III. ROTATIONAL MOTION & MECHANICS OF RIGID BODY

A. Fixed-axis rotation

By definition, the distance between any two points in a Rigid Body is constant. Let the axis of rotation be fixed and its direction be given by the unit vector $\hat{\mathbf{n}}$. The rotation angle θ is the natural generalized coordinate. The velocity of any point of the rigid body is,

$$\dot{\mathbf{r}}_a = \frac{d\theta\left(\hat{\mathbf{n}}\times\mathbf{r}_a\right)}{dt} = \varpi\times\mathbf{r}_a \tag{67}$$

Here the origin of the coordinate system belongs to the axis of rotation, and the vectorial angular velocity is introduced as $\varpi = \hat{\mathbf{n}}\dot{\theta}$.(can be done only in 3D).

- **Moment of inertia.** Kinetic energy can be expressed as:

$$T = \sum_a \frac{m_a\dot{\mathbf{r}}_a^2}{2} = \left(\sum_a m_a r_{a\perp}^2\right)\frac{\varpi^2}{2} = \frac{I_{\hat{\mathbf{n}}}\varpi^2}{2}. \tag{68}$$

  $I_{\hat{\mathbf{n}}}$ is called *moment of inertia* with respect to the given axis:

$$I_{\hat{\mathbf{n}}} = \sum_a m_a r_{a\perp}^2 = \sum_a m_a\left[r_a^2 - \left(\hat{\mathbf{n}}\cdot\mathbf{r}_a\right)^2\right]. \tag{69}$$

  By definition, I depends on the orientation of the axis of rotation, and its distance from the center of mass, R_{cm}:

$$T = \left(\sum_a m_a\right)\frac{\dot{\mathbf{R}}_{cm}^2}{2} + \sum_a \frac{m_a\left(\dot{\mathbf{r}}_a - \dot{\mathbf{R}}^{(cm)}\right)^2}{2} = \frac{\left(MR_{cm}^2 + I_{\hat{\mathbf{n}}}^{(cm)}\right)\varpi^2}{2} \quad\Longrightarrow\quad I_{\hat{\mathbf{n}}} = MR_{cm}^2 + I_{\hat{\mathbf{n}}}^{(cm)}. \tag{70}$$

  Here $M = \sum_a m_a$ is the total mass, and $I_{\hat{\mathbf{n}}}^{(cm)}$ is the moment of inertia with respect to the same-orientation axis coming through the CM.

- **Equation of motion.** *Generalized momentum* conjugated to θ is the component of angular momentum parallel to $\hat{\mathbf{n}}$:

$$p_\theta = \frac{\partial\mathcal{L}}{\partial\dot{\theta}} = \frac{\partial}{\partial\varpi}\sum_a\left(\varpi\times\mathbf{r}_a\right)\cdot\frac{\partial\mathcal{L}}{\partial\dot{\mathbf{r}}_a} = \hat{\mathbf{n}}\cdot\mathbf{L} \tag{71}$$

  Since $\mathcal{L} = T - U$,

$$L_{\hat{\mathbf{n}}} = \frac{\partial T}{\partial\varpi} = I_{\hat{\mathbf{n}}}\varpi \tag{72}$$

  The generalized force is called torque:

$$f_\theta = \frac{\partial\mathcal{L}}{\partial\theta} = \frac{\partial}{\partial\left(\delta\theta\right)}\sum_a\left(\delta\theta\mathbf{n}\times\mathbf{r}_a\right)\cdot\frac{\partial\mathcal{L}}{\partial\mathbf{r}_a} = \hat{\mathbf{n}}\cdot\sum_a\mathbf{r}_a\times\mathbf{f}_a \equiv \tau_{\hat{\mathbf{n}}} \tag{73}$$

  Thus, Lagrange equation is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(I_{\hat{\mathbf{n}}}\varpi\right) = \tau_{\hat{\mathbf{n}}} \tag{74}$$

B. Fixed-point rotation

If only one point of the body is fixed, the direction of the angular velocity may be arbitrary. One can the kinetic energy in terms of vector ϖ

$$T = \frac{\varpi^2}{2}\sum_a m_a\left[r_a^2 - \left(\hat{\mathbf{n}}\cdot\mathbf{r}_a\right)^2\right] = \frac{\varpi\cdot\hat{\mathbf{I}}\cdot\varpi}{2} \tag{75}$$

Here $\hat{\mathrm{M}}$ is called inertia tensor:

$$I_{ij} = \sum_a m_a \left[r_a^2 \delta_{ij} - r_{ai} r_{aj}\right] \tag{76}$$

As one can see, $I_{\hat{\mathbf{n}}} = \hat{\mathbf{n}} \cdot \hat{\mathrm{M}} \cdot \hat{\mathbf{n}}$, and

$$\hat{\mathbf{I}} = M\left[R^{(cm)2} - R_i^{(cm)} R_j^{(cm)}\right] + \hat{\mathbf{I}}^{(cm)} \tag{77}$$

Let $\mathbf{\Theta} = (\delta\theta_x, \delta\theta_y, \delta\theta_z)$ be the vector of infinitesimal rotations ($\varpi = \dot{\mathbf{\Theta}}$). One can obtain the equations of motion in a standard manner:

$$\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{\Theta}}} = \mathbf{L} = \hat{\mathbf{I}} \cdot \varpi \tag{78}$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{\Theta}} = \sum_a \mathbf{r}_a \times \mathbf{f}_a \equiv \tau \tag{79}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \hat{\mathbf{I}} \cdot \varpi = \tau \tag{80}$$

C. Euler equations

Any vector "nailed" to the rotating reference frame is transformed by rotation as $\dot{\mathbf{A}} = \varpi \times \mathbf{A}$. Therefore,

$$\left(\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}\right)_{Lab} = \left(\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}\mathbf{t}}\right)_{\mathbf{r}} + \varpi \times \mathbf{L} = \hat{\mathbf{I}} \cdot \dot{\varpi} + \varpi \times \hat{\mathbf{I}} \cdot \varpi = \tau \tag{81}$$

Here $(\mathrm{d}/\mathrm{d}t)_r$ is the time derivative at the rotating Reference Frame. Note that $(\mathrm{d}\varpi/\mathrm{d}t)_{Lab} = (\mathrm{d}\varpi/\mathrm{d}t)_r + \varpi \times \varpi = (\mathrm{d}\varpi/\mathrm{d}t)_r$. We obtain Euler equations:,

$$\hat{\mathbf{I}} \cdot \frac{\mathrm{d}\varpi}{\mathrm{d}t} = \tau - \varpi \times \hat{\mathbf{I}} \cdot \varpi. \tag{82}$$

If $\hat{\mathbf{I}}$ is diagonalized and I_1, I_2, I_3 are its eigenvalues, than this system of equations can be rewritten as:

$$I_1 \dot{\varpi}_1, = \tau_1 + \varpi_2 \varpi_3 (I_2 - I_3); \tag{83}$$

$$I_2 \dot{\varpi}_2 = \tau_2 + \varpi_3 \varpi_1 (I_3 - I_1); \tag{84}$$

$$I_3 \dot{\varpi}_3 = \tau_3 + \varpi_1 \varpi_2 (I_1 - I_2). \tag{85}$$

- **Precession of a free symmetric top.** If $\tau = \mathbf{0}$ (free motion), and $I_1 = I_2$ (symmetric spinning top), we obtain $\varpi_3 = const$, and

$$\dot{\varpi}_1 = -\Omega \varpi_2; \tag{86}$$

$$\dot{\varpi}_2 = \Omega \varpi_1, \tag{87}$$

where $\Omega = \varpi_3 (I_3/I_1 - 1)$. The solution to these equations is precession of the angle ϖ with frequency Ω (in rotating reference frame):

$$\varpi_1 = \varpi' \cos \Omega t, \qquad \varpi_2 = \varpi' \sin \Omega t \tag{88}$$

In stationary raference frame, precession rate is differtent since ϖ_1 and ϖ_2 are defined in with respect to the principle axes 1 and 2 , and those axes are rotating about the axis 3 one with angular velocity ϖ_3. Therefore, the projection of angular velocity of precession $\mathbf{\Omega}_{pr}$, onto the direction of the third axis, $\hat{n}_3$, is:

$$\hat{n}_3 \cdot \mathbf{\Omega}_{pr} = \Omega + \varpi_3 = \frac{I_3 \varpi_3}{I_1} = \frac{\hat{n}_3 \cdot \mathbf{L}}{I_1} \qquad \text{hence, } \mathbf{\Omega}_{pr} = \frac{\mathbf{L}}{I_1} \tag{89}$$

- **Asymmetric tops.** If $I_1 > I_2 > I_3$, the character of the motion can be determined by geometrical analysis. The conservation laws can be written as:

$$L_1^2 + L_2^2 + L_3^2 = const = \mathbf{L}^2 \tag{90}$$

$$\frac{L_1^2}{I_1} + \frac{L_2^2}{I_2} + \frac{L_3^2}{I_3} = const = 2E \tag{91}$$

The first equation describes a sphere of radius L , the other one is an ellipsoid. If E is fixed and $\mathbf{L}^2$ is increasing, the first intersection of the two surfaces occurs when $L^2 = 2EI_3$. After that, the trajectory corresponds to precession around the direction of minimal moment of inertia, I_3. At $L^2 = 2EI_2$ the orbits change the orientation, and precession occurs around the axis of maximum moment of inertia, I_1. The rotation about the intermediate axis is unstable!

D. Heavy Spinning Top

Consider a symmetric spinning top ($I_1 = I_2 \neq I_3$) under the influence of a uniform gravitational force, mg. Our generalized coordinates will be Euler angles (θ, ϕ, ψ). Here θ and ϕ are the azimuthal and polar angles which define the orientation of certain axis z' of the rigid body, in regular spherical coordinates. For the symmetric top, we choose z' to be the axis of symmetry. The third Euler angle ψ parameterizes the axial rotation around z'.

The Lagrangian can be written as

$$\mathcal{L} = \frac{I_\perp}{2}\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) + \frac{I_\parallel}{2}\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 - mgl\cos\theta \tag{92}$$

Here l is the distance between the fixed point of the body and the center of mass. $\dot{\psi}$, $\dot{\phi}$ and $\dot{\theta}$ corresponds to *rotation*, *precession* and *nutation* , respectively. Since $\partial\mathcal{L}/\partial\phi = \partial\mathcal{L}/\partial\psi = 0$, we can identify two conservation laws:

$$\frac{\partial\mathcal{L}}{\partial\dot{\psi}} = L_\parallel = I_\parallel\left(\dot{\psi} + \dot{\phi}\cos\theta\right) = const, \tag{93}$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = L_\phi = I_\perp\dot{\phi}\sin^2\theta + L_\parallel\cos\theta = const \quad\Longrightarrow\quad \dot{\phi} = \frac{L_\phi - L_\parallel\cos\theta}{I_\perp\sin^2\theta} = \Omega\frac{p_0 - \cos\theta}{\sin^2\theta}, \tag{94}$$

where $\Omega = L_\parallel/I_\perp = \left(I_\parallel/I_\perp\right)\varpi_\parallel$, and $p_0 = L_\phi/L_\parallel$. One more conserved quantity, as usual, is energy:

$$E = \frac{I_\perp}{2}\left[\dot{\theta}^2 + \Omega^2\frac{(p_0 - \cos\theta)^2}{\sin^2\theta}\right] + \frac{L_\parallel^2}{2I_\parallel} + mgl\cos\theta. \tag{95}$$

It is convenient to describe *nutation* in terms of variable $p = \cos\theta$:

$$\dot{p}^2 = 2\omega_0^2\left(1 - p^2\right)(p_E - p) - \Omega^2(p_0 - p)^2. \tag{96}$$

Here $p_E = \left(E - L_\parallel^2/2I_\parallel\right)/mgl$, and $\omega_0 = \sqrt{mgl/I_\perp}$ is the frequency of small oscillations for $\Omega = 0$, i.e. when the top becomes a regular physical pendulum. In a typical case, the right hand side has two roots in the physical interval of p: $-1 \leq p_1 \leq p_2 \leq +1$. The period of nutation can be found as:

$$T = 2\int_{p_1}^{p_2}\frac{dp}{\sqrt{2\omega_0^2\left(1 - p^2\right)(p_E - p) - \Omega^2(p_0 - p)^2}}, \tag{97}$$

The corresponding precession rate can be found as $\dot{\phi} = \Omega(p_0 - p)/\left(1 - p^2\right)$. Note that it changes sign at $p = p_0$. In general, the above result can be expressed in terms of an elliptic integral.

- **Example 1.** Determine the minimal Ω at which the potential energy maximum ($p=+1$) is a stable point. Since $\dot{p}=0$ at $p=1$, we obtain $p_0=1$. All the kinetic energy at $p=1$ is only due to spinning, therefore $p_E=\left(E-L_{\|}^2/2I_{\|}\right)/mgl=mgl/mgl=1$. Hence,

$$\dot{p}^2=2\omega_0^2\left[1+p-\frac{\Omega^2}{2\omega_0^2}\right](p-1)^2\,. \tag{98}$$

One can see the point $p=1$ gets unstable when $p+1-\Omega^2/2\omega_0^2\geq 0$, i.e. for $\Omega\leq\Omega_c=2\omega_0$. After that, a periodic nutation occurs between $p=1$ and $p=\Omega^2/2\omega_0^2-1$

- **Example 2.** Consider a motion of a fast spinning top ($\Omega\gg\omega_0$), whose axis is released with no initial nutation or precession ($\dot{\phi}=\dot{\theta}=0$) from the original position $\cos\theta=p^*$. Since $\dot{\phi}=\dot{\theta}=0$ initially, we conclude that $p_E=p_0=p^*$. This gives:

$$\dot{p}^2=(p_0-p)\left[2\omega_0^2\left(1-p^2\right)-\Omega^2\left(p_0-p\right)\right]\simeq\Omega^2\left(p_0-p\right)\left(p-p_0+2\Delta\right)=\Omega^2\left[\Delta^2-\left(p-p_0+\Delta\right)^2\right], \tag{99}$$

where $\Delta=\left(\omega_0/\Omega\right)^2\left(1-p_0^2\right)$. Upon our favorite substitution, $p=p_0+\Delta\left[\cos\vartheta-1\right]$, we obtain:

$$t=\int\frac{dp}{\dot{p}}=\frac{\vartheta}{\Omega}+const \tag{100}$$

Given the initial condition ($p=p^*$), $\vartheta=\Omega t$. Therefore,

$$\cos\theta\left(t\right)=p\left(t\right)=p_0+\frac{\omega_0^2}{\Omega^2}\left(1-p_0^2\right)\left[\cos\Omega t-1\right]. \tag{101}$$

$$\dot{\phi}=\frac{\Omega\left(p_0-p\right)}{1-p^2}\simeq\frac{mgl}{L_{\|}}\left[\cos\Omega t-1\right]\qquad\Longrightarrow\qquad\left\langle\dot{\phi}\right\rangle=\frac{mgl}{2L_{\|}} \tag{102}$$

E. Non-inertial Reference Frames

A non-inertial reference frame can be characterized by velocity of its origin, $\mathbf{v}_0\left(t\right)$, and the angular velocity $\mathbf{\Omega}\left(t\right)$. Its local motion is, $\mathbf{v}\left(\mathbf{r},t\right)=\mathbf{v}_0\left(t\right)+\mathbf{\Omega}\left(t\right)\times\mathbf{r}$. If $\mathbf{r}\left(t\right)$ is the position of a particle in this reference frame,

$$\mathcal{L}=\frac{m\left(\dot{\mathbf{r}}+\mathbf{v}\left(\mathbf{r}\right)\right)^2}{2}-U\left(\mathbf{r}\right)=\frac{m\dot{\mathbf{r}}^2}{2}+m\dot{\mathbf{r}}\cdot\mathbf{v}+\frac{m\mathbf{v}^2}{2}-U\left(\mathbf{r}\right)$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\mathbf{r}}}=m\left(\ddot{\mathbf{r}}+\frac{\partial}{\partial t}\left(\mathbf{v}_0\left(t\right)+\mathbf{\Omega}\left(t\right)\times\mathbf{r}\right)+\mathbf{\Omega}\left(t\right)\times\dot{\mathbf{r}}\right);$$

$$\frac{\partial\mathcal{L}}{\partial\mathbf{r}}=m\frac{\partial\left(\dot{\mathbf{r}}\times\mathbf{\Omega}\right)\cdot\mathbf{r}}{\partial\mathbf{r}}+\frac{m}{2}\frac{\partial\left(\mathbf{\Omega}\times\mathbf{r}\right)^2}{\partial\mathbf{r}}-\frac{\partial U\left(\mathbf{r}\right)}{\partial\mathbf{r}}=m\dot{\mathbf{r}}\times\mathbf{\Omega}-\frac{\partial}{\partial\mathbf{r}}\left[U\left(\mathbf{r}\right)-\frac{m\Omega^2\mathbf{r}_\perp^2}{2}\right]$$

Lagrange equation gives:

$$m\ddot{\mathbf{r}}=-m\mathbf{a}+2m\dot{\mathbf{r}}\times\mathbf{\Omega}+m\Omega^2\mathbf{r}_\perp-\mathbf{F}$$

Here $\mathbf{a}=\partial\mathbf{v}\left(\mathbf{r}\right)/\partial t$, the second term in r.h.s. is called Coriolis force, the third term is centrifugal one, and $\mathbf{F}$ is a regular mechanical force.