# THE POINCARE-HOPF THEOREM

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Abstract. Mapping degree, intersection number, and the index of a zero of a vector field are defined. The Poincare-Hopf theorem, which states that under reasonable conditions the sum of the indices of a vector field equals the Euler characteristic of the manifold, is proven. Some consequences are discussed.

## 1. Mapping Degree

In this note everything will be assumed to be smooth, and manifolds will be assumed to be compact and orientable. Many of the results hold in a broader context, but since the Poincaré-Hopf Theorem requires a compact and orientable manifold, we restrict ourselves to this simpler context. We will denote the dimension of a manifolds with superscripts, so $M^m$ is a manifold of dimension $m$.

Given a map $f : M \to N$ between two manifolds of the same dimension, and a regular value $y$ of $f$, we define the degree (sometimes called the Brouwer degree or mapping degree) of $f$ at $y$ by

$$\deg(f, y) = \sum_{x \in f^{-1}(y)} \operatorname{sign}(df)_x.$$

Note that since $y$ is a regular value of $f$, the $f^{-1}(y)$ is an isolated subset of a compact manifold, and is hence finite. The following technical lemmas will help us to prove that degree does not depend on the point $y$, and is invariant under homotopy of $f$.

**Lemma 1** (Transversality Lemma)**.** *Suppose $f : M \to N$, $P$ a closed sub-manifold of $N$, $C$ a closed subset of $M$, and $f|_C \pitchfork P$. Then there exists a $g : M \to N$ that equals $f$ on a neighbourhood of $C$ with $g \pitchfork P$. $g$ may be chosen to be homotopic to $f$ and arbitrarily close to $f$.*

*Proof.* This fact is very well known, but the authors have not yet found a reference for it. The interested reader is encouraged to contact the first author for a rigorous proof. □

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This lemma is in fact enormously powerful! Two manifolds are said to be in general position if they are transverse. The following lemma, says that any two manifolds can always be put into general position.

**Corollary 2** (General Position Lemma). *Suppose that $f : M \to N$, and $P$ a closed sub-manifold of $N$. There is a map $g : M \to N$ homotopic to and arbitrarily close to $f$, so that $g \pitchfork P$. Since being an embedding is a stable property, if $f$ is an embedding, than $g$ can also be chosen to be an embedding.*

*Proof.* Apply the Transversality Lemma with $C = \emptyset$. □

We now return to mapping degree, proving the fundamental result which shows that it is invariant under homotopy, and that will show that the index of a zero of a vector fields is well defined.

**Lemma 3** (Extension Lemma). *Assume that $M^{n+1}$ is a connected and with boundary, $N^n$ is connected, $f : \partial M \to N$, and $y$ is a regular value for $f$. If $f$ can be extended to $F : M \to N$, then $\deg(f, y) = 0$.*

*Proof.* Suppose that $f$ is such a map, and $F : M \to N$ is such an extension. By the Transversely Lemma with $C = \partial M$, we can assume without loss of generality that $F \pitchfork \{y\}$, that is to say that $y$ is a regular value of $F$. Now, $f^{-1}(y)$ is a finite set of points in $\partial M$, and $F^{-1}(y)$ consists of a finite set of circles and line segments, and the endpoints of the line segments all lie in $\partial M$. Furthermore, every point in $f^{-1}(y)$ is the endpoint of some line segment of $F^{-1}(y)$, and the circles in $F^{-1}(y)$ do not intersect $\partial M$. Thus the points of $f^{-1}$ occur in pairs, each pair being the endpoints of a curve $\alpha : [0, 1] \to M$. So it suffices to show that

$$\mathrm{sign}((df)_{\alpha(0)}) + \mathrm{sign}((df)_{\alpha(1)}) = 0,$$

where $\mathrm{sign}((df)_p)$ is defined to be $\mathrm{sign}(\det((df)_p))$.

We can choose $n$ vector fields $v_1(t), \cdots, v_n(t)$ along $\alpha$ so that $v_i(0)$ and $v_i(1)$ are tangent to $\partial M$ for each $i$, and $\{\alpha'(t), v_1(t), \cdots, v_n(t)\}$ is an oriented basis for $T_{\alpha(t)}M$ for $t \in [0, 1]$ (Parallel transport can be used to get these vector fields, as well as more hands on arguments, using the fact that $\alpha$ is an immersion, and local coordinates). From now on, every mention of a basis will be understood to be ordered. Since $F \pitchfork \{y\}$, the vectors

$$\{dF(\alpha'(t)), dF(V_1(t)), \cdots, dF(V_n(t))\}$$

must span $T_yN$ for each $t$. But $dF(\alpha'(t)) = 0$, and $T_yN$ is $m$-dimensional; so we conclude that

$$\{dF(V_1(t)), \cdots, dF(V_n(t))\}$$

is a basis for $T(y)$ for all $t$. It follows that

$$\{dF(V_1(0)), \cdots, dF(V_n(0))\} \text{ and } \{dF(V_1(1)), \cdots, dF(V_n(1))\}$$

have the same orientation. Similarly, the orientation of

$$\{\alpha'(t), v_1(t), \cdots, v_n(t)\}$$

is constant. But $\alpha(0)$ points outward from $\partial M$, and $\alpha(1)$ points inwards, so $\{v_1(0), \cdots, v_n(0)\}$ and $\{v_1(1), \cdots, v_n(1)\}$ must have opposite signs. ($n$ vectors in the tangent space on the boundary are defined to have positive orientation if they have positive orientation when supplemented on the left with a vector pointing into the manifold.) So $\det d(f)_{\alpha(0)}$ and $\det d(f)_{\alpha(1)}$ have the same sign if we use any given basis for $T_yN$, and the bases $\{v_1(0), \cdots, v_n(0)\}$ and $\{v_1(1), \cdots, v_n(1)\}$ for $T_{\alpha(0)}\partial M$ and $T_{\alpha(1)}\partial M$. But one of these basis is oriented and the other not, so we get

$$\operatorname{sign}((df)_{\alpha(0)}) + \operatorname{sign}((df)_{\alpha(1)}) = 0$$

as desired. (Of course, when written this way, it is assumed that determinants are taken in oriented bases.) □

**Corollary 4.** *If $g_0$ and $g_1$ are smoothly homotopic maps that have a common regular value $y$, then $\deg(g_0, y) = \deg(g_1, y)$.*

*Proof.* Let $G : M \times [0,1] \to N$ be a homotopy, so $G(\cdot, 0) = g_0$ and $G(\cdot, 1) = 1$. The result now follows by applying the previous lemma, since

$$\partial(M \times [0,1]) = M \times \{1\} - M \times \{0\}$$

and thus

$$\deg(F|_{\partial(M\times[0,1])}, y) = \deg(g_1, y) - \deg(g_0, y).$$

□

In light of this, we can speak of $\deg(f, y)$ whenever there is a $g$ homotopic to $f$ such that $y$ is a regular value of $g$: We just set $\deg(f, y) = \deg(g, y)$, and $y$ no longer needs to be a regular value for $f$, as long at is a regular value for $g$. The above lemma shows that this is well defined. The general position theorem gives that there is always a $g$ homotopic to $f$ so that $y$ is a regular value of $g$. Thus $\deg(f, \cdot)$ is globally defined.

**Proposition 5.** *If $M^m$ and $N^m$ are connected, then $\deg(f, y)$ is independent of $y \in N$.*

*Proof.* We will show that $\deg(f, y)$ is a locally constant function of $y$. Without loss of generality $y$ is a regular value of $f$ (Otherwise homotope $f$). Pick a neighbourhood $U$ of $y$ so that $f^{-1}(U)$ consists of finitely many disjoint open sets $V_1, \cdots, V_k$ and $f|_{V_i}$ is a diffeomorphism for each $i$. Then $\text{sign}(df)_x$ is constant on $V_i$, and we get that $\deg(f, x) = \sum_{i=1}^{k} \text{sign}((df)|_{V_i})$ for any $x \in U$. Thus $\deg(f, \cdot)$ is locally constant, and hence constant on the connected manifold $M$. $\square$

Given this proposition, we talk about the degree of a map and write $\deg(f)$. This degree can be shown to be equivalent to the degree defined in terms of volume forms and de Rham cohomology (see [3] or [1]).

## 2. Oriented Intersection Number

Suppose that $f : M \to N$ is transverse to $P$, a closed sub-manifold of $N$, and that $\dim P + \dim M = \dim N$. Then $f^{-1}(P)$ is a compact manifold of dimension 0, and is hence a finite collection of points. For any point $x \in f^{-1}(P)$, $T_{f(x)}P + (df)_x T_x M = T_{f(x)}M$. We define the oriented intersection number $I(f, P)_x$ to be 1 if a positively oriented a basis of $T_{f(x)}P$ plus a positively oriented basis for $(df)_x T_x M$ give an oriented basis for $T_x M$ and -1 otherwise. The oriented intersection number of $f$ and $P$ is defined to be

$$I(f, P) = \sum_{x \in f^{-1}(P)} I(f, P)_x.$$

This is simply a sort of signed sum of the number of points of intersection of $f(M)$ and $P$. When $f$ is an embedding, we often simply write $I(M, P)$. We will not need the following proposition, but we prove it to develop intuition for oriented intersection number.

**Proposition 6.** $I(M^m, P^p) = (-1)^{mp} I(P^p, M^m)$.

*Proof.* We must compare the direct sum orientations of $T_x(M) \oplus T_x(P)$ and $T_x(P) \oplus T_x(M)$. Write down bases, and observe that to convert one to the other requires $mp$ transpositions. $\square$

**Proposition 7.** *If $f, g : M \to N$ are both transverse to $P$ and are homotopic, then $I(f, P) = I(g, P)$.*

*Proof.* This proof is exactly the same as the proof that degree is homotopy invariant. $\square$

It follows, as in the discussion of degree, that $I(M, P)$ can always be defined even when $M$ and $P$ are not transverse: we just homotope $M$ a bit before taking its intersection number with $P$.

## 3. Vector Fields, The Index of a Zero

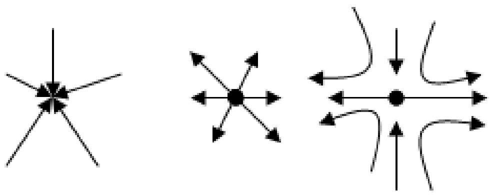
A vector field on a manifold $M^m$ is a map $V : M \to TM$ such that $\pi(V(x)) = x$ for all $x \in M$, where $\pi : TM \to M$ is the canonical projection. In fact, $V$ is easily seen to be an embedding, so if we define $\sigma(V) = \text{range}(V)$, then $\sigma(V)$ is a sub-manifold of $M$. In particular, the zero vector field has $\sigma(0) = M$, and in general, the intersections of $M$ with $\sigma(V)$ correspond to the zeros of $V$.

The map $V$ is then a diffeomorphism from $M$ to $\sigma(V)$, thus $\sigma(V)$ inherits an orientation from $V$. We very often abuse notation, and think of $V$ in some local coordinates as a map from $M$ to $\mathbb{R}^m$. When we do this, if $\{(v_1), \cdots, (v_m)\}$ is an oriented basis for $T_xM$, then $\{(v_1, 0), \cdots, (v_m, 0)\}$ is an oriented basis for $T_{(x,0)}M \subset T_{(x,0)}TM$, and $\{(v_1, (dV)_x v_1), \cdots, (v_m, (dV)_x v_m)\}$ is an oriented basis for $T_{(x,V(x))}\sigma(V)$.

Suppose that $x$ is an isolated zero of $V$, and fix some local coordinates near $x$. Pick a closed disk $D$ centered at $x$, so that $x$ is the only zero of $V$ in $D$. Then we define the index of $x$ for $V$, $\text{Ind}_x(V)$, to be the degree of the map

$$u : \partial D \to S^{m-1}, u(z) = \frac{V(z)}{|V(z)|}.$$

The reader is urged to try to verify the following indices.



index 1 index 1 index -1

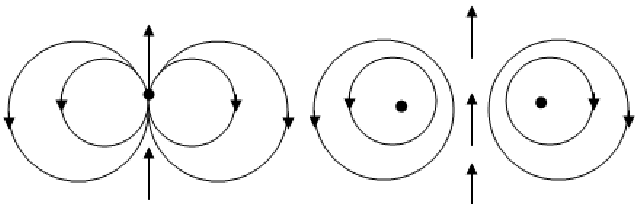
**Proposition 8.** $\text{Ind}_x(V)$ *does not depend on the choice of* $D$ *or the choice of local coordinates.*

*Proof.* If we pick another disk $D'$ centered at $x$, we can find a disk $D_0$ contained in both $D$ and $D'$. Then the Extension Lemma gives that the degrees of $u : D \to S^{m-1}$ and $u : D_0 \to S^{m-1}$ are the same, and also that the degrees of $u : D' \to S^{m-1}$ and $u : D_0 \to S^{m-1}$ are the same. Thus index does not depend on the choice of $D$. Index does not depend on the choice of local coordinates either, but we will omit the proof of that fact. The proof involves proving that an orientation preserving diffeomorphism of the disk is homotopic to the identity. See Burns and Gidea ([1], p.293) for a detailed proof. $\square$

**Lemma 9.** *If $D$ is a disk containing zeros $x_1, \cdots, x_k$ of $V$, then the degree of $V(x)/|V(x)|$ on $\partial D$ is equal to the sum of the indices of $V$ at the $x_i$.*

*Proof.* Put small spheres around each zero, note that the vector field is non-zero on the region in between the small spheres and $\partial D$, and apply the Extension Lemma. □

A zero $x$ of $V$ is called non-degenerate if $(dV)_x : T_xM \to T_xM$ is non-singular. If $V$ is a vector field with finitely many zeros, we can modify $V$ near each of its zeros, to get a new vector field $V'$ having the same sum of indices, still with finitely many zeros, but having only non degenerate zeros. The following lemma makes this precise, and will allow us to assume that our vector fields have only non-degenerate zeros whenever they have only finitely many zeros.



**Lemma 10.** *Suppose that $x$ is a zero of $V$ and $U$ is a small neighborhood of $x$ in $M$ containing no other zero of $V$. Let $D$ be a closed disk in $U$, whose interior contains $x$. There exists a vector field $V_1$ that equals $V$ outside of $D$, with only finitely many zeros in $D$, all of which are non-degenerate. Any such $V_1$ has $\sum_{x\in V^{-1}(0)} \operatorname{Ind}_x(V) = \sum_{V_1(x)=0} \operatorname{Ind}_x(V_1)$.*

*Proof.* We assume that $U$ is contained in some chart, so we can work in local coordinates. Pick a bump function $\rho$ that is 1 near $x$ and 0 outsize of $D$, and define

$$V_1(z) = V(z) + \rho(z)a,$$

where $a \in \mathbb{R}^k$. If $a$ is small enough, $V_1$ can be zero on where $p = 1$. Now, if $-a$ is a regular value of $V$, we get that all the zeros of $V_1$ will be non-degenerate, and if $a$ is also small enough, $V_1$ will still have finitely many zeros. Since $V$ and $V_1$ both restrict to the same function on $\partial D$, the above lemma gives

$$\begin{aligned} \sum_{x\in V^{-1}(0)\cap D} \operatorname{Ind}_x V &= \deg(V|_{\partial D}/|V|\partial D|) \\ &= \deg((V_1)_{\partial D}/|(V_1)_{\partial D}|) \\ &= \sum_{x\in V_1^{-1}(0)\cap D} \operatorname{Ind}_x V \end{aligned}$$

□

**Corollary 11** (The Splitting Lemma)**.** *If $V$ is a vector field with only finitely many zeros, there exists a vector field $W$ with only finitely many zeros, all of which are non-degenerate, so*

$$\sum_{x\in V^{-1}(0)} \operatorname{Ind}_x V = \sum_{x\in W^{-1}(0)} \operatorname{Ind}_x W.$$

The following two lemmas, combined with the homotopy invariance of oriented intersection number, will give the astounding fact that the sum of the indices of a vector field with only finitely many zeros does not actually depend on the vector field.

**Lemma 12.** *If $x$ is a non-degenerate zero of $V$, then*

$$\operatorname{Ind}_x V = \operatorname{sign}((dV)_x).$$

*We conclude that if $V$ has only finitely many zeros, all of which are non-degenerate, then*

$$\sum_{x\in V^{-1}(0)} \operatorname{Ind}_x V = \sum_{x\in V^{-1}(0)} \operatorname{sign}((dV)_x).$$

*Proof.* In local coordinates, $V$ is just a map from $R^n$ to $R^n$, and we see that $x$ is non-degenerate if and only if $V$, viewed in this fashion, is a local diffeomorphism. Furthermore, we can see that $\operatorname{Ind}_x V$ is 1 if $V$ preserves orientation, and -1 if it reverses orientation. To see this, pick a basis for $T_yS^{m-1}$, where $S^{m-1}$ is a small sphere around $X$. If we supplement this basis by a vector pointing outward from $S^{m-1}$, we get a basis for $T_yM$. $V$ is a local diffeomorphism, and $V$ will send this outward pointing vector to another vector pointing outward. Since outward facing vectors can be used to determine if a basis of $T_yS^{m-1}$ is oriented, $V$ preserves the orientation of $S^{m-1}$ if and only if it preserves orientation of the ambient space. □

**Lemma 13.** *If $V$ has only finitely many zeros, all of which are non-degenerate, then $\sigma(V)$ is transverse to $M$ in $TM$ and*

$$I(M, \sigma(V)) = \sum_{x\in V^{-1}(0)} \operatorname{sign}((dV)_x).$$

*Proof.* Say $x \in M$, and that $(x, 0)$ is a point of intersection of $M$ and $V$. Let $\{v_1, \cdots, v_m\}$ be an oriented basis for $T_xM$. So

$$\{(v_1, 0), \cdots, (v_m, 0)\}$$

is an oriented basis for $T_(0,0)xM$, the tangent space to $M \subset TM$. We also have, by the definition of our natural orientation of $V$, that

$$\{(v_1, (dV)_x v_1), \cdots, (v_m, (dV)_x v_m)\}$$

is an oriented basis for $V$. We need to prove that

$$\{(v_1, 0), \cdots, (v_m, 0), (v_1, (dV)_x v_1), \cdots, (v_n, (dV)_x v_m)\}$$

is a basis for $T_{(x,0)}(TM)$ (to show that the intersection is transverse), and decide when it is oriented. This is a basis (and is oriented) if and only if

$$\{(v_1, 0), \cdots, (v_n, 0), (0, (dV)_x v_1), \cdots, (0, (dV)_x v_n)\}$$

is a basis (and is oriented), because, just as subtracting one row from another doesn't change the determinant of a matrix, subtracting one vector from another doesn't change span (or orientation if the vectors form a basis). Since $(dV)_x$ is non singular, we get that this is a basis, and $M$ and $\sigma(M)$ are transverse. The orientation on $T_{(x,0)}(TM)$ is such that the above basis is oriented if and only

$$\{(dV)_x v_1, \cdots, (dV)_x v_m\}$$

is oriented, that is to say, if $\operatorname{sign}(dV)_x = 1$. So

$$I(M, V) = \sum_{x \in M \cap V} I(M, V)_x = \sum_{x \in V^{-1}(0)} \operatorname{sign}(dV)_x.$$

□

**Theorem 14.** $\sum_{x \in V^{-1}(0)} \operatorname{Ind}_x V$ *does not depend on the vector field* $V$*, as long as* $V$ *has only finitely many zeros.*

*Proof.* The Splitting Lemma allows us to assume that $V$'s zeros are all degenerate. So

$$\sum_{x \in V^{-1}(0)} \operatorname{Ind}_x V = \sum_{x \in V^{-1}(0)} \operatorname{sign}((dV)_x) = I(M, V).$$

But $V$ is homotopic to $M$, through the homotopy $V_t(x) = tV(x)$. (Recall that the zero vector field is equal to $M$, as a sub-manifold of $TM$.) But intersection number is homotopy invariant, so $I(M, V) = I(M, M)$. This does not depend on $V$! □

## 4. Poincaré-Hopf Theorem

In the next section, we will construct a vector field on $M$ whose sum of indices will be $\chi(M)$, the Euler characteristic of $M$. This, combined with our previous work, will then easily give the celebrated Poincaré-Hopf Theorem.

**Theorem 15** (Poincaré-Hopf Theorem)**.** *If $V$ is vector field on $M$ with only finitely many zeros, then $\sum_{x\in V^{-1}(0)} \operatorname{Ind}_x V = \chi(M)$.*

*Proof of the Poincaré-Hopf Theorem.* The sum of indices does not depend on the vector field, and we will construct a nice vector field $V_T$ whose sum of indices is equal to the $\chi(M)$, for some triangulation $T$ of $M$. So if $V$ is as above,

$$\sum_{x\in V^{-1}(0)} \operatorname{Ind}_x V = \sum_{x\in V_T^{-1}(0)} \operatorname{Ind}_x V_T = \chi(V).$$

$\square$

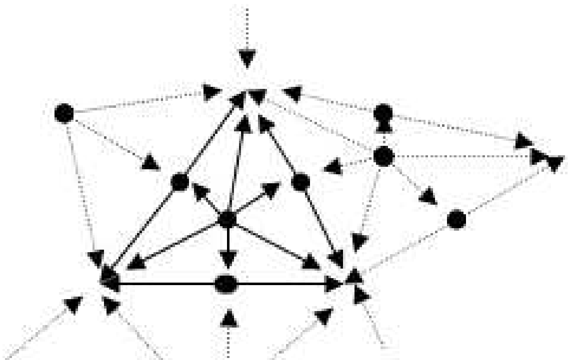
**Corollary 16.** *$\chi(M)$ does not depend on the choice of triangulation $T$ of $M$.*

## 5. Euler Characteristic

Consider a triangulation $T.$ on a compact surface $S.$. We define the Euler characteristic $\chi(M).$ as the number of vertices plus the number of faces minus the number of edges of the triangulation. It is a classical result that the Euler characteristic is a topological invariant, and by that we mean that its value in independent of which triangulation is used. In this way we think of the Euler characteristic to be a function of the surface $\chi(S).$ rather than just a particular triangulation.

We want to generalize the idea of Euler characteristic to manifolds of arbitrary dimension, from the dimension 2 case which corresponds to surfaces. Note that vertices are 0-dimensional simplices, edges are 1-dimensional simplices, and faces are 2-dimensional simplices. Let $s_i.$ be the number of $i$-dimensional simplices of a triangulation $T.$ of a surface $S.$, for $i = 0, 1, 2.$. Then $\chi(S) = \chi(T) = s_0 - s_1 + s_2.$. We generalize this to a compact manifold $M.$ of dimension $m.$ by $\chi(M) = \sum_{i=0}^{m} (01)^i s_i$, where the $s_i.$s are the number of $i.$-dimensional simplices of a given triangulation of $M$. It is a well known, but difficult to prove fact that every differentiable manifold has a triangulation, which may be chosen to be finite it the manifold is compact.

**Example 17** (A vector field which computes the Euler characteristic)**.**



Let $T$ be a triangulation on a compact manifold $M^m$. We can create a vector field $V_T \in TM$ with zeros in the centre of each simplex of $T$, in a way so that the flow lines of the vector field point from the centres of higher dimensional simplices towards the lower dimensional simplices. See the two dimensional example above.

In particular, each vertex of the triangulation is a zero of the vector field, and this zero acts as a sink, so it has an index of 1. At the centre of every edge is a zero where most of the field lines point towards it, except for 1 dimension of field lines pointing outwards (towards the vertices). These outward pointing lines correspond to a negative eigenvalue of the vector field's derivative $dV_T$. Since the index of the zero is equal to the sign of this derivative, we have that the index is equal to $-1$ for this zero. Similarly, the zero at the centre of an $i$-dimensional simplex will have exactly $i$ negative eigenvalues, so its index will be $(-1)^i$. Summing up the indices of all of the zeros gives

$$\begin{aligned}\sum_{x \in V_T^{-1}(0)} \operatorname{Ind}_x V_T &= \sum_{i=0}^{m} \sum_{\{i\text{-dimensional simplices S}\}} \operatorname{Ind}_{c_S} V_T \\ &= \sum_{i=0}^{m} s_i(-1)^i = \chi(M)\end{aligned}$$

where $c_S$ is the centre point of the simplex $S$.

## 6. Conclusion

The Poincaré-Hopf Theorem is a very useful tool for computing Euler characteristics. For example, if $M$ is a parallelizable manifold (that is, a manifold with trivial tangent bundle), then $M$ has lots of nowhere vanishing vector fields, and thus $M$'s Euler characteristic is 0. In particular, the Euler characteristic of any Lie group is 0, since Lie groups are parallelizable!

The Hopf Degree Theorem states that two maps of a compact, connected, oriented manifold $M^m$ into $S^k$ are homotopic if and only if they have the same degree. This can be used to prove the following surprising result.

**Theorem 18.** *A compact, oriented manifold $M$ possesses a no-where vanishing vector field if and only if its Euler characteristic is zero.*

*Proof.* The idea of the proof is to start off with a vector field with only finitely many zeros, and use an isotopy to move all the zeros to single coordinate chart. Then the Hopf Degree theorem and a converse to the Extension Lemma is used to homotope the vector field to one without zeros, while leaving it constant outside the coordinate chart. See Guillemin and Pollack ([2], p.144) for details, as well as the outline of proofs of the Hopf degree theorem and the converse to the Extension Lemma. □

Since every manifold $M^{2k+1}$ of odd dimension has zero Euler characteristic ($\chi(E) = I(M, M) = (-1)^{2k+1} I(M, M) = -\chi(E)$), this proves that every odd dimensional manifold has a no-where vanishing vector field. In fact, this theorem represents a complete answer to the question: When does a manifold posses a no-where vanishing vector field?

## References

[1] K. Burns and M. Gidea, *Differential Geometry and Topology With a View to Dynamical Systems*. Chapman and Hall/CRC, 2005.
[2] V. Guillemin and A. Pollack, *Differential Topology*. Prentice-Hall, 1974.
[3] R. Bott and L. Tu, *Differential Forms in Algebraic Topology*. Springer, New York, 1982.