Ph-507. Homework 3 (due: Wednesday, February 13).

PROBLEM 3-1 (5 pts)

a) Two asteroids *of the same mass* collide inelastically and form a single object. At the moment of the collision, they are moving in the same direction. Originally, one of the asteroids had a *circular* orbit with period T_1, while the period of the other was T_2. Find the period T of the orbital motion of the newly formed object.

b) Two asteroids of the same mass are orbiting Sun. They share the same orbit, but move in opposite directions. At some moment, they collide at the perihelion of the orbit, and form a single body. How much time will it take for this new object to fall on the Sun? The original orbit had period T and eccentricity ε.

PROBLEM 3-2 (3 pts)

At $t = 0$, the distance between two objects of masses m and $3m$, is minimal and equal to $r_{\min}$. Find out how this distance changes with time, if the relative velocity of the particles approaches v_0, when $r \to \infty$.

PROBLEM 3-3 (3 pts)

Consider a confined motion of a particle in gravitational field with $1/r^2$ correction:

$$U(r) = \left(-\frac{GM}{r} + \frac{\alpha}{r^2}\right) m,$$

Because of the correction, the principle axes of the elliptical orbit rotate with certain angular velocity Ω (called precession rate). Find Ω.

PROBLEM 3-4 (4 pts)

Episode I (phantom menace): *Long time ago, in a galaxy far, far away...* In a six-dimentional space, where the gravitational potential energy of masses m and M is given by

$$U(r) = -\Gamma \frac{Mm}{r^4},$$

a lonely planet was orbiting a lonely star. Eventually, the planet has left its circular orbit of radius R, and collapsed onto the star. Find the trajectory of the falling planet $r(\theta)$, in polar coordinates.

Solution to Problem 3-1

a) Let $-E_1$ and $-E_2$ be the total energies of the two asteroids. Since the orbit of the first one is circular, one can use the virial theorem to determine its kinetic and potential energies K_1, and U_1:

$$K_1 = E_1; \qquad U_1 = -2E_1$$

At the moment of the collision, potential energies of the two asteroids are the same, $U_2 = U_1$, therefore:

$$K_2 = -E_2 - U_2 = 2E_1 - E_2$$

We can now find the energy change in the collision ($\mu = m/2$ is the reduced mass, and $v_1 - v_2$ is the relative speed):

$$\Delta E = -\frac{\mu\left(v_1 - v_2\right)^2}{2} = \frac{1}{2}\left(\sqrt{K_1} - \sqrt{K_2}\right)^2 = -\frac{1}{2}\left(\sqrt{E_1} - \sqrt{2E_1 - E_2}\right)^2 = -E_1\left(\frac{3}{2} - \frac{1}{2}\frac{E_2}{E_1} - \sqrt{2 - \frac{E_2}{E_1}}\right)$$

Total energy after the collision is,

$$E = -E_1 - E_2 + \Delta E = -\frac{E_1}{2}\left(5 + \frac{E_2}{E_1} - 2\sqrt{2 - \frac{E_2}{E_1}}\right).$$

In Keplers problem energy is related to the period as: $T = k\left(E/m\right)^{-3/2}$($k$ is a constant). Therefore, $E_2/E_1 = \left(T_1/T_2\right)^{2/3}$, and the period of the new object can be written as:

$$T = k\left(\frac{E}{2m}\right)^{-3/2} = 8T_1\left(5 + \left(\frac{T_1}{T_2}\right)^{2/3} - 2\sqrt{2 - \left(\frac{T_1}{T_2}\right)^{2/3}}\right)^{-3/2} =$$

$$= \frac{8T_1T_2}{\left(5T_2^{2/3} + T_1^{2/3} - 2T_2^{1/3}\sqrt{2T_2^{2/3} - T_1^{2/3}}\right)^{3/2}}$$

b)After the asteroids have a head-on collision,, the new object has no kinetic enegy. Its distance from the Sun is

$$r = \left(1 - \varepsilon\right)a$$

We have to find the time needed to fall on the sun.from this point. The straight trajectory can be viewed as a degenerate helix with eccentricity 1. (its apohelion is r and perihelion is 0). Therefore, the major semiaxis of this "ellipse" is $a' = \left(1 - \varepsilon\right)a/2$. The fall time is just a half period which can be obtained from Kepler's law:

$$T_{fall} = T(a')/2 = \left(\frac{a'}{a}\right)^{3/2} T = \left(\frac{1 - \varepsilon}{2}\right)^{3/2} T$$

Solution to Problem 3-2

In the Center of Mass Reference Frame the energy of the system is,

$$E = \frac{\mu\dot{r}^2}{2} + \frac{\mu l^2}{2r^2} - \frac{Gm_1m_2}{r} = const = \frac{\mu v_0^2}{2},$$

where $\mu = m_1m_2/\left(m_1 + m_2\right)$ is the reduced mass. Since $\dot{r} = 0$ when $r = r_{\min}$,

$$\frac{l^2}{r_{\min}^2} = v_0^2 + \frac{8Gm}{r_{\min}} = v_0^2\left(1 + 2\alpha\right).$$

here $\alpha = 4Gm/v_0^2 r_{\min}$.

$$t\left(R\right) = \int\limits_{r_{\min}}^{R} \frac{dr}{\sqrt{\frac{E}{2\mu} + \frac{8Gm}{r} - \frac{l^2}{r^2}}} = \int\limits_{r_{\min}}^{R} \frac{rdr}{v_0\sqrt{r^2 + 2\alpha r r_{\min} - \left(1+2\alpha\right) r_{\min}^2}} = \int\limits_{r_{\min}}^{R} \frac{rdr}{v_0\sqrt{\left(r + \alpha r_{\min}\right)^2 - \left(1+\alpha\right)^2 r_{\min}^2}}$$

After substitution: $r + \alpha r_{\min} = r_{\min}\left(1+\alpha\right)\cosh\Psi$, the problem is solved:

$$t\left(\Psi\right) = \frac{r_{\min}}{v_0}\left[\left(1+\alpha\right)\sinh\Psi - \alpha\Psi\right] = \frac{1}{v_0}\left[\left(r_{\min} + \frac{4Gm}{v_0^2}\right)\sinh\Psi - \frac{4Gm}{v_0^2}\Psi\right],$$

$$R\left(\Psi\right) = \left(r_{\min} + \frac{4Gm}{v_0^2}\right)\cosh\Psi - \frac{4Gm}{v_0^2}.$$

Solution to Problem 3-3

$$U_{eff}\left(r\right) = \left(\frac{l^2 + 2\alpha}{2r^2} - \frac{GM}{r}\right)m = \left(\frac{\tilde{l}^2}{2r^2} - \frac{GM}{r}\right)m$$

where $\tilde{l} = \sqrt{l^2 + 2\alpha}$. We can now use the fact that U_{eff} is formally equivalent to the one in unperturbed Kepler-problem. The period of radial motion is known to be independent of $\tilde{l}$, $T = 2\pi a^{2/3}/\sqrt{GM}$. If the reduced angular momentum l were equal to $\tilde{l}$, the change in polar angle over the period (θ_T) would be 2π. Since physical l is different from $\tilde{l}$, we obtain

$$\theta_T = 2\int\limits_{r_{\min}}^{r_{\max}} \frac{l}{r^2}\frac{dr}{\dot{r}} = 2\frac{l}{\tilde{l}}\int\limits_{r_{\min}}^{r_{\max}} \frac{\tilde{l}dr}{r^2\dot{r}} = 2\pi\frac{l}{\tilde{l}}$$

The precession rate is:

$$\Omega = \frac{\theta_T - 2\pi}{T} = \frac{2\pi}{T}\left[\left(1 + \frac{2\alpha}{l^2}\right)^{-1/2} - 1\right] = \frac{\sqrt{GM}}{a^{2/3}}\left[\left(1 + \frac{2\alpha}{GMa\left(1-\varepsilon^2\right)}\right)^{-1/2} - 1\right]$$

Solution to Problem 3-4

$$U_{eff}\left(r\right) = \frac{m}{2}\left(\frac{l^2}{r^2} - \frac{2\Gamma M}{r^4}\right)$$

Steady (yet unstable) circular motion is only possible if $\partial U_{eff}\left(r\right)/\partial r = 0$, i.e.

$$R^2 = \frac{4\Gamma M}{l^2}$$

Since $E = U_{eff}\left(R\right) = ml^2/4R^2$,

$$\dot{r}^2 = \frac{2}{m}\left(E - U_{eff}\left(r\right)\right) = l^2\left(\frac{1}{2R^2} - \frac{1}{r^2} + \frac{R^2}{2r^4}\right) = \frac{1}{2}\left(\frac{l}{R}\right)^2\left(\frac{R^2}{r^2} - 1\right)^2$$

$$\theta = \int \frac{l}{r^2}\frac{dr}{\dot{r}} = \pm\int \frac{\sqrt{2}Rdr}{\left(R^2 - r^2\right)} = \pm\frac{1}{\sqrt{2}}\log\left(\frac{R/r + 1}{R/r - 1}\right) + \theta_0$$

$$r\left(\theta\right) = R\tanh\left(\pm\frac{\theta - \theta_0}{\sqrt{2}}\right)$$