Ph-507. Homework 6 (due: Friday, March 28).

PROBLEM 6-1 (3 pts.)

A uniform hollow sphere of radius R can freely rotate in all directions about certain point on its surface. The sphere is in the presence of Earth gravity g. While at the equilibrium position, it is set to spinning about its vertical axis,.with angular velocity ϖ. If you give the sphere a slight kick, the osillations will start and the plane of these oscillations will be rotating (i.e. precessing). Find the precession rate Ω.

PROBLEM 6-2 (5 pts)

Right after a symmetric spinning top (in gravity) has been kicked by a nutty professor, the nutation and precession rates had values $\dot{\theta} = 0$ and $\dot{\phi} = \omega_\phi$, respectively. The orientation of its symmetry axis was given by $\cos\theta = p^*$, and $\phi = 0$. Assuming that the spinning is fast, determine the sets of the initial conditions (ω_ϕ, p^*), for which (a) the direction of precession is constant, (b) there is no net precession. Express the results in terms of the following parameters of the top: frequency of its free oscillations $\varpi_0 = \sqrt{mgl/I_\perp}$, spinning rate $\varpi_\parallel$ and "aspect ratio" $\alpha = I_\parallel/I_\perp$.

PROBLEM 6-3 (4 pts.)

Estimate the minimal speed with which you would have to ride a regular bicycle on the Moon (wearing a light space suit). The "ride" means a steady motion with no need of active balancing (e.g. by moving your body). The gravitational acceleration on the Moon is $g_m \approx 1.6\ m/s^2$

PROBLEM 6-4 (3 pts.)

Find the normal frequencies of a five-atomic linear molecule. Assume all the four bonds to have the same rigidity constant k and all the atoms to have the same mass m. Consider only longitudinal modes (i.e. those along the molecular axis).

Solution to Problem 6-1

Moment of inertia of the hollow sphere with respect to its center of mass, is:

$$I^{(cm)} = \frac{2MR^2}{3}\delta_{ij}.$$

By using the parallel axis theorem, we obtain:

$$I_\perp = MR^2 + \frac{2MR^2}{3} = \frac{5MR^2}{3}, \qquad I_\parallel = \frac{2MR^2}{3}.$$

In terms of $p = \cos\theta$, and polar angle ϕ , the equations of motion of this spinning top are:

$$\dot{\phi}(p) = \frac{I_\parallel \varpi}{I_\perp}\frac{(p_0 - p)}{1-p^2};$$

$$\dot{p}^2 = \frac{2MgR}{I_\perp}\left(1-p^2\right)(p_E - p) - \left(\frac{I_\parallel \varpi}{I_\perp}\right)^2 (p_0 - p)^2.$$

Since initially $p = -1$, $\dot{p}^2$ should be non-negative at that point. This is only possible when $p_0 = -1$. We can now find the precession rate:

$$\Omega = \dot{\phi} = \frac{I_\parallel \varpi}{I_\perp}\frac{(-1-p)}{1-p^2} \approx -\frac{\varpi}{5}.$$

Solution to Problem 6-2.

The equations of motion of the heavy spinning top are:

$$\dot{\phi}(p) = \alpha\varpi_\parallel \frac{(p_0 - p)}{1-p^2}$$

$$\dot{p}^2 = 2\omega_0^2\left(1-p^2\right)(p_E - p) - \alpha^2\varpi_\parallel^2 (p_0 - p)^2$$

When the spinning is fast, the roots of the right hand side (turning points of nutation, $p_2 \geq p_1$), are very close.

a) If $\dot{\phi}$ does not change its sign, p_0 lies outside of the interval (p_1, p_2). The boundary of this regime corresponds to the situation when p_0 coincides with one of the roots, which implies that $p_E = p_0$. Since the initial position, p^* also corresponds to $\dot{p} = 0$, we obtain:

$$\dot{p}^2(p^*) = 2\omega_0^2\left(1-p^{*2}\right)(p_0 - p^*) - \alpha^2\varpi_\parallel^2 (p_0 - p^*)^2 = 0$$

This equation has two solutions: $p_0 = p^*$, and

$$(p_0 - p^*) = \frac{2\omega_0^2}{\alpha^2\varpi_\parallel^2}\left(1-p^{*2}\right)$$

This corresponds to

$$\omega_\phi = \dot{\phi}(p^*) = \alpha\varpi_\parallel \frac{(p_0 - p^*)}{1-p^{*2}} = \frac{2\omega_0^2}{\alpha\varpi_\parallel}, \quad \text{or} \quad 0.$$

Note that $\omega_\phi^{(0)} = mgl/L_\parallel = \omega_0^2/\alpha\varpi_\parallel$ (which corresponds to steady, nutation–free precession), lies between the two obtained solutions. Therefore, the constant direction of precession corresponds to the following interval:

$$0 < \omega_\phi < \frac{2\omega_0^2}{\alpha\varpi_\parallel}.$$

b) For a fast spinning top, the nutation is well described by harmonic oscillations:

$$p = \bar{p} + (p^* - \bar{p}) \cos \alpha \varpi_{\|} t.$$

Here $\bar{p} = (p_1 + p_2)/2$ is the midpoint between the two roots. The average precession is given by:

$$\left\langle \dot{\phi}(p) \right\rangle = \alpha \varpi_{\|} \left\langle \frac{(p_0 - p(t))}{1 - p^2(t)} \right\rangle \simeq \alpha \varpi_{\|} \frac{(p_0 - \bar{p})}{1 - \bar{p}^2}.$$

Therefore, there will be no net precession when $\bar{p} = p_0$. In other words, the two roots, p_1 and p_2, are symmetric with respect to p_0, i.e. p_0 is the maximum of $\dot{p}^2(p)$:

$$\frac{d}{dp}\left(1 - p^2\right)(p_E - p)\Big|_{p_0} = 0 \qquad \Longrightarrow \qquad (p_E - p_0) = -\frac{\left(1 - p_0^2\right)}{2p_0}.$$

In the vicinity of $p = p_0$, $\dot{p}^2(p)$ is a parabola:

$$\dot{p}^2 \simeq -\frac{\omega_0^2\left(1 - p_0^2\right)^2}{p_0} - \alpha^2 \varpi_{\|}^2 (p_0 - p)^2.$$

Now we can use the fact that p^* is the root of the polynomial at the right hand side:

$$p_0 - p^* = \pm \frac{\omega_0}{\alpha \varpi_{\|}} \frac{\left(1 - p_0^2\right)}{\sqrt{-p_0}} \simeq \pm \frac{\omega_0}{\alpha \varpi_{\|}} \frac{\left(1 - p^{*2}\right)}{\sqrt{-p^*}}.$$

Finally, we determine the initial precession rate (p^* must be negative):

$$\omega_\phi = \dot{\phi}(p^*) = \alpha \varpi_{\|} \frac{(p_0 - p^*)}{1 - p^{*2}} = \pm \frac{\omega_0}{\sqrt{-p^*}}.$$

Solution to Problem 6-3.

The wheel rotation stabilize the motion of a bicycle due to gyroscopic effect. In order for this effect to be relevant the corresponding kinetic energy of rotation should be comparable to the gravitational potential energy, i.e:

$$m_w v^2 \sim M g_m h$$

Here m_w and M are masses of the wheel and the rider+bicycle, respectively. h is the height of the Center of mass above the ground. We obtain:

$$v \sim \sqrt{\frac{M}{m_w} g_m h} \sim \sqrt{\frac{100kg}{10kg} 1.6m/s^2 \cdot 1m} \approx 4m/s \approx 10mph$$

We have chosen $M \sim 100kg$, $m_w \sim 10kg$, and $h \approx 1m$. The result for Earth would be $v \sim 10m/s \approx 20mph$, which is reasonable.

Solution to Problem 6-4.

Symmetric modes: $(x, y, 0, -y, -x)$;

$$T = \frac{m}{2}\left(2\dot{x}^2 + 2\dot{y}^2\right)$$

$$U_{sym} = \frac{k}{2}\left[2(x - y)^2 + 2y^2\right]$$

$$\begin{pmatrix} 1-\frac{m}{k}\varpi^2 & -1 \\ -1 & 2-\frac{m}{k}\varpi^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=0$$

$$\left(\frac{m}{k}\varpi^2\right)^2-3\left(\frac{m}{k}\varpi^2\right)+1=0$$

$$\varpi_{1,2}^2=\frac{3k}{2m}\left(1\pm\frac{\sqrt{5}}{3}\right)$$

Anti-symmetric modes: $(x, y, -2(x+y), y, x)$ (upon exclusion of translation of the molecule as a whole).

$$T=\frac{2m}{2}\left(3\dot{x}^2+3\dot{y}^2+4\dot{x}\dot{y}\right)$$

$$U_{anti}=\frac{k}{2}\left[2\left(x-y\right)^2+2\left(2x+3y\right)^2\right]=\frac{10k}{2}\left[x^2+2y^2+2xy\right]$$

$$\begin{pmatrix} 1-3\frac{m}{k}\frac{\varpi^2}{5} & 1-2\frac{m}{k}\frac{\varpi^2}{5} \\ 1-2\frac{m}{k}\frac{\varpi^2}{5} & 2-3\frac{m}{k}\frac{\varpi^2}{5} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=0$$

$$\left(\frac{m}{k}\frac{\varpi^2}{5}\right)^2-\left(\frac{m}{k}\varpi^2\right)+\frac{1}{5}=0$$

$$\varpi_{3,4}^2=\frac{5k}{2m}\left[1\pm\frac{1}{\sqrt{5}}\right]$$