Ph-507. Midterm Exam, March 10, 2008.

HINT: all three problems may be done with very modest amount of calculation. However, feel free to do the problems the way you like.

PROBLEM 1 ***(5pts)***

A particle of mass m with total energy E, is scattered by central field

$$U(r) = \frac{k}{r^2}.$$

It was found that at certain finite values of impact parameter s there is no apparent scattering. Find those values, and sketch the corresponding trajectories. The potential is *not weak*.

PROBLEM 2 (**5** ***pts.)***

A space station has a circular orbit about certain planet. The magnitude of its linear momentum in the reference frame of the planet is p_0. Find the minimal impulse that the station's engines must generate to change the period of its orbital motion by factor α . Assume the mass of the station to remain constant.

PROBLEM 3 (**5** ***pts.)***

The Lagrangian of certain particle is

$$L = \frac{m\dot{x}^2}{2} - \left[U_0(x) + \delta U(x)\right]$$

Assume that the solution for the unperturbed problem, $\delta U(x) = 0$, is known. Derive the expression for the period $T(E)$, which takes into account all perturbations, up to the n-th order in δU.

Solution to Problem 1

$$\Delta\theta = 2\int\limits_{r_{\min}}^{\infty} \frac{1}{\sqrt{1-(s^2+k/E)/r^2}}\frac{s\mathrm{d}r}{r^2} = 2\int\limits_{r_{\min}}^{\infty} \frac{1}{\sqrt{1-S^2/r^2}}\frac{s\mathrm{d}r}{r^2}$$

Here $s^2+k/E=S^2$. Use the fact that $\theta_m=\pi$ for a free particle ($U=0$):

$$\Delta\theta = \frac{2s}{S}\int\limits_{r_{\min}}^{\infty} \frac{1}{\sqrt{1-S^2/r^2}}\frac{S\mathrm{d}r}{r^2} = \frac{\pi s}{\sqrt{s^2+k/E}}$$

We now have to find s at which $\Delta\theta = \pi + 2\pi N.$:

$$1+\frac{k}{s^2E} = \frac{1}{(1+2N)^2}, \text{ therefore } s = \frac{1}{N+1/2}\sqrt{-\frac{kN(1+N)}{E}}$$

Note that solution only exists for an attractive potential, $k<0$. It corresponds to the trajectory which makes N complete turns about the scatering center, and returns to the original direction of the flight.

Solution to Problem 2

The period is the function of the total energy. Its change by the factor α implies that the energy changes by $\alpha^{-2/3}$:

$$K+U = \alpha^{-2/3}(K_0+U_0)$$

Note that the impulse from engine will change the kinetic but not potential energy ($U=U_0$) , and that according to virial theorem $U_0=-2K_0$.

Therefore,

$$K = K_0\left(2-\alpha^{-2/3}\right)$$

This means that the engine should change the momentum of the station from p_0 to $p_0\sqrt{2-\alpha^{-2/3}}$. The minimal impulse needed to do so is,

$$P = p_0\left|\sqrt{2-\alpha^{-2/3}}-1\right|$$

Solution to Problem 3

$$T = \left(\int\limits_{x_1+\delta x_1}^{x_2+\delta x_2'} \frac{\sqrt{2m}\mathrm{d}x}{\sqrt{E-U(x)-\delta U(x)}}\right) = \frac{\sqrt{2m}}{\prod\limits_{k=1}^{n}(k-1/2)}\frac{\partial^n}{\partial E^n}\int\limits_{x_1}^{x_2}(E-U(x)-\delta U(x))^{n-1/2}\,\mathrm{d}x$$

$$\frac{\sqrt{2m}}{\prod\limits_{k=1}^{n}(k-1/2)}\frac{\partial^n}{\partial E^n}\int\limits_{x_1}^{x_2}\sum_{m=0}^{n}\frac{(-1)^m}{m!}\delta U^m(x)\frac{\partial^m}{\partial E^m}(E-U(x))^{n-1/2}\,\mathrm{d}x = \sum_{m=0}^{n}\frac{(-1)^m}{m!}\frac{\partial^m}{\partial E^m}\int\limits_{x_1}^{x_2}\frac{\sqrt{2m}\delta U^m(x)}{\sqrt{E-U}}\mathrm{d}x$$

$$= \sum_{m=0}^{n}\frac{(-1)^m}{m!}\frac{\partial^m}{\partial E^m}T_0\left\langle \delta U^m\right\rangle$$