

Ph-507. Homework 4 (due: Friday, February 22).

PROBLEM 4-1 (3 pts)

Episode II: *A long time ago, in a galaxy far, far away...* In 2D space, the gravitational potential energy should have a logarithmic form. In particular, for a system of N particles,

$$U(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N) = \frac{\gamma}{2} \sum_{j \neq k} m_j m_k \log |\mathbf{r}_j - \mathbf{r}_k|.$$

Here the summation is performed over all pairs of different particles ($a, b = 1, \dots, N$), and the double-counting is cancelled by the $1/2$ factor. γ is the "gravitational constant".

a) What is the time-averaged kinetic energy of this system? (Hint: try to do "rescaling" of the coordinates, and follow the logic of virial theorem);

b) 2D astronomer Relpok has found that the period of the radial motion of a planet depends on its minimal and maximal distances from the star (R_{\min}, R_{\max}), according to the following law:

$$T = R_{\max}^\alpha F\left(\frac{R_{\min}}{R_{\max}}\right).$$

What is the scaling exponent α ? How much does F vary when R_{\min}/R_{\max} changes from 1 (circular orbit), to 0.

PROBLEM 4-2 (3 pts)

Consider a particle of mass m scattered by the following attractive potential:

$$U(r) = -\frac{k}{r^\beta},$$

a) Suppose you know the differential cross-section, as a function of deflection angle α : $d\sigma/d\Omega = f_0(\alpha)$, for certain energy E_0 . Apply scaling ideas to find the cross-section for the same particle with an arbitrary energy E .

b) For $\beta \geq 2$, find the cross-section of the process in which particle gets "sucked" by the potential, as a function of the total energy E . Hint: you have to find the range of impact parameters for which the particle will end up at $r = 0$.

PROBLEM 4-3 (2 pts)

An electron with energy E elastically collides with a positron, originally at rest. Determine the differential cross-section, as a function of the electron deflection angle.

PROBLEM 4-4 (3 pts)

The distribution of dust in Solar system is roughly uniform. This results in the following *small* correction to regular gravitational potential energy of a planet of mass m :

$$\delta U(r) = \frac{2\pi}{3} G m \rho r^2$$

Here ρ is the average density of the dust. Find the precession rate of the planet due to this correction. Assume that the period of the orbital motion T , and eccentricity of the orbit ϵ , are known. Hint: use perturbative techniques.

PROBLEM 4-5 (2 pts)

A particle is scattered by the following potential:

$$U(r) = U_0 \exp\left(-\frac{r^2}{2\sigma_0}\right),$$

The total energy of the particle (E) is large compared to U_0 . At which deflection angle a peak in scattering intensity ("rainbow effect") is expected?

Solution to Problem 4-1

a) When we perform "rescaling" of all the spatial coordinates, $\mathbf{r} \rightarrow \lambda \mathbf{r}$, the "gravitational" potential energy acquires an additive term:

$$U(\lambda \mathbf{r}_1, \dots, \lambda \mathbf{r}_N) = U(\mathbf{r}_1, \dots, \mathbf{r}_N) + \frac{\gamma \log \lambda}{2} \sum_{j \neq k} m_j m_k$$

If we choose $\lambda = 1 + \varepsilon$ ($\varepsilon \ll 1$), $\log \lambda \approx \varepsilon$ and:

$$\frac{\gamma \varepsilon}{2} \sum_{j \neq k} m_j m_k = U((1 + \varepsilon) \mathbf{r}_1, \dots, (1 + \varepsilon) \mathbf{r}_N) - U(\mathbf{r}_1, \dots, \mathbf{r}_N) = \varepsilon \sum_a \mathbf{r}_a \cdot \frac{\partial}{\partial \mathbf{r}_a} U(\mathbf{r}_1, \dots, \mathbf{r}_N). \quad (1)$$

Since the total energy is finite, motion in logarithmic potential is always confined. Therefore, the average kinetic energy in 2D system of gravitating particles is **constant**:

$$\langle T \rangle = \frac{1}{2} \left\langle \sum_a \mathbf{r}_a \cdot \frac{\partial}{\partial \mathbf{r}_a} U(\mathbf{r}_1, \dots, \mathbf{r}_n) \right\rangle = \frac{\gamma}{2} \sum_{j \neq k} m_j m_k = \frac{\gamma}{2} \left[\left(\sum_j m_j \right)^2 - \sum_j m_j^2 \right]$$

b) A simultaneous transformations of coordinates $\mathbf{r} \rightarrow \lambda \mathbf{r}$, and time $t \rightarrow \lambda^\alpha t$, will preserve the equations of motion if kinetic and potential energies are transformed in the same way. Since U remains the same, up to an additional constant term, kinetic energy should be invariant as well:

$$T' = \sum_a \frac{m_a}{2} \left(\frac{\lambda}{\lambda^\alpha} \frac{d\mathbf{r}_a}{dt} \right)^2 = \lambda^{2(1-\alpha)} T \implies \alpha = 1.$$

For $R_{\min} = R_{\max} = R$ (circular orbit). Effective potential has minimum at R :

$$\frac{\partial U_{eff}}{\partial R} = \frac{\partial}{\partial R} \left(\frac{L^2}{2mR^2} + \gamma mM \log R \right) = -\frac{L^2}{mR^3} + \frac{\gamma mM}{R} = 0 \implies \frac{L^2}{mR^2} = \gamma mM.$$

Consider an orbit very close yet different from circular one. We can expand the potential energy in terms of $x = r - R$:

$$\delta U = \frac{1}{2} \left(\frac{\partial^2 U_{eff}}{\partial r^2} \right)_{r=R} x^2 = \frac{1}{2} \left(\frac{3L^2}{mR^4} - \frac{\gamma mM}{R^2} \right) x^2 = \frac{\gamma mM}{R^2} x^2$$

The total energy is analogous to that of a linear oscillator, which allows us to find the period:

$$E = \frac{m\dot{x}^2}{2} + \frac{2\gamma mM}{2R^2} x^2 \implies T_1 = \frac{2\pi R}{\sqrt{2\gamma M}}.$$

For $R_{\min} = 0$ $L = 0$, and the period is twice the time of falling from distance $R_{\max} = R$ to the center:

$$T_0 = 2 \int_0^R \frac{dr}{\sqrt{2(E - \gamma M \log r)/m}} = 2 \int_0^R \frac{dr}{\sqrt{2\gamma M \log(R/r)}},$$

after substitution $\log(R/r) = x^2$,

$$T_0 = 2 \int_0^1 \frac{R dx e^{-x^2}}{\sqrt{2\gamma M} x} = \frac{4R}{\sqrt{2\gamma M}} \int_0^\infty e^{-x^2} dx = R \sqrt{\frac{2\pi}{\gamma M}}.$$

Thus,

$$T(R_{\min}, R_{\max}) = R_{\max} \sqrt{\frac{2}{\gamma M}} f\left(\frac{R_{\min}}{R_{\max}}\right) \text{ where } f(1) = \pi, \quad f(0) = \sqrt{\pi}.$$

Solution to Problem 4-2

a) Consider a particle of energy E_0 , and a scaling transformation ($\mathbf{r} \rightarrow \lambda\mathbf{r}$, $t \rightarrow \lambda^\gamma t$) which preserves the shape of its trajectory. Since $U \rightarrow \lambda^{-\beta}U$, both the kinetic and the total energy E will be transformed by the same factor:

$$E = \lambda^{-\beta}E_0 \implies \lambda = \left(\frac{E_0}{E}\right)^{1/\beta}.$$

Upon the rescaling, the impact parameter and the cross-section change as $s \rightarrow \lambda s$, and $d\sigma \rightarrow \lambda^2 d\sigma$, respectively. The deflection angle α is scale-invariant. Therefore,

$$\frac{d\sigma}{d\Omega}(E, \alpha) = \lambda^2 \frac{d\sigma}{d\Omega}(E_0, \alpha) = \left(\frac{E_0}{E}\right)^{2/\beta} f_0(\alpha).$$

b) The effective potential has the following form:

$$U_{eff} = \frac{Es^2}{r^2} - \frac{k}{r^\beta}.$$

In order to be "captured", the particle must be able to reach the center, $r = 0$. Therefore, the total energy must exceed the maximum of U_{eff} . The maximum exists for $\beta > 2$:

$$U_{eff}^{(\max)} = U_{eff} \left[\left(\frac{\beta k}{2Es^2} \right)^{\frac{1}{\beta-2}} \right] = \frac{\beta-2}{\beta} \left[\frac{2}{\beta k} (\sqrt{E}s)^\beta \right]^{\frac{2}{\beta-2}} < E.$$

Thus,,

$$s(E) = \sqrt{\frac{\beta}{\beta-2} \left(\frac{(\beta-2)k}{2E} \right)^{1/\beta}} \implies \sigma = \pi s^2(E) = \frac{\pi\beta}{\beta-2} \left(\frac{(\beta-2)k}{2E} \right)^{2/\beta}.$$

Solution to Problem 4-3

If v is initial speed of the electron, $V_{cm} = v/2$. The speed with respect to CM is also $v/2$, therefore:

$$\cot \alpha_L = \frac{V_{cm} + (v/2) \cos \alpha}{(v/2) \sin \alpha} = \cot \frac{\alpha}{2}$$

For the Coulomb potential,

$$s = \frac{e^2}{2E_{rel}} \cot \frac{\alpha}{2} = \frac{e^2}{E} \cot \alpha_L.$$

Here $E_{rel} = \mu v^2/2 = E/2$. We can now find the scattering cross-section, as

$$\frac{d\sigma}{d\Omega} = \frac{s}{\sin \alpha_L} \left| \frac{ds}{d\alpha_L} \right| = \frac{e^4 \cos \alpha_L}{E^2 \sin^4 \alpha_L}$$

Solution to Problem 4-4

Use the perturbative approach:

$$\delta\theta = \frac{d}{dL} \int_0^T \delta U(r(t)) dt = \frac{2\pi}{3} G\rho \sqrt{-\frac{m}{2E}} \frac{d}{dl} \int_{a(1-\epsilon)}^{a(1+\epsilon)} \frac{2r^3 dr}{\sqrt{\epsilon^2 a^2 - (r-a)^2}}.$$

After making the substitution, $r = a(1 + \epsilon \cos \Psi)$, we obtain:

$$\delta\theta = \frac{2\pi}{3} G\rho \sqrt{-\frac{m}{2E}} \frac{d\epsilon^2}{dl} \frac{d}{d\epsilon^2} \int_0^{2\pi} a^3 (1 + \epsilon \cos \Psi)^3 d\Psi = \pi G\rho T a^2 \frac{d\epsilon^2}{dl}.$$

Since $\epsilon^2 = 1 - l^2/GMa$,

$$\frac{d\epsilon^2}{dl} = \frac{d\epsilon^2}{\sqrt{GMa}d\sqrt{1-\epsilon^2}} = -\frac{2\sqrt{1-\epsilon^2}}{\sqrt{GMa}}.$$

Finally, the precession rate of the orbit is,

$$\Omega = \frac{\delta\theta}{T} = -2\pi\rho a^{3/2}\sqrt{1-\epsilon^2}\sqrt{\frac{G}{M}}.$$

Solution to Problem 4-5

For $U_0 \ll E$, the perturbative approach can be used:

$$\alpha(s) \simeq -\frac{1}{2E} \frac{\partial}{\partial s} \int_{-\infty}^{+\infty} U(\sqrt{x^2+s^2}) dx = -\frac{U_0}{2E} \frac{\partial}{\partial s} \int_{-\infty}^{+\infty} \exp\left(-\frac{x^2+s^2}{2\sigma_0}\right) dx = \frac{\sqrt{\pi}}{(2\sigma_0)^{3/2}} \frac{U_0 s}{E} \exp\left(-\frac{s^2}{2\sigma_0}\right),$$

The rainbow effect is expected when $d\alpha/ds = 0$, i.e. $s^2 = \sigma_0$. The corresponding deflection angle is,

$$\alpha_{\max} = \sqrt{\frac{\pi}{2e}} \frac{U_0}{2E} \approx 0.38 \frac{U_0}{E}.$$