a face which is maximal by inclusion.

Definitions (cont.)

A *triangulation* of a manifold M is a simplicial complex whose geometric realization is homeomorphic to M.





Torus Triangulation Graphic: `mathematica.stackexchange.com/questions/57829/torus-triangulation`

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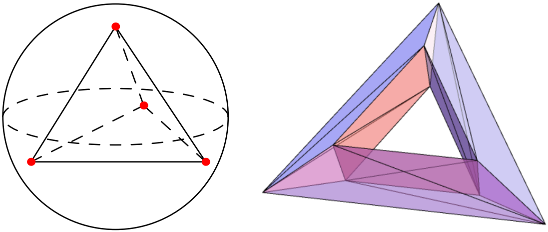

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A triangulation Δ of dimension d is said to be *balanced* provided that there exists a proper $(d+1)$-coloring; i.e., there exists $\kappa : V \to \{1, 2, \ldots, d+1\}$ such that if $\{a, b\}$ is an edge in Δ, $\kappa(a) \neq \kappa(b)$.

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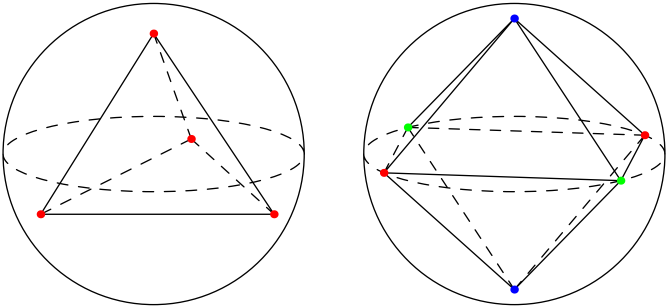

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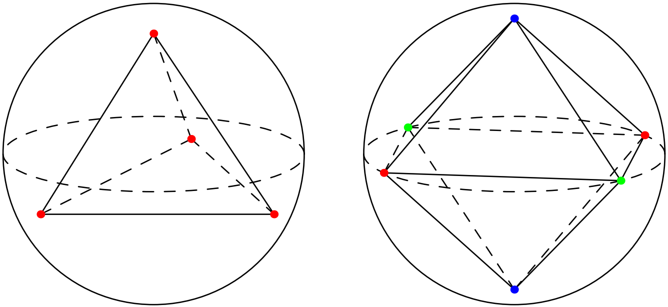

Note that the boundary complex of the $(d+1)$-cross-polytope, denoted ∂C_{d+1}, is the minimal balanced triangulation of $\mathbb{S}^d$.

Main Construction

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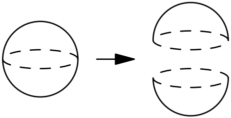
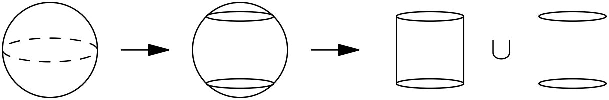


$$\mathbb{S}^2 \times \mathbb{S}^{d-3} \cong (D^2 \times \mathbb{S}^{d-3}) \cup (D^2 \times \mathbb{S}^{d-3})$$



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Two questions:

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2. How do we glue the two copies together?

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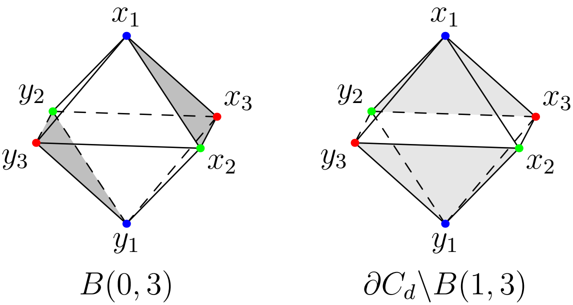
∂C_d contains d pairs of antipodal vertices: $\{x_1, y_1, x_2, y_2, \ldots, x_d, y_d\}$.
A facet in ∂C_d is of the form $\{u_1 u_2 \ldots u_n\}$, where $u_i \in \{x_i, y_i\}$. Define a *switch* to be when u_i and u_{i+1} are of different labels.
Then, we define $B(i, d)$ to be the set of facets with at most i switches.



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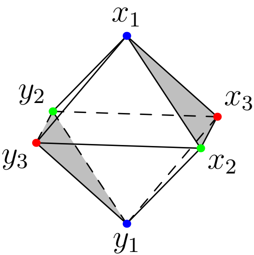
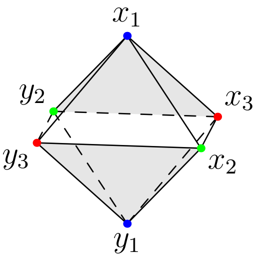


$B(0, 3)$



$\partial C_d \backslash B(1, 3)$

Klee and Novik (2011) showed that $B(1, d)$ triangulates $D^2 \times \mathbb{S}^{d-3}$.

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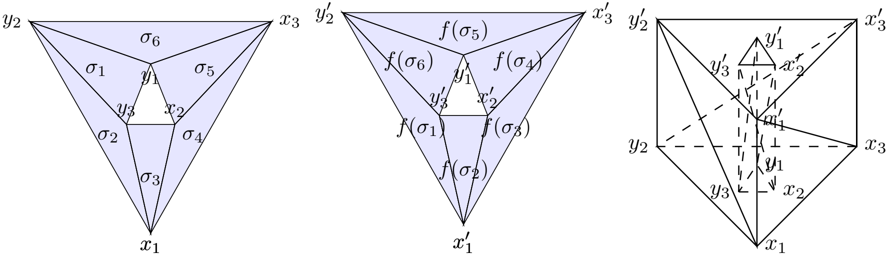
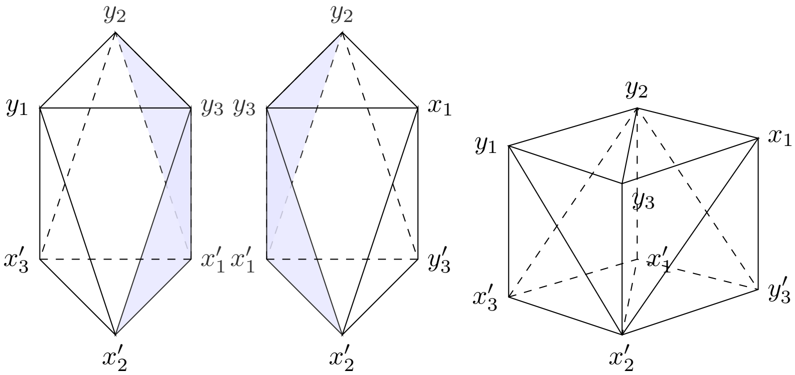
Identify faces σ and $f(\sigma)$ and connect using a cross-polytope, then glue cross-polytopes together to create a "tubular neighborhood".



$\Diamond$-Connected-Sum

How do we glue the cross-polytopes together? Define the $\Diamond$-connected sum as follows:



Properties of Triangulation

We were able to show certain properties of the triangulation Σ:

- $f_0 = 4d$
- $f_1 = 4d(2d-3)$
- $f_{d-1} = (d+2)2^d - 8d$
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We also developed a program to generate this construction for arbitrary d in Python/Sage.

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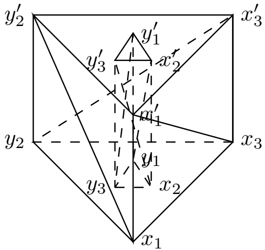

In the case $d = 3$, this produces two disjoint 3-cross-polytopes triangulating $S^2 \times S^0$, which is two disjoint 2-spheres, so it must be vertex minimal.

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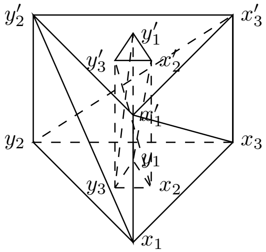

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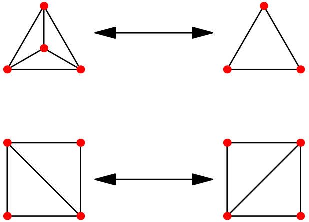
In general, we may be able to do this in less vertices.

Bistellar Flips and Cross Flips

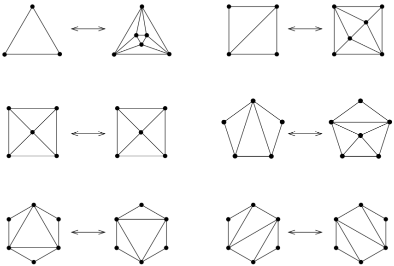
Pachner (1987) showed that any two d-dimensional triangulations of a closed combinatorial manifold may be connected by a sequence of *bistellar flips*, where a subcomplex is replaced with its complement in the boundary of the $(d+1)$-simplex.



$$d = 2$$

Bistellar Flips and Cross Flips (cont.)

Izmestiev, Klee, and Novik (2015) showed that an analogous result is true for balanced complexes, if we instead use subcomplexes satisfying certain properties in the boundary of the $(d+1)$-cross-polytope.



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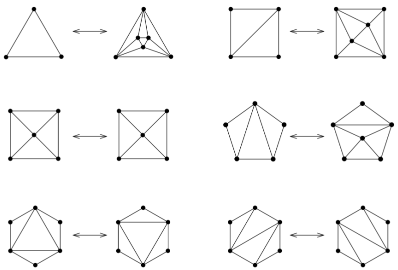

From [1].

This allows us to computationally search for vertex-minimal triangulations.

References

S. Klee and I. Novik: Centrally symmetric manifolds with few vertices, *Advances in Mathematics* **229** (2012), 487-500.

I. Izmestiev, S. Klee, and I. Novik: Simplicial moves on balanced complexes, *Advances in Mathematics* **320C** (2017), 82-114.