Ph-507. Homework 2 (due: Friday, February 1).

PROBLEM 2-1 (3 pts)

Considetr a relativistic particle in an external potential, $U = k\,|x|$:

$$\mathcal{L} = -mc^2\sqrt{1 - \frac{\dot{x}^2}{c^2}} - k\,|x|$$

Find the period of its oscillations as a function of amplitude x_0. Note that the Lagrangian is NOT quadractic in v.

PROBLEM 2-2 (2 pts)

a)A waterslide has a shape given by the following 3D curve: $x\,(z) = z\cos\left(2\pi z/\lambda\right)$, $y\,(z) = z\sin\left(2\pi z/\lambda\right)$ Here the positive direction of z is taken to be downward, parameter λ is unknown, and this spiral path makes N total turns (assume $N \gg 1$). Find the overall travel time down this slide, if the initial and final speeds are zero and v, respectively. Neglect dissipation.

PROBLEM 2-3 (7 pts)

A particle of mass m is confined by potential $U(x)$. Find the period of its oscillations, as a function of the total energy E.

(a) $U(x) = U_0\tan^2\left(x/\xi\right)$;

(b) $U(x) = U_0\left[\mathrm{e}^{-2x/\xi} - 2\mathrm{e}^{-x/\xi}\right]$;

(c) $U(x) = -U_0\left[\left(x/\xi\right)^2 - 1\right]^2$. Express the result in terms of elliptic integral $K\,(z)$.

PROBLEM 2-4 (3 pts)

a) Due to the presence of ions, electrostatic interactions are normally "screened" in water. For example, electrostatic potential near uniformly charged flat surface (e.g. biological membrane) obeys the following Poisson-Boltzmann equation:

$$\frac{d^2\Psi}{dz^2} = \xi^{-2}\sinh\Psi$$

Here z is the direction normal to the surface, ξ is the so-called Debye screening length and Ψ is the potential, in certain units $(k_B T/e)$. By *using the analogy to a mechanical problem*, find the solution to this equation $\Psi\,(z)$, which decays to 0 at the infinity, and has value $\Psi = \Psi_0$ near the charged surface (at $z = 0$).

Solution to Problem 2-1

$$E = \frac{\partial \mathcal{L}}{\partial \dot{x}} - \mathcal{L} = \frac{mc^2}{\sqrt{1-\frac{\dot{x}^2}{c^2}}} + k\,|x| = const = mc^2 + kx_0$$

Therefore,

$$\frac{\dot{x}}{c} = \sqrt{1-\left(\frac{1}{1+k\left(x_0-|x|\right)/mc^2}\right)^2} == \pm\tanh\psi$$

Here

$$\cosh\psi = \frac{k}{mc^2}\left(x_0-|x|\right)+1$$

$$T = 4\int_0^{x_0}\frac{dx}{|\dot{x}|} = \frac{4mc}{k}\int_0^{\psi_{\max}}\frac{d\cosh\psi}{\tanh\psi} = \frac{4mc}{k}\sinh\psi_{\max} = \frac{4mc}{k}\sqrt{\left(1+\frac{kx_0}{mc^2}\right)^2-1} = \frac{4}{c}\sqrt{x_0\left(x_0+\frac{2mc^2}{k}\right)}$$

Solution to Problem 2-2

$$E = m\dot{z}^2\left(1+\frac{2\pi^2z^2}{\lambda^2}\right) - mgz = const = 0$$

the total height is λN , and from the energy conservation law

$$mg\lambda N = \frac{mv^2}{2}$$

$$T = \int_0^{\lambda N}\frac{dz}{\dot{z}} = \int_0^{\lambda N}\frac{dz}{\sqrt{gz}}\sqrt{1+\frac{2\pi^2z^2}{\lambda^2}} \approx \frac{2N^{3/2}}{3}\sqrt{\frac{2\lambda}{g}} = \frac{2N}{3}\frac{v}{g}$$

Here we have taken into account that $2\pi^2N^2 \gg 1$.

Solution to Problem 2-3

a)

$$E = \frac{m\dot{x}^2}{2} + U_0\tan^2 x/\xi = const$$

$$T = 2\int_{x_{\min}}^{x_{\max}}\frac{dx}{\dot{x}} = \sqrt{2m}\int_{x_{\min}}^{x_{\max}}\frac{dx}{\sqrt{E-U_0\tan^2\left(x/\xi\right)}} = \xi\sqrt{\frac{2m}{\left(E+U_0\right)}}\int_{-1}^{1}\frac{du}{\sqrt{1-u^2}} = \pi\xi\sqrt{\frac{2m}{\left(E+U_0\right)}}.$$

here $u = \sqrt{1+U_0/E}\,\sin\left(x/\xi\right)$ and the turning points $(x_{\min},\, x_{\max})$, are $\pm\xi\sin^{-1}\left(\frac{E}{E+U_0}\right)$.

b) Since the particle is confined, $E<0$. Similarly to (a), we obtain:

$$T = \sqrt{\frac{2m}{|E|}}\int_{x_{\min}}^{x_{\max}}\frac{e^{x/\xi}dx}{\sqrt{\left(2e^{x/\xi}-1\right)U_0/|E| - e^{2x/\xi}}} = \xi\sqrt{\frac{2m}{|E|}}\int_{u_{\min}}^{u_{\max}}\frac{du}{\sqrt{\left(U_0/|E|-1\right)U_0/|E| - u^2}} = \pi\xi\sqrt{\frac{2m}{|E|}}.$$

here $u = e^{x/\xi} - U_0/|E|$.

c)

$$T = \sqrt{\frac{2m}{U_0}} \int_{x_{\min}}^{x_{\max}} \frac{dx}{\sqrt{U_0\left[(x/\xi)^2 - 1\right]^2 - |E|}} = \sqrt{\frac{2m}{U_0}} \int_{x_{\min}}^{x_{\max}} \frac{dx}{\sqrt{\left[1 - \sqrt{|E|/U_0} - (x/\xi)^2\right]\left[1 + \sqrt{|E|/U_0} - (x/\xi)^2\right]}}$$

Periodic solution exists for $-U_0 < E < 0$, and the turning points are $(x_{\min}, x_{\max}) = \pm\xi\sqrt{1 - \sqrt{|E|/U_0}}$. Upon substitution: $\sin\theta = \left(1 - \sqrt{|E|/U_0}\right)^{-1/2} x/\xi$, we obtain:

$$T = 2\xi\sqrt{\frac{2m}{\sqrt{U_0} + \sqrt{|E|}}} K\left(\frac{\sqrt{U_0} - \sqrt{|E|}}{\sqrt{U_0} + \sqrt{|E|}}\right),$$

where $K(z)$ is elliptic integral of the first kind.

Solution to Problem 2-4

PB equation can be interpreted as Lagrange equation, with z playing the role of time t and Ψ being the "generalized coordinate". The "Lagrangian" can be written as:

$$L = \frac{1}{2}\left(\frac{d\Psi}{dz}\right)^2 + \xi^{-2}\cosh\Psi,$$

The conserved quantity ("energy") is,

$$E = \frac{1}{2}\left(\frac{d\Psi}{dz}\right)^2 - \xi^{-2}\cosh\Psi = const.$$

Since $\Psi = 0$ and $\Psi' = 0$ for $z \longrightarrow \infty$, we conclude that $E = -\xi^{-2}$. Therefore,

$$z(\Psi) = \xi \int_{\Psi_0}^{\Psi} \frac{d\psi}{\sqrt{2(\cosh\psi - 1)}} = \frac{\xi}{2} \int_{\Psi_0}^{\Psi} \frac{d\psi}{\sinh(\psi/2)} = \xi \log\left(\frac{1 + \exp(\psi/2)}{1 - \exp(\psi/2)}\right)\Bigg|_{\Psi_0}^{\Psi},$$

$$1 - \exp\left(\frac{\Psi}{2}\right) = \left[1 + \exp\left(\frac{\Psi}{2}\right)\right]\exp(-z/\xi)\tanh(\Psi_0/4),$$

$$\Psi(z) = 2\log\left[\frac{1 + \exp(-z/\xi)\tanh(\Psi_0/4)}{1 - \exp(-z/\xi)\tanh(\Psi_0/4)}\right].$$