The Inverse Problem for Rational Equivariant Cohomology

Alexander Wang, Wang Ying, Juntai Zhou

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Abstract

Let G be a discrete group, and $\mathcal{A}$ be a commutative differential graded algebra over $\mathbb{Q}$. In this paper, we produce a model-category theoretic construction for a topological space P such that the equivariant cohomology $H^*_G(P)$ is isomorphic to $\mathcal{A}$.

Contents

1 Introduction

Given a topological space X, a common topological invariant to study is its singular homology chain complex $H_*(X)$. However, we can instead ask the inverse question: if we begin with a chain complex $C_\bullet$, can we find a topological space Y such that $H_*(Y) \cong C_\bullet$?

A traditional solution to this problem is through the use of Moore spaces, spaces $M(G, n)$ which have trivial reduced homology in all degree $k \neq n$ and homology G in degree n for sufficiently well-behaved groups G. These spaces can then be glued together using a wedge sum to obtain a space with the desired homology in each degree. However, this construction is somewhat unsatisfying as it requires restrictions on the desired homology groups.

The cohomological version of this problem was solved by Sullivan [9] in 1977 using rational homotopy theory, where the torsion homotopy groups are ignored. He defined a spatial realization functor $\langle - \rangle$[1] taking commutative differential graded algebras over the field of rational numbers $\mathbb{Q}$ to simplicial sets, for which the geometric realization would have the desired cohomology. We discuss this construction in Section 3.

If we now additionally allow our topological space to admit a group action, we can instead consider a cohomology theory which respects the group action, known as equivariant cohomology. We can instead pose the inverse question in the context of equivariant cohomology. In this paper, we survey the knowledge involved, and solve the problem in the equivariant case. In particular, we will introduce our set-up at the end of section 4 after introducing the relevant background material.

1.1 Convention

We use underlined text and script fonts to denote categories. The category $\underline{\text{CDGA}}$ refers to the category of non-negatively graded, commutative differential algebras over $\mathbb{Q}$, unless otherwise stated.

Weak equivalences are denoted by $\xrightarrow{\sim}$, fibrations $\twoheadrightarrow$, and cofibrations $\hookrightarrow$.

2 Acknowledgements

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3 Preliminaries

3.1 Simplicial Sets and Geometric Realization

Definition 3.1. Let Δ denote the simplicial category, consisting of objects which are ordered sets $[n] = \{0, 1, 2, \ldots, n\}$ and morphisms which are order-preserving functions.

Definition 3.2. A simplicial set is a contravariant functor $\Delta \to \underline{\text{Set}}$.

In particular, we consider $\Delta_n = \text{Hom}_\Delta(-, [n])$ to be the standard n-simplex. For a simplicial set $X : \Delta^{\text{op}} \to \underline{\text{Set}}$, we can consider the graded sequence of sets $X_n = X([n])$ for $n = 0, 1, \ldots$. These sets, together with the morphisms, completely determine the simplicial set.

We define the **cofaces** $d^i : [n-1] \to [n]$ to be

$$d^i(k) = \begin{cases} k & k < i \\ k+1 & k \geq i. \end{cases}$$

and we define the **codegeneracies** $s^j : [n+1] \to [n]$ analogously to be

$$s^j(k) = \begin{cases} k & k \leq i \\ k-1 & k > i. \end{cases}$$

These maps must satisfy the cosimplicial identities given below

- $d^j d^i = d^i d^{j-1}$ for $i < j$
- $s^j s^i = s^i s^{j+1}$ for $i \leq j$

[1]Some literature takes the spatial realization functor as the composition of the $\langle - \rangle$ and geometric realization, namely the functor $|\langle - \rangle| : \underline{\text{CDGA}^{\text{op}}} \to \underline{\text{Top}}$.

- $s^j d^i = \begin{cases} \text{id} & i = j \text{ or } i = j+1 \\ d^i s^{j-1} & i < j \\ d^{i-1} s^j & i > j+1. \end{cases}$

We can use this to reformulate our definition of a simplicial set.

Definition 3.3. A **simplicial set** is a sequence of sets X_n for $n \in \mathbb{N} \cup \{0\}$ together with **face maps** $d_i : X_n \to X_{n-1}$ for $i \in [n]$ and **degeneracies** $s_j : X_n \to X_{n+1}$ for $j \in [n]$, with the face maps and degeneracies satisfying the conditions

- $d_i d_j = d_{j-1} d_i$ for $i < j$
- $d_i s_j = s_{j-1} d_i$ for $i < j$
- $d_i s_j = \begin{cases} \text{id} & i = j \text{ or } i = j+1 \\ s_{j-1} d_i & i < j \\ s_j d_{i-1} & i > j+1. \end{cases}$

We refer to the elements $x \in X_n$ as an n-**simplex**. An element $x \in X_n$ is said to be **degenerate** provided that $x \in \text{im}(s_j)$ for some $s_j : X_{n-1} \to X_n$.

Since simplicial sets provide a powerful method to study geometric invariants, we will often like to be able to realize simplicial sets as geometric objects, and so we provide the following definitions.

Definition 3.4. Define the geometric n-simplex as

$$\Delta^n_{\text{geom}} = \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1}_{\geq 0} | \sum_{i=0}^{n} t_i = 1 \right\}$$

Definition 3.5. Let X be a simplicial set. Then, the **geometric realization** of X is given by

$$|X| = \left(\coprod_{n \geq 0} X_n \times \Delta^n_{\text{geom}} \right) / \sim$$

where $\sim$ is the equivalence relation given by

$$(d_i(x), t) \sim (x, d^i(t)) \qquad (s_j(x), u) \sim (x, s^j(u))$$

for $x \in X_n$, and $s, t \in \Delta^{n-1}$.

Similarly, given a topological space Y, we can define the **singular simplicial set**, $\text{Sing}(Y)$, to have n-simplices as follows

$$\text{Sing}(Y)_n = \text{Hom}_{\underline{\text{Top}}}(\Delta^n_{\text{geom}}, Y)$$

with face and degeneracy maps given by precomposition. Then, it is true that these functors form an adjunction given below. A detailed proof of the adjunction can be found in [2].

$$\underline{\text{sSet}} \underset{\text{Sing}(-)}{\overset{|-|}{\rightleftarrows}} \underline{\text{Top}}.$$

There are two properties of $| - |$ we will use later:

Proposition 3.6 ([7])**.** The geometric realization $|K|$ of a simplicial set K is a CW complex with $n-$skeleton $|K(n)|$ and $n-$cells the non-degenerate $n-$simplicies $\sigma \in N_\bullet K_n$. The attaching map for σ is the restriction of q_K to $\{\sigma\} \times \partial \Delta^n$.

Proposition 3.7 ([4])**.** The functor | - | commutes with finite limits, in particular pullbacks.

3.2 Adjunction

Definition 3.8. Let $\mathcal{C}, \mathcal{D}$ be categories, and let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be covariant functors. Then, F and G are said to be **adjoint** provided that for all $X \in \mathrm{Obj}(\mathcal{C})$ and for all $Y \in \mathrm{Obj}(\mathcal{D})$, we have that

$$\mathrm{Hom}_{\mathcal{D}}(X, GY) \cong \mathrm{Hom}_{\mathcal{C}}(FX, Y)$$

where this bijection is natural. When this is the case, we say that F is **left adjoint** to G and G is **right adjoint** to F, and this adjunction is denoted by

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}, \quad \perp$$

Definition 3.9. Let $\mathcal{C}, \mathcal{D}$ be categories, and let $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ be a pair of adjoint functors, with L left adjoint to R, and let $X \in \mathrm{Obj}(\mathcal{C})$ and $Y \in \mathrm{Obj}(\mathcal{D})$. Then, by the adjunction, we have a natural bijection

$$\Psi : \mathrm{Hom}_{\mathcal{D}}(LX, LX) \to \mathrm{Hom}_{\mathcal{C}}(X, RLX)$$

Since $\mathrm{id}_{\mathcal{D}} \in \mathrm{Hom}_{\mathcal{D}}(LX, LX)$, we define the **unit** of the adjunction to be $\Psi(\mathrm{id}_{\mathcal{D}}) \in \mathrm{Hom}_{\mathcal{C}}(X, RLX)$.
Dually, we can define the **counit** of the adjunction to be the map in $\mathrm{Hom}_{\mathcal{C}}(LRX, X)$ corresponding to $\mathrm{id}_{\mathcal{D}} \in \mathrm{Hom}_{\mathcal{D}}(RX, RX)$.

3.3 Commutative Differential Graded Algebras

The algebraic structure we use in the construction of the space is the commutative differential graded algebra.

Definition 3.10. A **differential graded algebra** is a graded algebra $\mathcal{A}$ together with a degree 1 map $d : A \to A$ satisfying

1. $d^2 = 0$
2. $d(ab) = (da)b + (-1)^{\deg(a)}(db)$

The category <u>CDGA</u> refers to the category of non-negatively commutative graded differential algebras over the field of rational numbers $\mathbb{Q}$, with morphisms given by level-wise maps which commute with the differential.

We will often refer to an object in <u>CDGA</u> as a cdga. The structure of chain complexes gives rise to a weaker kind of equivalence between them:

Definition 3.11. A **weak equivalence** f between two cdga's $\mathcal{A}$ and $\mathcal{A}'$ is a cdga map such that the underlying chain map of f induces an isomorphism between the homology of the underlying chain complexes of $\mathcal{A}$ and $\mathcal{A}'$. We also call such cgda maps **quasi-isomorphisms**.

Remark 3.12. We say that A is weakly equivalent to B if there is a zig-zag of weak equivalences between them:

$$A \xleftarrow{\sim} X_1 \xrightarrow{\sim} \cdots \xleftarrow{\sim} X_n \xrightarrow{\sim} B.$$

4 Sullivan's Realization Functor

In this section we give a brief review of the spatial realization functor Sullivan developed in [9] and the related properties.

4.1 Differential forms and de Rham's theorem on general spaces

To extend the definitions of forms to more general spaces, we need several axioms:

1. First, we must have a notion of cells, and forms on a cell.
2. The spaces are made up inductively of cells of increasing dimension. Cells have boundaries which are attached by admissible maps to the inductive space. The forms on a space is built up inductively by extending over a new cell an inductive form pulled back to the boundary of that new cell.
3. There is an "integration" map from forms to ordinary cochains on the space.
4. The extension property: any form in the boundary of a cell extends over the entire cell.

Example 4.1. CW-complexes; simplicial sets.

Remark 4.2. With this setup, we have a generalized de Rham theorem: form cohomology is isomorphic to ordinary (space) cohomology.

Theorem 4.3 (Sullivan [9])**.** For any such notion of cell, space, form, and integration map, if forms on cells satisfy the de Rham theorem, so do forms on spaces.

Proof. Repeatedly apply the axioms, induction, and five-lemma. □

As an example, we want to define the de Rham algebra of a simplicial set X. We first need the simplicial commutative cochain algebra Ω^*, and then take the degree-wise maps from X to Ω^*, which have the desired cdga structure.

Definition 4.4. The simplicial commutative cochain algebra $\Omega^* = \{\Omega^n\}_n$ is defined by:

$$\Omega_n = \frac{\mathbb{Q}[t_0, \cdots, t_n] \otimes \wedge(y_0, \cdots, y_n)}{\left(\sum_i t_i - 1, \sum_j y_j\right)},$$

where t_i are of degree 0, and y_j are of degree 1, and $dt_i = y_i, dy_j = 0$.

The differential on Ω^* is defined by

$$d(f) = \sum_i \frac{\partial f}{\partial t_i} y_i$$

for $f \in \dfrac{\mathbb{Q}[t_0, \cdots, t_n]}{\sum_i t_i - 1}$.

Definition 4.5. For a simplicial set X, the de Rham algebra of X is defined as

$$\Omega^n(X) := \mathrm{Hom}_{\underline{\mathrm{sSet}}}(X, \Omega^n).$$

Remark 4.6. Intuitively, Ω^* can be regarded as the functor that sends a simplicial set to the (polynomial) forms on it.

Consequently, Theorem 4.3 establishes a weak-equivalence (3.11) between functors:

Theorem 4.7. There is a weak equivalence from the de Rham functor to the rational cochain complex functor:

$$\Omega^*(X) \xrightarrow{\sim} C^*(X; \mathbb{Q}).$$

4.2 Spatial Realization

Let $\mathcal{A}$ be a cdga. We introduce the spatial realization functor $\langle - \rangle$, the right adjoint of the functor Ω^*.

Definition 4.8. An $\mathcal{A}-$differential system is a cdga map of $\mathcal{A}$ into the forms on X.

Definition 4.9. The spatial realization $\langle \mathcal{A} \rangle$ is defined by the sets of all $\mathcal{A}-$differential systems on standard simplices $\Delta^0, \Delta^1, ...$, which form a simplicial set with face maps, degeneracy maps induced by the ones on differential forms.

Using the simplicial cdga Ω^*, we can rewrite the definition above as:

Definition 4.10. $\langle - \rangle : \underline{\text{CDGA}} \to \underline{\text{sSet}}$ is the functor specified by $\langle \mathcal{A} \rangle := \underline{\text{CDGA}}(\mathcal{A}, \Omega^*)$.

The simplicial set $\langle \mathcal{A} \rangle$ is a basic object in rational homotopy theory. However, for our interest, we would only present the following property:

Theorem 4.11. There is a natural map $\mathcal{A} \to \mathbb{Q}-$polynomial forms on $\langle \mathcal{A} \rangle$ which induces an isomorphism of cohomology over $\mathbb{Q}$.

Remark 4.12. By the de Rham theorem 4.3, $H^*(\langle \mathcal{A} \rangle, \mathbb{Q}) = H^*(\mathcal{A})$.

Theorem 4.13 ([1])**.** If a cdga $\mathcal{A}$ is of finite type, then the unit map $\mathcal{A} \to \Omega^* \langle \mathcal{A} \rangle$ is a weak equivalence.

Remark 4.14. Although $\Omega^*(-)$ is a left adjoint functor, it preserves homotopy pullback of finite type objects. A proof of this fact could be found below Proposition 15.8 in [3].

4.3 Rational Homotopy Theory

Let S be a simplicial set. We now discuss the nature of the unit map $S \to \langle \Omega^*(S) \rangle$, which lays in the rational homotopy theory.

For a simply connected finite complex X, the homotopy groups $\pi_n(X)$ can be decomposed as $\mathbb{Z}^r \oplus T$, where r is the rank and T is a finite torsion subgroup. However, most of time we do not know much about T. This challenge motivates the rational homotopy theory, the central idea of which is to ignore the torsion homotopy groups. By the work of Sullivan, for any simply connected space X, there exists a rational simply connected space $X_{\mathbb{Q}}$ which does not have torsions. Moreover, X and $X_{\mathbb{Q}}$ are rationally homotopic equivalent.

The counit map we are looking at, $S \to \langle \Omega^*(S) \rangle$, is precisely the rationalization of S, i.e., it sends S to the its rationally homotopic equivalent space $S_{\mathbb{Q}}$, a rational space [2]. We now briefly review some essential concepts in rational homotopy theory, including rational homotopy equivalence, rational spaces, and rationalization of simply connected spaces. A more detailed introduction can be found in the note [1].

Definition 4.15. Let X and Y be simply connected spaces. A map $f : X \to Y$ is called a rational homotopy equivalence (denoted $X \xrightarrow{\sim_{\mathbb{Q}}} Y$) if it satisfies any of the following equivalent conditions:

1. The induced map $f_* : \pi_*(X) \otimes \mathbb{Q} \to \pi_*(Y) \otimes \mathbb{Q}$ is an isomorphism.
2. The induced map $f_* : H_*(X; mathbbQ) \to H_*(Y; mathbbQ)$ is an isomorphism.

Similar to the Definition 3.11, we say that two simply connected spaces X and Y are rationally homotopic equivalent if there is a zig-zag of rational homotopy equivalences bewteen them. We use $X \sim_{\mathbb{Q}}$ to indicate the rational homotopic equivalence.

Definition 4.16. A simply connected space X is called rational if it satisfies any of the following equivalent conditions:

1. The homotopy groups $\pi_k(X)$ are uniquely divisible for all k.
2. The homology groups $H_k(X)$ are uniquely divisible for all $k > 0$.

Theorem 4.17. Every simply connected space admits a rationalization.

The proof is based on the induction on the Postnikov tower of X using the Eilenberg-Maclane spaces, and could be found in Belgrund's notes.

[2]Due to the equivalence of $\underline{\text{sSet}}$ and $\underline{\text{Top}}$, we use simplicial set and topological spaces interchangeably.

5 Classifying Space and Equivariant Cohomology

This section follows the note [8], where proofs and a more detailed introduction to principal bundles could be found.

Definition 5.1. Let G be a group, B a topological space. A principal G-bundle over B is a continuous G-equivariant map of topological spaces, $p : P \to B$, such that G acts trivially on B, and p satisfies the local triviality condition. We denote the set of isomorphism classes of principal $G-$bundles over B as $P_G(B)$.

Remark 5.2. The group G acts freely and transitively on P and p factors through a homeomorphism $P/G \to B$.

Lemma 5.3. Let $p : E \to B$ be a fiber bundle with fiber F, and let $f_0 : X \to B$ and $f_1 : X \to B$ be homotopic maps. Then the pull-back bundles are isomorphic as principal G-bundles, i.e. $f_0^*(E) \cong f_1^*(E)$.

Let X, Y be two spaces. $[X, Y]$ denotes the set of the homotopy classes of maps from X to Y.

Corollary 5.4. Let $p : E \to B$ be a principal $G-$bundle over a connected space B. Then for any space X the pullback construction gives a well defined map $\rho : [X, B] \to P_G(X)$.

$$\begin{array}{ccc} P & \longrightarrow & X \\ \downarrow & & \downarrow \\ E & \longrightarrow & B \end{array}$$

To classify principal $G-$bundles, the following definition is made:

Definition 5.5. A principal $G-$bundle $p : EG \to BG$ is called universal if the map $\rho : [X, BG] \to P_G(X)$ is a bijection for every space (CW-complex) X. In this case the base space of the universal bundle BG is called a classifying space for G, and $EG \to BG$ is called the universal bundle.

Remark 5.6. There are many other important properties of BG. For example, $\pi_n(G) \cong \pi_{n+1}(BG)$. In particular, if G is discrete, then BG is the Eilenberg–MacLane space.

There are several constructions of the classifying space, such as Milnor's construction and the simplicial construction. In this note, we will use the simplicial construction of BG and EG, so we present it here for referrence:

Definition 5.7. Given a group G, the nerve of G, $N_\bullet G$, is the simplicial set with $N_n G = G^n$ with face and degeneracy maps given by

$$s_j(g_1, ..., g_n) = (g_1, ..., g_j, 1, g_{j+1}, ..., g_n)$$
$$d_i(g_1, ..., g_n) = \begin{cases} (g_2, ..., g_n), i = 0 \\ (g_1, ..., g_i g_{i+1}, ..., g_n), 0 < i < n \\ (g_1, ..., g_{n-1}), i = n. \end{cases}$$

Definition 5.8. Given a group G, we can form the group $\widetilde{G}$ with the object G and a unique morphism $g \to h$ for each $(g, h) \in G \times G$. Define $N_n\widetilde{G}$ as G^{n+1} with face and degeneracy maps given by

$$s_i(g_1, ..., g_n) = (g_1, ..., g_j, g_j, g_{j+1}, ..., g_n)$$
$$d_i(g_1, ..., g_n) = (g_1, ..., g_{i-1}, g_{i+1}, ..., g_n).$$

Theorem 5.9. The classifying space BG and EG are the geometric realization of $N_\bullet G$ and $N_\bullet\widetilde{G}$, respectively:

$$BG = |N_\bullet G|$$
$$EG = |N_\bullet \widetilde{G}|.$$

However, EG and BG are unique up to homotopy equivalence. This gives a well-defined notion of equivariant cohomology:

Definition 5.10. Let G be a topological group acting on X. The Borel construction of $G-$equivariant cohomology of X is $H^*_G(X) := H^*(X \times_G EG)$, where $X \times_G EG := (X \times EG)/G$.

Remark 5.11. Choosing any base point $x \in X$, there is a map $H^*_G(x) = H^*(BG) \to H^*_G(X)$, so $H^*_G(X)$ is a $H^*(BG)-$algebra.

Proposition 5.12. Given any space X with a map $X \to BG$, let P be the pullback as in corollary 5.4, then $H^*_G(P) \cong H^*(X)$ as $H^*(BG)-$algebras.

Proof. We establish a homotopy equivalence between $X \times EG$ and $P \times_G EG$: Denote the maps $f : P \to X$, $\varphi : P \to EG$. Note that both f and φ are $G-$equivariant. Define

$$\begin{aligned} F : P \times_G EG &\to X \text{ by} \\ (p, t) &\mapsto f(p), \end{aligned}$$

and

$$\begin{aligned} G : X &\to P \times_G EG \text{ by} \\ x &\mapsto (\tilde{x}, \varphi(\tilde{x})), \end{aligned}$$

where $\tilde{x}$ is a lift of x. Since P has a transitive $G-$action, it is easy to check that F, G are well-defined. Clearly $F \circ G =$Id, and $G \circ F(p, t) = (p, \varphi(p))$. Since EG is contractible, $G \circ F \simeq$Id. □

Sullivan's spatial realization recovers a space with a given (co)homology. To generalize this to the equivariant case, we assume that we are given an algebra $\mathcal{A}$ under $H^*(BG)$. By proposition 5.12, we can reduce our original question to constructing a space X with a map $X \to BG$ and a quasi-isomorphism $X \to |\langle \mathcal{A} \rangle|$. It turns out that the most natural attempt works: we can construct a map $BG \to |\langle \Omega^*(N_\bullet G) \rangle|$, and the pullback gives the desired space. The proof uses model categories, which will be presented in the following sections.

$$\begin{array}{ccc} X & \longrightarrow & |\langle \mathcal{A} \rangle| \\ \downarrow & & \downarrow \\ BG & \longrightarrow & |\langle \Omega^*(N_\bullet G) \rangle| \end{array}$$

One technical concern is that to use the theorem of classification of principal bundles, we might need to assume that X is a CW-complex. However, we can do everything in simplicial setting and take the geometric realization in the end, thus indeed yielding a CW-complex X by Theorem 3.6.

6 Model Categories

In order to bring the good properties of classifying spaces to arbitrary chain complexes, some model categorical tools are required. We briefly introduce the basic definitions and theorems in this section. We direct readers to [6] for more robust treatment of model categories.

Definition 6.1 ([6])**.** A model structure on a category $\mathcal{C}$ consists of three distinguished subcategories of $\mathcal{C}$ called weak equivalences, cofibrations, and fibrations, and two functorial factorizations (α, β) and (γ, δ) satisfying the following axioms:

1. (2-of-3) Let f and g are morphisms of $\mathcal{C}$ such that gf is defined. If two of f, g, and gf are weak equivalences, then so is the third.

2. If f is a retract of g and g is a weak equivalence, cofibration, or fibration, then so is f.

3. If f is both a weak equivalence and cofibration, we call it a trivial cofibration, dually for trivial fibration.[3] The trivial cofibrations have the left lifting property with respect to fibrations, and cobrations have the right lifting property with respect to trivial fibrations.

[3] also called acyclic cofibrations and fibrations

4. For any morphism f, $\alpha(f)$ is a cobration, $\beta(f)$ is a trivial fibration, $\gamma(f)$ is a trivial cofibration, and $\delta(f)$ is a fibration.

Definition 6.2. A model category is a category $\mathcal{C}$ with all small limits and colimits together with a model structure.

Remark 6.3. The explicit definition of the these three sets of morphisms can be given quite arbitrarily, and the only requirement is that they satisfy the axioms above. We will now use the projective model structure on the category CDGA.

Definition 6.4. The projective model structure on CDGA is defined by:

1. Weak equivalences are those in the sense of 3.11.
2. Fibrations are degreewise surjective cdga maps.

We also need a model structure on sSet.

Definition 6.5. The classical model structure on sSet is given by:

1. Weak equivalences are weak homotopy equivalences.
2. Cofibrations are level-wise injections.

In the language of model categories, we prove that the functor Ω^* preserves weak equivalences:

Lemma 6.6. The functor $\Omega^*(-)$ sends weak equivalences in the classical model structure on sSet to those in the projective model structure on CDGA^{op}; namely, it takes weak homotopy equivalences to quasi-isomorphisms.

Proof. Let $X \xrightarrow{\sim} Y$ in sSet. By definition of the classical model structure, the geometric realization of X and Y are weak homotopic equivalent, which means $\pi_n(X) \simeq \pi_n(Y$ for all n. By the Hurewicz theorem, $H_n(X) \simeq H_n(Y)$ for all n, meaning that $C^*(X)$ is quasi-isomorphic to $C^*(Y)$ in CDGA. But it follows from 4.7 that for any $X \in$ sSet, $\Omega^*(X)$ is quasi-isomorphic to $C^*(X)$; hence $\Omega^*(X) \xrightarrow{\sim} \Omega^*(Y)$. □

Now we introduce a special kind of model categories called (left/right) proper model categories.

Definition 6.7. Let $\mathcal{M}$ be a model category. $\mathcal{M}$ is called right proper if the pullback of a weak equivalence along a fibration is a weak equivalence, as shown in the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{\sim} & A \\ \downarrow & \ulcorner & \downarrow \text{fibration} \\ B & \xrightarrow[\sim]{} & C \end{array} \quad .$$

Dually, $\mathcal{M}$ is a left proper model category if the pushout of a weak equivalence along a cofibration is a weak equivalence. $\mathcal{M}$ is called a proper category if it is both right and left proper.

We now show that CDGA with the projective model structure is a right proper model category.

Definition 6.8. An object X in a model category is called fibrant if the unique morphism $X \to 1$ is a fibration, where 1 denotes the terminal object.

Remark 6.9. Every object in CDGA is fibrant. The proof follows from the fact that in CDGA the terminal object is the zero chain complex, and the fibrations are degreewise surjections.

Theorem 6.10. A model category in which all objects are fibrant is right proper.

Corollary 6.11. CDGA is a right proper model category.

Remark 6.12. Recall that weak equivalences are defined to be morphisms whose underlying chain maps are quasi-isomorphisms, and the fact that right properness preserves pullbacks of weak equivalences along fibrations will be used in the later proof. In fact, CDGA and sSetare both left and right proper.

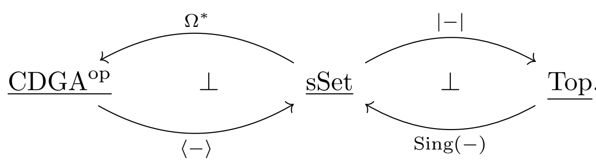
7 Construction of the Space

We prove our main theorem in this section.

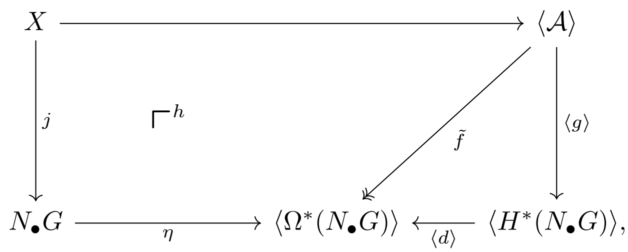
Theorem 7.1. Given a cdga map $\mathcal{A} \to H^*(BG)$, there is a topological space P that has a free and transivitive $G-$action on it and $H^*_G(P, \mathbb{Q}) \simeq H^*(\mathcal{A}) \simeq \mathcal{A}$. Moveover, P is a CW-complex.

Proof. We first develop the categorical intersection between CDGA, sSet, and Top in this section. Recall that Top is the category of topological spaces and continuous maps, sSet of objects simplicial sets and the maps between simplicial sets, and CDGA of commutative differential algebras over the field $\mathbb{Q}$ and algebra homomorphisms.

Recall that we have two pairs of adjunctions as follows:



Given a cdga $\mathcal{A}$, a discrete group G whose classifying space is of finite type, and a map $g : H^*(N_\bullet G) \to \mathcal{A}$, we can form the following homotopy pullback diagram in sSet:



in which the map η given by the spatial realization adjunction, d by the de Rham theorem 4.2 and formality, and $\tilde{f}$ by replacing the map $f = \langle d \rangle \cdot \langle g \rangle$ by a weakly equivalent fibration. We claim that X, the ordinary pullback of $\langle \mathcal{A} \rangle$ and $N_\bullet G$ over $\langle \Omega^*(N_\bullet G) \rangle$, presents a homotopy pullback. This follows from the following proposition and the fact that *sSet* with the classical model structure is right proper (cf. Remark 6.11):

Proposition 7.2 ([5])**.** In the diagram $A \to C \leftarrow B$, if the model category is right proper and one of the maps is a fibration, then the ordinary pullback $A \times_C B$ represents the homotopy pullback $A \times^h_C B$.

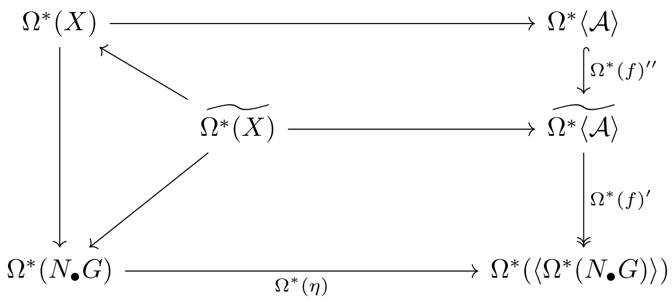
Proposition 7.3. In $\underline{\text{CDGA}}^{\text{op}}$, $\Omega^*(N_\bullet G)$ is weakly equivalent to $\Omega^*(\langle H^*(N_\bullet G) \rangle)$.

Proof. By Remark 3.12 it suffices to show that there is a zig-zag of weak equivalences between $\Omega^*(N_\bullet G)$ and $\Omega^*(\langle H^*(N_\bullet G) \rangle)$. Assume that $N_\bullet G$ is of finite type. By Theorem 4.13 the unit map η is a weak equivalence, and by Theorem 6.6 the map $\Omega^*(\eta) : \Omega^* \langle \Omega^*(N_\bullet G) \rangle \to \Omega^*(N_\bullet G)$ is a weak equivalence.

It follows from the de Rham Theorem (cf. 4.7) that $d : \Omega^*(N_\bullet G) \xrightarrow{\sim} H^*(N_\bullet G)$ is a weak equivalence. Again by Theorem 4.13 we have a zig-zag of weak equivalences between $\Omega^* \langle \Omega^*(N_\bullet G) \rangle$ and $\Omega^*(\langle H^*(N_\bullet G) \rangle)$. $\square$

Remark 7.4. The map $\Omega^*(\eta)$ is a weak equivalence also follows from the fact that $\langle \Omega^*(N_\bullet G) \rangle$ is the rationalization of $N_\bullet G$, and $\Omega^*(-)$ sends rational homotopy equivalence to quasi-isomorphisms by Lemma 6.6. In the construction of the space, we only need $\Omega^*(\eta)$ to be a weak equivalence.

By Remark 4.14, we have the homotopy pullback diagram of $\Omega^*(A)$ and $\Omega^*(N_\bullet G)$ over $\Omega^* \langle \Omega^*(N_\bullet G) \rangle$. in $\underline{\text{CDGA}}^{\text{op}}$ (the outer rectangle):



We factor $\Omega^*(\widetilde{f})$ into $\Omega^*(f)'' \circ \Omega^*(f)'$, where $\Omega^*(f)'$ is a fibration and $\Omega^*(f)''$ is a trivial cofibration. Then we take another ordinary pullback of $\widetilde{\Omega^*(A)}$ and $\Omega^*(N)$ over $\Omega^*\langle\Omega^*(N_\bullet G)\rangle$, namely $\widetilde{\Omega^*(X)}$. Again by Proposition 7.2, $\widetilde{\Omega^*(X)}$ presents a homotopy pullback of $\widetilde{\Omega^*(A)}$ and $\Omega^*(N_\bullet G)$ over $\Omega^*(\langle\Omega^*(N_\bullet G)\rangle)$(the lower right trapezium).

Since $\Omega^*(\eta)$ is a weak equivalence (cf. Proposition 7.3), it follows from the right properness of $\underline{\mathrm{CDGA}^{\mathrm{op}}}$ that the map $\Omega^*(X) \to \widetilde{\Omega^*(A)}$ is a weak equivalence. Hence we have a zig-zag of weak equivalences between $\Omega^*(X)$ and $\Omega^*\langle\mathcal{A}\rangle$ (where the rightmost map is just the identity):

$$\Omega^*(X) \xleftarrow{\sim} \widetilde{\Omega^*(X)} \xrightarrow{\sim} \widetilde{\Omega^*\langle\mathcal{A}\rangle} \xleftarrow{\sim} \Omega^*\langle\mathcal{A}\rangle \xrightarrow{\sim} \Omega^*\langle\mathcal{A}\rangle.$$

By Definition 5.1, G acts trivially on X. Now we pull back the universal bundle $p : N_\bullet\widetilde{G} \to N_\bullet G$ along $j : X \to N_\bullet G$, and get $P = N_\bullet\widetilde{G} \times_{N_\bullet G} X$. Then by Proposition 3.7 we have the following double pullback diagram in $\underline{\mathrm{Top}}$, in which G acts freely on $|P|$ (Remark 5.2), and $H^*_G(|P|, \mathbb{Q}) \cong H^*(|X|, \mathbb{Q}) \cong \mathcal{A}$ (cf. Proposition 5.12 and Remark 4.12). Furthermore, it follows from Proposition 3.6 that $|P|$ is a CW-complex:

$$\begin{array}{ccccc} |P| & \longrightarrow & |X| & \longrightarrow & |\langle\mathcal{A}\rangle| \\ \downarrow & & \downarrow & & \downarrow \\ EG & \longrightarrow & BG & \longrightarrow & |\langle\Omega^*(N_\bullet G)\rangle|. \end{array}$$

□

8 Further questions

In the non-equivariant case of the problem, there exists a constructive solution for producing spaces with specified homology through gluing together Moore spaces using a wedge sum. A further direction of research is to construct an analog of Moore spaces and wedge sums for equivariant cohomology. In particular, this problem is difficult as defining a wedge sum which respects the group action requires gluing two spaces with a G-action at a point, but the equivariant cohomology construction requires a fixed-point free action, so there may be no suitable choice of point. One possible idea for a remedy is to glue the spaces along a G-orbit within each space with trivial equivariant cohomology, and applying an equivariant variation of the Mayer-Vietoris sequence on the resulting decomposition.

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