THE CREMONA GROUP: LECTURE 1

Birational maps of $\mathbb{P}^n$. A birational map from $\mathbb{P}^n$ to $\mathbb{P}^n$ is specified by an $(n+1)$-tuple $(f_0, \ldots, f_n)$ of homogeneous polynomials of the same degree, which can be assumed to not have any common factor. The map

$$[z_0, \ldots, z_n] \mapsto [f_0(z_0, \ldots, z_n), \ldots, f_n(z_0, \ldots, z_n)]$$

is defined on the locus where at least one f_i is nonzero. Since the f_i don't have any common factor, this indeterminacy locus has codimension at least 2.

Alternatively, a birational map from $\mathbb{P}^n$ to $\mathbb{P}^n$ is specified by an n-tuple of of elements of rational functions $F_i \in \mathbb{C}(Z_1, \ldots, Z_n)$. To go from the $(n+1)$-tuple of polynomials to the n-tuple of rational functions, replace $(f_0, ..., f_n)$ with $(f_1/f_0, f_2/f_0, ..., f_n/f_0)$ evaluated at $z_0 = 1$. In the opposite direction, given $(F_1, ..., F_n)$, consider $(1, F_1, ..., F_n)$ evaluated at $Z_i = z_i/z_0$ and then clear denominators.

Most birational maps from $\mathbb{P}^n$ to itself defined by polynomials f_i of degree d have topological degree d^n (the generic point has d^n preimages). However, many have topological degree 1 and are hence invertible. An invertible birational map is called a birational automorphism, and the group of all such is the Cremona group of $\mathbb{P}^n$,

$$\mathrm{Cr}(n) = \mathrm{Aut}(\mathbb{C}(Z_1, \ldots, Z_n)).$$

An element of the Cremona group f such that both f and f^{-1} have empty indeterminacy locus is called biregular.

The Cremona group contains the group of biregular automorphisms $\mathrm{Aut}(\mathbb{P}^n) = PGL(n+1, \mathbb{C})$ (see Hartshorne 7.1.1 for a proof of this equality). When $n = 1$ we have $\mathrm{Cr}(1) = PGL(2, \mathbb{C})$, but when $n = 2$ already the Cremona group is dramatically larger than $PGL(3, \mathbb{C})$. For example, it contains the group $\mathrm{Aut}(\mathbb{C}^2)$ of polynomial automorphisms of $\mathbb{C}^2$, which contains

$$(Z_1, Z_2) \mapsto (Z_1 + p(Z_2), Z_2)$$

for all polynomials p. In particular, $\mathrm{Cr}(2)$ is not finite dimensional.

The standard quadratic involution. An important example is the map σ defined by

$$[z_0, z_1, z_2] \mapsto [1/z_0, 1/z_1, 1/z_2] = [z_1 z_2, z_0 z_2, z_0 z_1].$$

It collapses the lines $z_0 = 0, z_1 = 0, z_2 = 0$ to the points $[1,0,0], [0,1,0], [0,0,1]$ respectively. σ is defined away from the three points $[1,0,0], [0,1,0], [0,0,1]$, but we may compute

$$\begin{aligned} \lim_{t\to 0} \sigma([1, tz_1, tz_2]) &= \lim_{t\to 0} [t^2 z_1 z_2, tz_2, tz_1] \\ &= [0, z_2, z_1]. \end{aligned}$$

These features are representative of general birational maps: they may fail to be injective, and their indeterminacy can be resolved with blow ups. The exceptional set is given by the Jacobian of the determinant; this set is contracted to a smaller dimensional variety, and away from this set the map is a local isomorphism.

Monomial maps. Another example of a birational map of $\mathbb{P}^2$ is given by

$$(Z_1, Z_2) \mapsto (Z_1^2 Z_2, Z_1 Z_2),$$

which has inverse

$$(Z_1, Z_2) \mapsto (Z_1 Z_2^{-1}, Z_1^{-1} Z_2^2).$$

In fact there is a group homomorphism $GL(n, \mathbb{Z}) \to \mathrm{Cr}(n)$. For $n = 2$ this maps a 2 by 2 matrix (a_{ij}) to

$$(Z_1, Z_2) \mapsto (Z_1^{a_{11}} Z_2^{a_{21}}, Z_1^{a_{12}} Z_2^{a_{22}}).$$

This action is obtained from the usual linear action on $\mathbb{C}^n$ by exponentiation. Indeed, if $Z_i = \exp(Q_i)$, then the action on the Q_i is the usual linear action.

One can rephrase this discussion as follows. Set $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, and note $(\mathbb{C}^*)^n \subset \mathbb{P}^n$. The group of biregular automorphisms of $(\mathbb{C}^*)^n$ contains $GL(n, \mathbb{Z})$, and since $(\mathbb{C}^*)^n$ is birational to $\mathbb{P}^n$ this gives a subgroup of $\mathrm{Cr}(n)$.

Monomial automorphisms preserve the locus where $|Z_i| = 1$ for all i. This is an n real dimensional torus. If the monomial map is given by $A \in GL(n, \mathbb{Z})$, then the action on this torus is the usual linear action of A on $\mathbb{R}^n/\mathbb{Z}^n$. If A has an eigenvalue of absolute value greater than 1, the dynamics of this action is very chaotic.

Degree. The degree of the map

$$[z_0, \ldots, z_n] \mapsto [f_0(z_0, \ldots, z_n), \ldots, f_n(z_0, \ldots, z_n)]$$

is given by the common degree of the f_i, as always assuming the f_i have no common factor. Note this algebraic degree is usually bigger than 1, even though the generic point in $\mathbb{P}^2$ will have a unique preimage.

The image under a degree d map of a line $\mathbb{P}^1 \subset \mathbb{P}^n$ will by a curve (an algebraic variety of complex dimension 1). For example, the image of $z_0 = z_1 + z_2$ under the standard quadratic involution

$$[z_0, z_1, z_2] \mapsto [z_1 z_2, z_0 z_2, z_0 z_1]$$

is defined by $(z_1 - z_0)(z_2 - z_0) = z_0^2$.

The degree of this curve, for generic $\mathbb{P}^1 \subset \mathbb{P}^n$, is the degree of the rational map. The fact that a birational automorphism can send lines to curves of arbitrarily high degree is another indication that birational automorphisms can have complicated dynamics. The degree of the square of a map is not always the square of the degree, as is clear for σ since the square is the identity.

Algebraic structure. The set of $(f_0, \ldots, f_n)$ giving birational automorphisms of degree at most d is a quasi-projective variety. The set of birational maps of degree exactly d can be given the structure of an algebraic variety, but Blanc-Furter recently prove that the set of birational maps of degree at most d cannot.

The Cremona group has a Zariski topology, but is not an algebraic variety of infinite dimension.

History. Favre describes the history of the Cremona group in three periods.

- 1860-1920: The Cremona group is studied by Cremona, Noether, De Jonquières, Castelnuovo, Enriques, etc, and is one of the central objects in algebraic geometry. One of the highlights of this period is Noether's theorem that Cr(2) is generated by $PGL(3, \mathbb{C})$ and the standard involution σ. (By contrast, Cr(n) with $n > 2$ cannot be generated by finitely many algebraic families of rational maps. In particular, it cannot be generated by elements of bounded degree.)
- 1930-1990: A presentation for the Cremona group is obtained. A key problem is the classification of finite subgroups. (This has now been accomplished for Cr(2), but it is still unknown if every finite group is a subgroup of Cr(4)!) The study of the iteration of a birational map begins to be studied.
- 1990-present: Understand Cr(2) by understanding its finitely generated subgroups.

Analogy to algebraic groups. The very first question one should ask about $\mathrm{Cr}(n)$ is whether it is an algebraic group. The answer is negative: It is infinite dimensional, and is not even an infinite dimensional variety. However, its algebraic subgroups, such as $PGL(n+1, \mathbb{C})$, are much studied. The maximal dimensional abelian algebraic subgroups are all conjugate to the diagonal matrices $\Delta_n \subset PGL(n+1, \mathbb{C})$.

Δ_n is invariant under conjugation by monomial transformations. Despite the fact that it is not literally true, $\mathrm{Cr}(n)$ might be thought of as a linear algebraic group with maximal torus Δ_n and Weyl group $GL(n, \mathbb{Z})$.

One oddity is that there are abelian algebraic subgroups that are maximal under inclusion but have dimension less than n.

Perhaps finitely generated subgroups of $\mathrm{Cr}(n)$ behave like finitely generated subgroups of linear groups? The most famous restriction on such groups is the Tits Alternative.

Theorem A. *Every finitely generated subgroup of a linear group either contains a non-abelian free subgroup, or has a finite index solvable subgroup.*

Analogy to diffeomorphism groups. It is natural to guess that $\mathrm{Cr}(n)$ might share some properties with the group of all C^∞ diffeomorphisms of $\mathbb{P}^n$. A key property of such groups is the following result of Thurston from the 1970s.

Theorem B. *The group of diffeomorphisms isotopic to the identity of a compact manifold is simple.*

Many of our favorite linear algebraic groups, such as $SL(n, \mathbb{C})$, are also simple groups.

The Zimmer program studies homomorphisms from lattices in higher rank Lie groups (for example $SL(n, \mathbb{Z}), n > 2$) into diffeomorphism groups. This is also an active area of study when the diffeomorphism groups are replaced by Cremona groups.

Analogy to mapping class groups. The mapping class group can be defined as the group $\mathrm{Out}(\pi_1(\Sigma_g))$ of outer automorphisms of the fundamental group of a closed surface. For any finitely generated group G, there is a short exact sequence

$$1 \to \mathrm{Inn}(G) \to \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1.$$

If G has trivial center, we have $\mathrm{Inn}(G) = G$. Let $X(G, n)$ denote the character variety of homomorphisms $G \to GL(n, \mathbb{C})$ up to conjugation by $GL(n, \mathbb{C})$. The group $\mathrm{Out}(G)$ acts on the affine variety $X(G, n)$ by automorphisms.

This setup can be used to show that every mapping class group is a subgroup of the group of birational transformations of some variety.

The mapping class group can also be defined as the group of isotopy classes of surface diffeomorphisms, and one of the key results is the Nielsen-Thurston classification.

Theorem C. *Every element of the mapping class group is either pseudo-Anosov, reducible, or finite order.*

The generic mapping class is pseudo-Anosov, and iteration of pseudo-Anosovs distorts geometric objects like loops on the surface exponentially quickly. The other cases are characterized by preserving some geometric structure, and iteration causes less distortion.

Mapping class groups satisfy the Tits Alternative.

Overview of results. To what extent does $\mathrm{Cr}(n)$ behave similarly to the above related classes of groups? Surprisingly, *much* more is true of $\mathrm{Cr}(2)$ than $\mathrm{Cr}(n)$ with $n > 2$.

A recent Annals paper of Cantat gives

Theorem A'. *Every finitely generated subgroup of* $\mathrm{Cr}(2)$ *either contains a non-abelian free subgroup, or has a finite index solvable subgroup.*

Blanc and Zimmerman show

Theorem B'. $\mathrm{Cr}(n)$ *has no closed normal subgroups.*

However, a recent Acta paper of Cantat and Lamy shows

Theorem B''. $\mathrm{Cr}(2)$ *has many normal subgroups.*

The Nielsen-Thurston classification is related to the classification of isometries of hyperbolic space. Each such isometry is either loxodromic, parabolic, or elliptic. These correspond to a mapping class being pseudo-Anosov, reducible, or finite order. One can define loxodromic, parabolic, or elliptic for elements of $\mathrm{Cr}(2)$, and show

Theorem C'. *Every element of* $\mathrm{Cr}(2)$ *is either loxodromic, parabolic, or elliptic.*

Most interesting is that the these cases can be characterized by the degree growth of iterates, a measurement of how quickly the map distorts geometric objects. Elliptic implies finite order, and loxodromic implies that a geometric structure (a fibration) is preserved.

Some of these theorems are proven using an action of $\mathrm{Cr}(2)$ on an infinite dimensional hyperbolic space, which is somewhat analogous to the action of the mapping class group on Teichmüller space. Hyperbolic spaces arise from real vector spaces with non-degenerate bilinear forms of signature $(1, n-1)$. In the Hodge decomposition, the $(1,1)$ part of the cohomology of a complex surface always has a bilinear form (cup product) that is of degree $(1, n-1)$.