IV. OSCILLATIONS

A. Free oscillations

- **Linearized Lagrangian.** Let $\vec{q}=0$ be a fixed point, i.e.:

$$F_i = -\frac{\partial U}{\partial q_i} = 0 \tag{103}$$

Then, in its vicinity potential energy can be approximated by a quadratic form,

$$U(q) \approx U_0 + \frac{\partial U}{\partial q_i} q_i + \frac{1}{2}\frac{\partial^2 U}{\partial q_j \partial q_i} q_j q_i = \frac{1}{2} V_{ij} q_j q_i + const. \tag{104}$$

Here $V_{ij} = \partial^2 U / (\partial q_j \partial q_i)$. The fixed point is stable when V_{ij} is positive-definite. When the kinetic energy is quadratic (or bilinear) in $\dot{q}_i$, the overall Lagrangian can be written as,

$$\mathcal{L} = \frac{1}{2}\mu_{ij}\dot{q}_j\dot{q}_i - \frac{1}{2}V_{ij}q_j q_i, \tag{105}$$

Which results in linear equations of motion:

$$\mu_{ij}\ddot{q}_i + V_{ij} q_i = 0 \tag{106}$$

- **Normal modes.** We are looking for the solutions to the equations of motion, in the form

$$q_i(t) = q_i^{(\varpi)} \exp(\mathrm{i}\varpi t).$$

Such solutions are called normal modes (eigenmodes) of the system. They must satisfy the following Eq:

$$\left[\hat{V} - \varpi^2 \hat{\mu}\right] \vec{q}^{(\varpi)} = 0 \tag{107}$$

A nonzero solutions of this system of linear homogeneous equations exist only for a limited set of frequencies ϖ_n (resonant frequencies). They must be roots of the Secular Equation:

$$det\left(\hat{V} - \varpi_n^2 \hat{\mu}\right) = 0. \tag{108}$$

After these frequencies are determined, one can find the corresponding eigen vectors, by solving Eq. ((**?**)). **Important:** if certain roots of the secular equations correspond to $\varpi_n^2 < 0$, this means that $\varpi_n = \pm i\gamma$ is imaginary, and the point is unstable. The corresponding normal mode determines the "unstable direction": it either growth or decays exponentially with time:

$$q_i(t) = q_i^{(n)} \exp(\gamma t), \quad \text{or} \quad q_i(t) = q_i^{(n)} \exp(-\gamma t) \tag{109}$$

- **General Solution.** Normal modes of system constitute a *complete and orthogonal* basis. In other words, the general solution can be written as their linear combination:

$$q_i(t) = Re\sum_n C_n \vec{q}^{(n)} \exp(\mathrm{i}\varpi_n t) = \frac{1}{2}\sum_n C_n q_i^{(n)} \exp(\mathrm{i}\varpi_n t) + C_n^* \vec{q}^{(n)} \exp(-\mathrm{i}\varpi_n t), \quad Re(\varpi_n) \geq 0. \tag{110}$$

Here $\vec{q}^{(n)}$ and ϖ_n are the normal modes and their frequencies. Note that the complex conjugated modes must have amplitudes that are conjugated too. That is because $q_i(t)$ is real. Therefore, one can keep only half of the modes, e.g. only those with $Re(\varpi_n) \geq 0$. The modes with negative frequencies will be automatically present. One can see from the above equation that the velocities are given by:

$$\dot{q}_i(t) = -Im\sum_n \varpi_n C_n q_i^{(n)} \exp(\mathrm{i}\varpi_n t), \quad Re(\varpi_n) \geq 0. \tag{111}$$

- **Initial conditions.** Suppose, at $t = 0$, $q_i = q_i^{(init)}$ and $\dot{q}_i = u_i^{(init)}$. Due to orthogonality, the amplitudes of the normal modes can be determined by projecting initial vectors q_i and $\dot{q}_i$ onto each mode $q_i^{(n)}$:

$$\vec{q}^{(n)} \cdot \vec{q}^{(init)} = \left(\vec{q}^{(n)}\right)^2 Re\left(C_n\right) \tag{112}$$

$$\vec{q}^{(n)} \cdot \vec{w}^{(init)} = -\left(\vec{q}^{(n)}\right)^2 Im\left(C_n\right) \tag{113}$$

Here the scalar product is defined in a standard way, $\vec{q} \cdot \vec{q}' \equiv \vec{q} \cdot \hat{\mu} \cdot \vec{q}' = \mu_{ij} q_i q_i'$. Therefore,

$$C_n = \frac{1}{\left(\vec{q}^{(n)}\right)^2} \left[\vec{q}^{(n)} \cdot \vec{q}^{(init)} - \mathrm{i}\frac{\vec{q}^{(n)} \cdot \vec{u}^{(init)}}{\varpi_n}\right] \tag{114}$$

Since the eigenvectors (normal modes) are defined up to an arbitrary constant, it is often convenient to normalize them so that $\left(\vec{q}^{(n)}\right)^2 = \mu_{ij} q_i^{(n)} q_j^{(n)} = 1$.

B. Dissipation and driving force.

1. 1D problem

This is our first deviation from Lagrangian mechanics. We introduce dissipative (frictional) forces. Consider the problem with dissipative term in one dimension:

$$m\ddot{x} + R\dot{x} + kx = 0 \tag{115}$$

Look for the principle frequencies:

$$m\left(-\varpi^2 + i\varpi\delta + \varpi_0^2\right) x_\varpi = 0 \tag{116}$$

$$\varpi = \frac{i\delta}{2} \pm \sqrt{\varpi_0^2 - \frac{\delta^2}{4}} \tag{117}$$

Two regimes:

- $\varpi_0 > \delta/2$: oscillations with frequency $\varpi_\delta = \sqrt{\varpi_0^2 - \delta^2/4}$ and characteristic decay rate δ

$$\varpi = \frac{i\delta}{2} \pm \varpi_\delta \tag{118}$$

$$x = Re\left(x_0 \exp\left[-\frac{\delta}{2} + \mathrm{i}\sqrt{\varpi_0^2 - \frac{\delta^2}{4}}\right] t\right) \tag{119}$$

- $\varpi_0 < \delta/2$: the overdumped regime. There are two purely imaginary roots:

$$\varpi = \frac{\mathrm{i}}{2}\left(\delta \pm \sqrt{\frac{\delta^2}{4} - \varpi_0^2}\right) = \mathrm{i}\gamma_\pm. \tag{120}$$

$$x = x_+ e^{-\gamma_+ t} + x_- e^{-\gamma_- t} \tag{121}$$

2. Oscillations under periodic force.

We now introduce the force term into the equations of motion.

$$m\ddot{x} + R\dot{x} + kx = f(t) \tag{122}$$

By performing the Fourier transform of the right- and left- hand sides, we obtain independent equations for all frequency modes:

$$\left(\varpi_0^2 - \varpi^2 + i\varpi\delta\right) x_\varpi = \frac{f_\varpi}{m}$$

This gives:

$$x_\varpi = \frac{f_\varpi}{m\left(\varpi_0^2 - \varpi^2 + i\varpi\delta\right)} = \left[\frac{\varpi_0^2 - \varpi^2}{\left(\varpi_0^2 - \varpi^2\right)^2 + \varpi^2\delta^2} - \frac{\mathrm{i}\varpi\delta}{\left(\varpi_0^2 - \varpi^2\right)^2 + \varpi^2\delta^2}\right] \frac{f_\varpi}{m}$$

The overall amplitude of the forced oscillations is,

$$|x_\varpi| = \frac{|f_\varpi|}{m\sqrt{\left(\varpi_0^2 - \varpi^2\right)^2 + \varpi^2\delta^2}}$$

It peaks at $\varpi = \varpi_0$ (resonance condition), at which point the response is purely imaginary (maximum energy dissipation). The characteristic width of the resonance peak is δ.

3. D>1 system with dumping

- **Free oscillations**

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q_i} + F_i^{(dis)} \tag{123}$$

A common, and very convenient form for $F_i^{(dis)}$ is the linear one (good for viscous forces in liquid, *but not for solid friction*):

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = \frac{\partial \mathcal{L}}{\partial q_i} - \frac{1}{2}\frac{\partial R_{ij}\dot{q}_i\dot{q}_j}{\partial \dot{q}_i} = \frac{\partial U}{\partial q_i} - R_{ij}\dot{q}_j \tag{124}$$

Here $R_{ij}\dot{q}_i\dot{q}_j/2$ is called a dissipative function, R_{ij} is a positive-definite tensor. The equation of motion near the fixed point is,

$$\mu_{ij}\ddot{q}_j + R_{ij}\dot{q}_j + V_{ij}q_j = 0 \tag{125}$$

Again, one can use the normal mode method. By substituting a

$$q_i(t) = q_i^{(n)} \exp\left(\mathrm{i}\varpi_n t\right),$$

into the equation of motion, we obtain the following secular equation:

$$det\left(\left(\mathrm{i}\varpi_n\right)^2 \hat{\mu} + \mathrm{i}\varpi_n \hat{R} + \hat{V}\right) = 0 \tag{126}$$

Since this is a polynomial with all real coefficients, all its solutions have the following form:

$$\varpi_n = \mathrm{i}\kappa_n \pm \tilde{\varpi}_n \tag{127}$$

One can prove that $\kappa_n > 0$, if $\hat{R}$ is positive-definite. Thus, the equations of motion for the system with dissipations has the following general solution:

$$q_i(t) = Re\sum_n C_n \vec{q}^{(n)} \exp\left(-\kappa_n t + \mathrm{i}\tilde{\varpi}_n t\right), \qquad Re\left(\varpi_n\right) \geq 0 \tag{128}$$

IMPORTANT: in general, these modes are not orthogonal!

- **Forced oscillations.** When the system is subjected to a driving, we have to solve the system of linear differential equations with non-zero right hand side:

$$\mu_{ij}\ddot{q}_j + R_{ij}\dot{q}_j + V_{ij}q_j = f_i \exp\left(\mathrm{i}\varpi t\right) \tag{129}$$

After the substitution:

$$\vec{q}\,(t) = \vec{q}\exp\left(\mathrm{i}\varpi t\right), \tag{130}$$

The problem reduces to a system of linear algebraic equations:

$$\left(-\varpi^2\hat{\mu} + \mathrm{i}\varpi\hat{R} + \hat{V}\right)\cdot\vec{q}_\varpi = \vec{f}_\varpi. \tag{131}$$

It is solved by inverting the matrix at the left hand side:

$$\vec{q}_\varpi = \left(-\varpi^2\hat{\mu} + \mathrm{i}\varpi\hat{R} + \hat{V}\right)^{-1}\cdot\vec{f}_\varpi. \tag{132}$$

C. Parametric resonance

Consider an oscillator with periodically varying parameters, e. g. a pendulum with varying length:

$$\ddot{x} + \omega_0^2\left(1 + h\cos\Omega t\right)x = 0 \tag{133}$$

We look for a solution in the general form:

$$x\,(t) = a\,(t)\exp\left(\mathrm{i}\varpi t\right) + a^*\,(t)\exp\left(-\mathrm{i}\varpi t\right) \tag{134}$$

Here $a\,(t)$ is supposed to be a slowly varying amplitude. After substituting this into the equation of motion, one obtains:

$$\left[2\mathrm{i}\omega\dot{a}\,(t) + \left(\omega_0^2 - \omega^2\right)a\,(t)\right]\exp\left(\mathrm{i}\varpi t\right) + c.c. = -\frac{h\omega_0^2}{2}\left[a^*\exp\left(\mathrm{i}\left(\Omega - \varpi\right)t\right) + a\exp\left(\mathrm{i}\left(\Omega + \varpi\right)t\right)\right] + c.c., \tag{135}$$

Here we have neglected the term $\ddot{a}\,(t)$ because $a\,(t)$ is assumed to be slow. The right and left hand sides may have the same frequency only if

$$\varpi = \frac{\Omega}{2}. \tag{136}$$

By collecting all the terms with frequency ω, we obtain:

$$2\mathrm{i}\omega\dot{a} = \left(\omega^2 - \omega_0^2\right)a - \frac{h\omega_0^2}{2}a^* \tag{137}$$

In terms of real and imaginary parts of the amplitude function ($a = a' + ia''$), this equation can be written as:

$$\frac{d}{dt}\begin{pmatrix} a' \\ a'' \end{pmatrix} = \frac{1}{2\omega}\begin{pmatrix} 0 & \left(\omega^2 - \omega_0^2\right) + h\omega_0^2/2 \\ -\left(\omega^2 - \omega_0^2\right) + h\omega_0^2/2 & 0 \end{pmatrix}\begin{pmatrix} a' \\ a'' \end{pmatrix}. \tag{138}$$

The matrix can be diagonalized. Its eigenvalues are given by equation:

$$4\omega^2\lambda^2 - \frac{h^2\omega_0^4}{4} + \left(\omega^2 - \omega_0^2\right)^2 = 0 \qquad \Longrightarrow \qquad \lambda_\pm = \pm\frac{1}{2\omega}\sqrt{\frac{h^2\omega_0^4}{4} - \left(\omega^2 - \omega_0^2\right)^2}. \tag{139}$$

Therefore,

$$a = a_+\exp\left(\lambda_+ t\right) + a_-\exp\left(\lambda_- t\right). \tag{140}$$

The amplitude will grow exponentially if λ is real. In other words, the system becomes linearly unstable at $x = 0$. This gives the following conditions for the *parametric resonance:*

$$h \geq 2\frac{\left|\omega^2 - \omega_0^2\right|}{\omega_0^2}\ ; \qquad \omega = \frac{\Omega}{2}. \tag{141}$$

$h_c = 2\left|\omega^2 - \omega_0^2\right|/\omega_0^2$ is called *bifurcation* point (in the context of dynamical systems, *bifurcations* are analogous to phase transitions). The eventual amplitude of the oscillations is determined by the nonlinear terms in the equation of motion.