Thm (Apisa-W) Let M be a $GL(2,\mathbb{R})$ orbit closure with

$$rk(M) \geq \frac{g}{2} + 1 \quad \leftarrow \text{high rank}$$

Then M is a component of a stratum or a **quadratic double**

- $1 \leq rk \leq g \quad \leftarrow$
- $2 \cdot rk \leq \dim$.

Lemma If M has high rank, it is not a locus of translation covers.

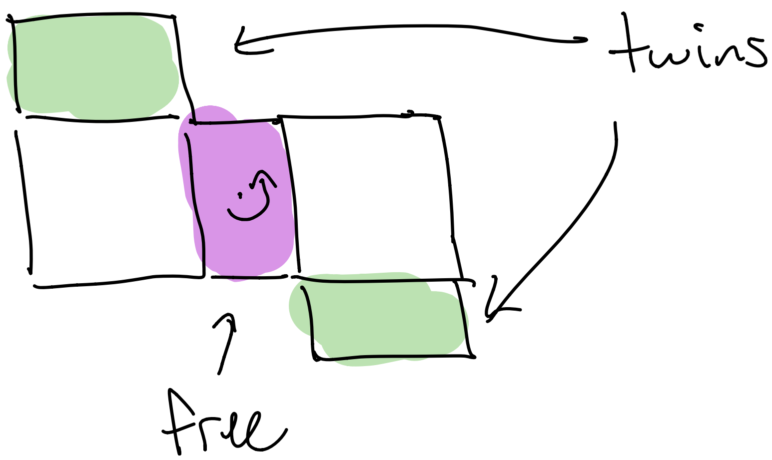
Pf Suppose M is a locus of covers of genus h.
A basic fact gives $rk(M) \leq h$.

Riemann-Hurwitz:

$$2-2g \leq 4-4h$$

$\iff h \le \frac{g}{2} + \frac{1}{2}$ $\square$

Recall M is <u>geminal</u> if every cylinder on every surface in M is <u>free</u> or has a <u>twin</u>



<u>Thm</u> (Apisa-W) A high rank geminal orbit is either a component of a stratum or a quad double

<u>Proof</u> Without high rank, there is a 3rd possiblity. Ruled out by Lemma. $\square$

Suffices to show:

* <u>Thm A</u> Every high rank orbit

closure is geminal.

<u>Aside</u>: In this talk, I'll pretend marked pts never appear.

<u>Thm</u> (Apisa-W) Consider a $GL(2,\mathbb{R})$ equivariant pt marking over a stratum $\mathcal{M}$ of $rk > \underline{1}$

Then

- $\mathcal{M}$ is <u>not</u> hyperelliptic. Then all marked pts are "free".

or

- $\mathcal{M}$ <u>is</u> hyperelliptic. Then all marked pts are free, or fixed, or appear in exchanged pairs

<u>Prop B</u>. Suppose $\mathcal{M}$ is high rank, not geminal, and $\underline{\dim(\mathcal{M}) > 2 \cdot \operatorname{rank}(\mathcal{M})}$. Then the boundary of $\mathcal{M}$ contains an orbit closure $\mathcal{M}'$ that

(1) consists of connected surfaces

(2) has $rk(M') = rk(M)$

(3) not geminal.

<u>**Prop C**</u> Suppose M has high rk, $rk \geq 3$, not geminal, $\dim(M) = 2\,\mathrm{rank}(M)$.
Then the boundary of M contains an orbit closure M' that

(1) consists of connected surface of genus $\leq g-2$.

(2) has $rk(M') = rk(M) - 1$

(3) not geminal.

<u>**Proof of Thm A assuming Props B & C**</u>

Let M be a counterexample of minimal dim.

$$\boxed{rk \geq \frac{g}{2} + 1}$$

Notes (1) high rk $\Rightarrow rk > 1$

(2) high rk & $rk = 2$
$\Rightarrow g = 2$

g	$\lceil \frac{g}{2} + 1 \rceil$
1	
2	2
3	[illegible]

Thm A is true in genus 2.
by McMullen, or newer arguments

So $\mathrm{rh}(M) \geq 3$.

Let M' be given by Props.

- If $\dim(M') > 2\,\mathrm{rh}(M)$
  $\Rightarrow \mathrm{rh}(M') = \mathrm{rh}(M)$
  $\mathrm{genus}(M') \leq \mathrm{genus}(M)$
  $\Rightarrow M'$ is high rank
- If $\dim(M') = 2\,\mathrm{rh}(M)$
  $\Rightarrow \mathrm{rh}(M') = \mathrm{rh}(M) - 1$
  $\mathrm{genus}(M') \leq \mathrm{genus}(M) - 2$

  $\Rightarrow M'$ is high rh.

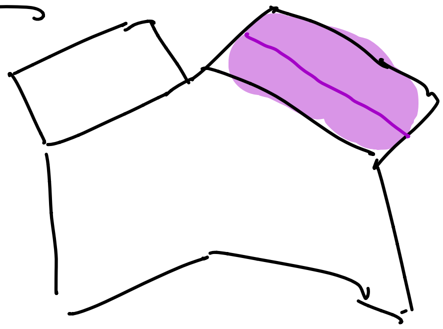
Contradicts minimality. $\square$

Easiest way to produce M' is to degenerate a cylinder.

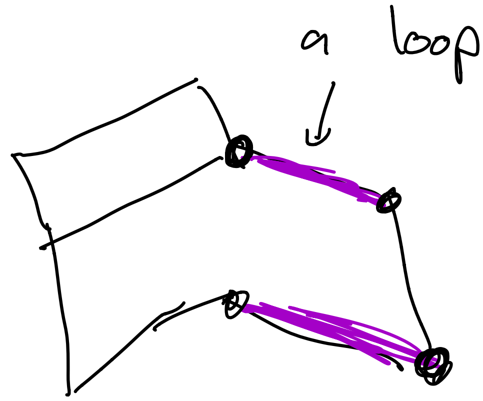
5	3
4	3

$\mathrm{rh} \geq \frac{g}{2} + 1$

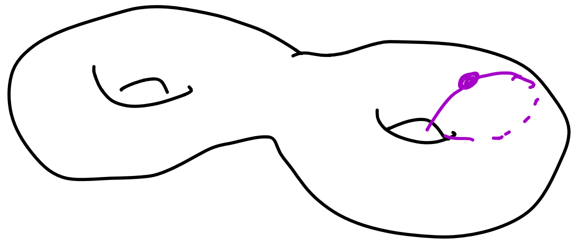
Ex 1



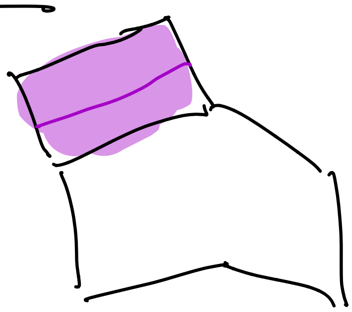
$\mathcal{H}(1,1)$



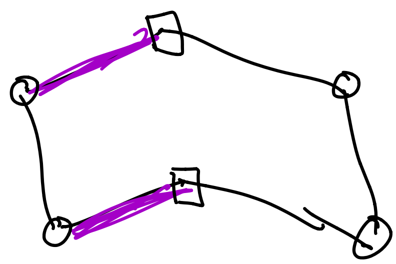
$\mathcal{H}(2)$



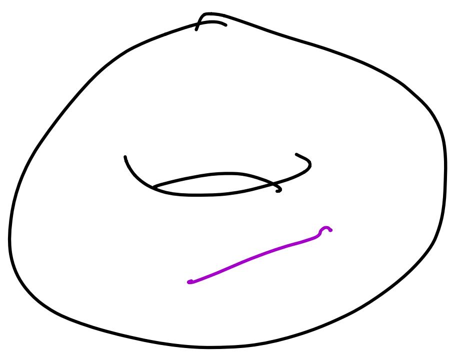
Ex 2



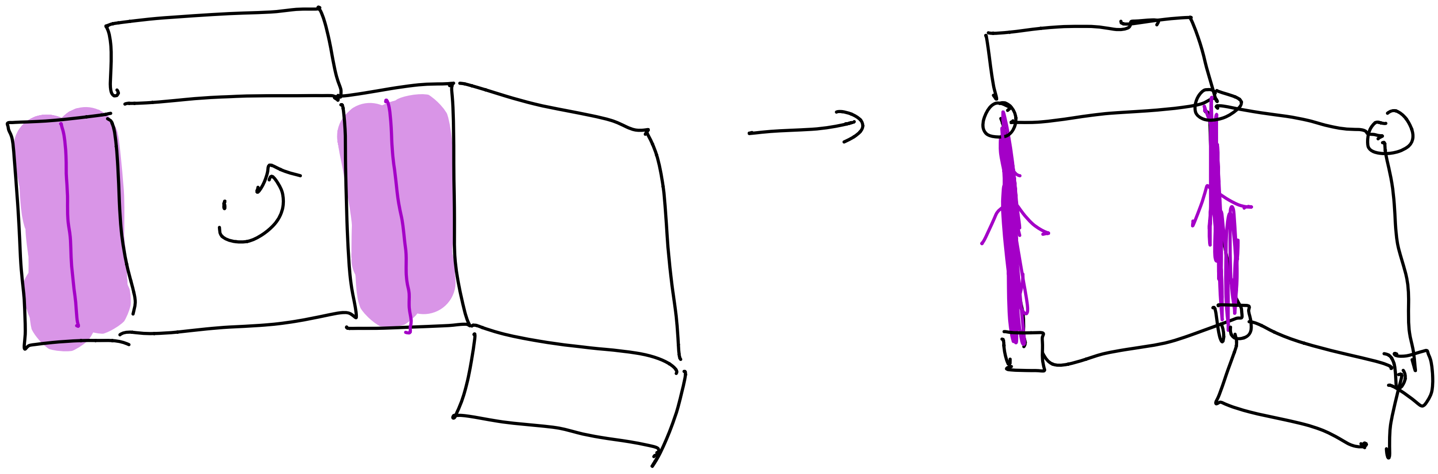
$\mathcal{H}(2)$



$\mathcal{H}(0,0)$

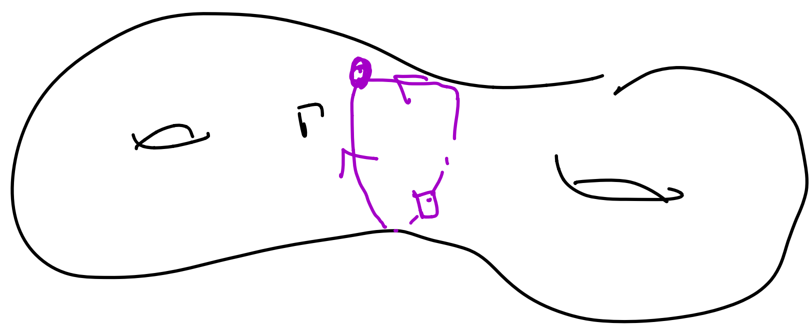


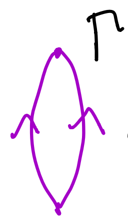
Ex 3



$\mathcal{H}(1,1)$

Recall A directed graph is acyclic if it has no directed loops. It is strongly connected if every edge is part of a directed loop.





← acyclic.

Cylinder Degeneration Dichotomy

(Apisa-Mirzakhani-W)

Suppose the result of doing a cylinder degen. to a surface in M gives a graph Γ of saddle connections on a surface in $\mathcal{M}' \subset \partial \mathcal{M}$. Then either

- $\mathrm{rk}(\mathcal{M}') = \mathrm{rk}(\mathcal{M})$ and Γ is strongly connected
- $\mathrm{rk}(\mathcal{M}') < \mathrm{rk}(\mathcal{M})$ and Γ is acyclic

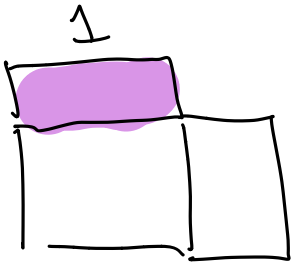
  Moreover, Γ is "$\mathcal{M}'$-rel-scalable"

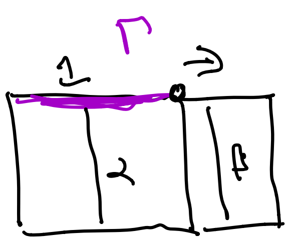
<u>Def</u> Γ is $\mathcal{M}'$-rel-scalable if there is a purely relative $v \in T(\mathcal{M}')$ whose evaluation on each edge gives the holonomy

<u>Recall</u> $T(\mathcal{M}') \subset H^1(X, \Sigma) = H_1(X, \Sigma)^*$

pure rel means $v(\gamma) = 0 \quad \forall \gamma \in H_1(X) \subset H_1(X, \Sigma)$

<u>Ex</u>





$\mathcal{H}(0,0)$

$v = \alpha^* - \beta^*$

Poincaré duals

__Lemma__ rel-scalable

① $\Rightarrow$ every loop in Γ has zero hol

② $\Rightarrow$ acyclic

__Proof__

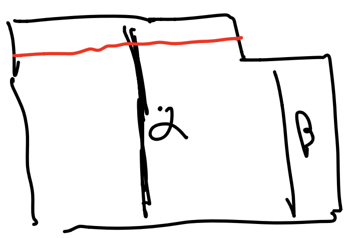
① Def of rel scalable $\Rightarrow$ if γ is a loop in Γ,

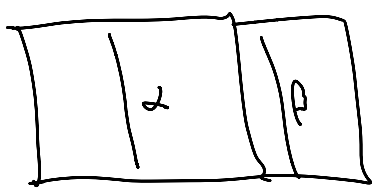
$\mathrm{hol}(\gamma) = v(\gamma)$.

But v rel $\Rightarrow v(\gamma) = 0$.

② A directed loop always goes in same dir. so $\mathrm{hol}(\gamma) \neq 0$. $\square$

In ex above





$v = \alpha^* - \beta^*$ not rel.

$v = \alpha^* - \beta^*$ rel

" "

v becomes pure rel

<u>Rk</u> If $\dim(M) = 2\,\mathrm{rk}(M)$

$$\dim(M') = \dim(M) - 1$$

$$2\,\mathrm{rk}(M') = 2\,\mathrm{rk}(M) - 2$$

$$\dim(M') > 2\,\mathrm{rk}(M')$$

<u>Next time</u>

- new ideas.
- primality, connected,
- double degenerations.

<u>Almost HR Conj</u>

$$\mathrm{rk}(M) = \frac{g}{2} + \frac{t}{2}$$