1: Jackson 6.4

(a) We first note, from section 5.10 in Jackson, the magnetic field and magnetic induction in a uniformly magnetized sphere is:

\[ \mathbf{H}_{in} = -\frac{1}{3} \mathbf{M} \quad \mathbf{B}_{in} = \frac{2\mu_0}{3} \mathbf{M} \tag{1} \]

In this case, we are given the magnetization \( \mathbf{M} = \frac{3m}{4\pi R^3} \hat{z} \), which gives:

\[ \mathbf{H}_{in} = -\frac{m}{4\pi R^3} \hat{z} \quad \mathbf{B}_{in} = \frac{2m\mu_0}{4\pi R^3} \hat{z} \tag{2} \]

In the presence of a magnetic field and for a conductor moving at velocity \( \mathbf{v} \), Ohm’s law becomes:

\[ \mathbf{J} = \sigma (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{3} \]

We are told that there is no current in the steady state, which gives the electric field:

\[ \mathbf{E} = -\mathbf{v} \times \mathbf{B} = -\omega \rho \hat{\phi} \times \left( \frac{2m\mu_0}{4\pi R^3} \right) \hat{z} = -\frac{\omega m \mu_0 \rho}{2\pi R^3} \hat{\phi} \tag{4} \]

Converting to spherical coordinates:

\[ \mathbf{E} = -\frac{\omega m \mu_0 r^2}{2\pi R^3} \sin^2(\theta) \hat{r} - \frac{\omega m \mu_0 r}{2\pi R^3} \sin(\theta) \cos(\theta) \hat{\theta} \tag{5} \]

We can now use Gauss’ law to find the induced charge:

\[ \frac{Q_{ind}}{\epsilon_0} = -\frac{\omega m \mu_0 r^3}{2\pi R^3} \int_0^{2\pi} \int_0^\pi r \sin^3(\theta) d\theta d\phi = -\frac{4\omega m \mu_0 r^3}{3R^3} \]

\[ \rightarrow Q_{ind} = -\frac{4\omega m (\mu_0 \epsilon_0) r^3}{3R^3} \tag{6} \]

The induced charge density is thus:

\[ \rho_{ind} = \frac{Q_{ind}}{\frac{4}{3} \pi r^3} = -\frac{\omega m}{\pi \epsilon_0 c^2 R^3} \tag{7} \]

(b) The dipole moment components are given by:

\[ p_x = \int x' \rho(r') d^3r' \quad p_y = \int y' \rho(r') d^3r' \quad p_z = \int z' \rho(r') d^3r' \tag{8} \]

Since the charge density is constant, the odd symmetry of the Cartesian components gives zero dipole moment components. We do not yet know the surface charge on the sphere, so the quadrupole moments can’t be calculated directly. Since we know the field inside the sphere explicitly however, we can simply solve the problem as a simple boundary value problem. Inside, we have an electric field:

\[ \mathbf{E} = -\frac{\omega m \mu_0 r^2}{2\pi R^3} \sin^2(\theta) \hat{r} - \frac{\omega m \mu_0 r}{2\pi R^3} \sin(\theta) \cos(\theta) \hat{\theta} \tag{9} \]
By inspection, the potential is:

\[
\varphi_{\text{in}} = -\frac{\omega m \mu_0 r^2}{6\pi R^3} \left[ P_0(\cos(\theta)) + P_2(\cos(\theta)) \right] + \text{constant}
\]

\[
= \left[ \text{constant} - \frac{\omega m \mu_0 r^2}{6\pi R^3} \right] P_0(\cos(\theta)) - \frac{\omega m \mu_0 R^2}{6\pi R} P_2(\cos(\theta))
\]  \hspace{1cm} (10)

where we used the relation \((\sin^2(\theta) = \frac{2}{3} [1 + P_2(\cos(\theta))]\). The general solution for the potential outside the sphere is given by:

\[
\varphi_{\text{out}} = \sum_{\ell} C_\ell r^{-(\ell+1)} P_\ell(\cos(\theta))
\]  \hspace{1cm} (11)

The potential must be continuous across the sphere’s surface, which gives:

\[
\left[ \text{constant} - \frac{\omega m \mu_0}{6\pi R} \right] P_0(\cos(\theta)) - \frac{\omega m \mu_0 R^2}{6\pi R} P_2(\cos(\theta)) = \sum_{\ell} C_\ell R^{-(\ell+1)} P_\ell(\cos(\theta))
\]  \hspace{1cm} (12)

By inspection, we see that the only two nonvanishing coefficients are \(C_0\) and \(C_2\):

\[
C_0 = \text{constant} - \frac{\omega m \mu_0}{6\pi} \quad C_2 = -\frac{\omega m \mu_0 R^2}{6\pi}
\]  \hspace{1cm} (13)

The potential outside the sphere is thus:

\[
\varphi_{\text{out}} = \left[ \text{constant} - \frac{\omega m \mu_0}{6\pi} \right] 1 - \frac{\omega m \mu_0 R^2}{12\pi r^3} (3\cos^2(\theta) - 1)
\]  \hspace{1cm} (14)

Converting to Cartesian coordinates (but keeping \(r = \sqrt{x^2 + y^2}\)):

\[
\varphi_{\text{out}} = \left[ \text{constant} - \frac{\omega m \mu_0}{6\pi} \right] \frac{1}{r} - \frac{\omega m \mu_0 R^2}{12\pi r^5} (3z^2 - r^2)
\]

\[
= -\frac{\omega m \mu_0 R^2}{4\pi r^5} z^2 + \frac{\omega m \mu_0 R^2}{12\pi r^5} x^2 + \frac{\omega m \mu_0 R^2}{12\pi r^5} y^2
\]  \hspace{1cm} (15)

where the first monopole term vanishes since we are told there is no net charge on the sphere. Comparing the potential to the multipole expansion (4.10), we find the quadrupole moments:

\[
Q_{11} = Q_{22} = \frac{\omega m \mu_0 R^2}{6\pi} \quad Q_{33} = -\frac{\omega m \mu_0 R^2}{2\pi}
\]  \hspace{1cm} (16)

This gives the wrong answer \(Q_{11} = Q_{22} = -\frac{Q_{33}}{3}\), which is strange since we obtain the right answer for the surface charge.

(c) The electric field inside and outside the sphere is:

\[
E_{\text{in}} = -\nabla \varphi_{\text{in}} = -\frac{\omega m \mu_0}{2\pi R^3} \sin^2(\theta) \hat{r} - \frac{\omega m \mu_0}{2\pi R^3} \sin(\theta) \cos(\theta) \hat{\theta}
\]  \hspace{1cm} (17)

\[
E_{\text{out}} = -\nabla \varphi_{\text{out}} = -\frac{\omega m \mu_0 R^2}{4\pi r^4} (3\cos^2(\theta) - 1) \hat{r} - \frac{\omega m \mu_0 R^2}{2\pi r^4} \cos(\theta) \sin(\theta) \hat{\theta}
\]  \hspace{1cm} (18)

The boundary condition for the normal electric field component at the sphere’s surface is:

\[
\sigma_f = \epsilon_0 \left[ E_{\text{out}}^r - E_{\text{in}}^r \right]
\]

\[
= -\frac{\omega m \mu_0 \epsilon_0}{2\pi R^2} \left[ \frac{3}{2} \cos^2(\theta) - \frac{1}{2} - \sin^2(\theta) \right]
\]

\[
= -\frac{\omega m}{2\pi c^2 R^2} \left[ \frac{5}{3} P_2(\cos(\theta)) - \frac{2}{3} \right]
\]

\[
= \frac{\omega m}{3\pi c^2 R^2} \left[ 1 - \frac{5}{2} P_2(\cos(\theta)) \right]
\]  \hspace{1cm} (19)
(d) The line integral of the electric field is simply the potential difference between two points. In this case, its the potential difference between the equator and pole:

\[
\mathcal{E} = \varphi_{\text{out}}(R, \pi/2) - \varphi_{\text{in}}(R, 0) \\
= -\frac{\omega m \mu_0}{12\pi R} \left( 3\cos^2\left(\frac{\pi}{2}\right) - 1 - 3\cos^2(0) + 1 \right) \\
= \frac{\omega m \mu_0}{4\pi R}
\]  

(20)

2: Jackson 6.5

(a) The electromagnetic momentum is given by (6.117):

\[
P = \mu_0 \varepsilon_0 \int_V \mathbf{E} \times \mathbf{H} \, d^3r \\
= -\mu_0 \varepsilon_0 \int_V (\nabla \varphi) \times \mathbf{H} \, d^3r
\]

Recalling the identity:

\[(\nabla f) \times \mathbf{v} = \nabla \times (f \mathbf{v}) - f(\nabla \times \mathbf{v}) \]  

(21)

We can rewrite the momentum:

\[
P = \mu_0 \varepsilon_0 \int_V \left[ \varphi (\nabla \times \mathbf{H}) - \nabla \times (\varphi \mathbf{H}) \right] d^3r \\
= \frac{1}{c^2} \left[ \int_V \varphi \mathbf{J} \, d^3r - \int_V \nabla \times (\varphi \mathbf{H}) \, d^3r \right] \\
= \frac{1}{c^2} \left[ \int_V \varphi \mathbf{J} \, d^3r - \oint (\hat{n} \times (\varphi \mathbf{H})) \, dS \right]
\]  

(22)

We thus see that if the product \( \varphi \mathbf{J} \) at the boundary of the volume is small compared to the volume integral inside the volume, the momentum simplifies to:

\[
P = \frac{1}{c^2} \int_V \varphi \mathbf{J} \, d^3r
\]  

(23)

(b) We can expand the potential around the current density location \( \mathbf{r}_\mathbf{J} \):

\[
\varphi = \varphi(\mathbf{r}_\mathbf{J}) + (\nabla \varphi(\mathbf{r}_\mathbf{J})) \cdot (\mathbf{r} - \mathbf{r}_\mathbf{J}) + \frac{1}{2} (\nabla \varphi(\mathbf{r}_\mathbf{J})) \cdot (\mathbf{r} - \mathbf{r}_\mathbf{J})^2 + \ldots
\]

\[
\approx -\mathbf{E}(\mathbf{r}_\mathbf{J}) \cdot (\mathbf{r} - \mathbf{r}_\mathbf{J})
\]  

(24)

where we arbitrarily set \( \varphi(\mathbf{r}_\mathbf{J}) = 0 \) as the reference and keep only the second term. Plugging this
approximation into the momentum result found in part (a):

\[ P = -\frac{1}{c^2} \int_V \left[ E(r_J) \cdot (r - r_J) \right] J d^3r \]

\[ = -\frac{1}{c^2} \sum_i E_i(r_J) \int_V (r - r_J) J_i d^3r \]

\[ = -\frac{1}{c^2} \sum_{i,j} E_j(r_J) \int_V [r - r_J]_i J_j d^3r \]

\[ = \frac{1}{2c^2} \sum_{i,j} E_j(r_J) \int_V \left[ (r - r_J)_i J_j - [r - r_J]_j J_i \right] d^3r \]

\[ = \frac{1}{2c^2} \epsilon_{ijk} E_j(r_J) \int_V \left[ (r - r_J)_i J_j - [r - r_J]_j J_i \right] d^3r \]

\[ = \frac{1}{2c^2} \left[ E(r_J) \times \int_V (r - r_J) \times J d^3r \right] \]

\[ = \frac{1}{c^2} E(r_J) \times m \quad (25) \]

(c) We set the origin at the current distribution location and \( \varphi(0) = 0 \) to obtain:

\[ \varphi = -E(r) \cdot r \quad (26) \]

The surface integral contribution neglected in (22) is now:

\[ -\frac{1}{c^3} \oint (\hat{n} \times (\varphi \mathbf{H})) dS = \frac{1}{c^3} \oint [E(r_J) \cdot r] [\hat{n} \times \mathbf{H}] dS \]

\[ = \frac{1}{4\pi c^2} \oint [E(r_J) \cdot r] [\hat{n} \times \mathbf{H}] dS \]

\[ = \frac{1}{4\pi c^2} \oint [E(r_J) \cdot r] \left[ \hat{n} \times \left( \frac{3\hat{n} \cdot \mathbf{m} - \mathbf{m}}{|\mathbf{x}|^3} \right) \right] dS \]

\[ = -\frac{1}{4\pi c^2} \oint [E(r_J) \cdot r] \frac{\hat{n} \times \mathbf{m}}{|\mathbf{x}|^3} dS \quad (27) \]

3: Jackson 6.14

(a) For harmonic fields, we write the fields in complex form:

\[ E(r, t) = E(r) e^{i\omega t} \quad B(r, t) = B(r) e^{i\omega t} \quad (28) \]

which recasts Maxwell’s equations (only Ampere’s law and Faraday’s law are relevant):

\[ \nabla \times \mathbf{E} = -i\omega \mathbf{B} \quad \nabla \times \mathbf{B} = i\omega \mu_0 \epsilon_0 \mathbf{E} \quad (29) \]

We know from elementary analysis of capacitors that the electric and magnetic fields have only \( \hat{z} \) and \( \phi \) components respectively. The problem now inspires us to expand the fields in a power series by \( \omega \):

\[ E(r, t) = e^{i\omega t} \sum_j E_j \omega^j \hat{z} \quad B(r, t) = e^{i\omega t} \sum_k B_k \omega^k \hat{\phi} \quad (30) \]
Plugging these expansions into Ampere’s law and Farady’s law (noting that azimuthal symmetry causes the $\phi$ derivatives to drop out):

$$- \sum_j \frac{\partial E_j}{\partial \rho} \omega^j = -i \sum_k B_k \omega^{k+1}$$

$$\sum_k \frac{1}{\rho} \frac{\partial (\rho B_k)}{\partial \rho} \omega^k = i \mu_0 \epsilon_0 \sum_j E_j \omega^{j+1}$$

Since each power of $\omega$ must be satisfied simultaneously, we obtain the relations:

$$\frac{\partial E_j}{\partial \rho} = i B_{j-1}$$

$$\frac{1}{\rho} \frac{\partial (\rho B_j)}{\partial \rho} = i \mu_0 \epsilon_0 E_{j-1}$$

The lowest order term for $E$ is the field generated by the charge $Q$ on the capacitor plates:

$$Q(t) = \int_0^t I_0 \cos(\omega t) dt = \frac{I_0}{\omega} \sin(\omega t)$$

which gives the field:

$$E = \sigma = \frac{Q}{\epsilon_0 \pi a^2} = \frac{I_0}{\omega \epsilon_0 \pi a^2} \sin(\omega t)$$

We thus find the lowest order coefficient for $E$:

$$E_{-1} = -\frac{i I_0}{\epsilon_0 \pi a^2}$$

where the $i$ in the numerator is due to the $\sin(\omega t)$ factor, which is only real if there is an extra factor of $i$. Equation (34) thus gives the lowest order coefficient for $B$:

$$B_0 = \frac{i \mu_0 \epsilon_0}{\rho} \int_0^\rho \rho E_{-1} d\rho$$

$$= \frac{i \mu_0 \epsilon_0 \rho}{\rho} \frac{1}{2} \frac{i I_0}{\epsilon_0 \pi a^2}$$

$$= \mu_0 \rho I_0 \frac{1}{2 \pi a^2}$$

We can thus apply (33) and (34) again twice to find the next order terms up to $j = 2$:

$$E_1 = i \int B_0 d\rho$$

$$= \frac{i \mu_0 I_0 \rho^2}{4 \pi a^2}$$

$$B_2 = \frac{i \mu_0 \epsilon_0}{\rho} \int \rho E_1 d\rho$$

$$= -\frac{\mu_0 I_0 (\mu_0 \epsilon_0)}{4 \pi a^2 \rho} \int \rho^3 d\rho$$

$$= -\frac{\mu_0 I_0 (\mu_0 \epsilon_0) \rho^3}{16 \pi a^2}$$
We can now plug the coefficients into the expansions:

$$\mathbf{E} \approx -\frac{iI_0}{\epsilon_0}\frac{1}{\pi a^2} + \frac{i\mu_0 I_0 \rho^2}{4\pi a^2}$$ \quad (41)

$$\mathbf{B} \approx \frac{\mu_0 I_0 \rho}{2\pi a^2} - \frac{\mu_0 I_0 \rho^3}{16\pi a^2 c^2 \omega^2}$$ \quad (42)

We find the real fields by including the $e^{i\omega t}$ time dependence and taking the real part:

$$\mathbf{E}(r, t) \approx \left[ -\frac{I_0}{\epsilon_0}\frac{1}{\pi a^2} - \frac{\mu_0 I_0 \rho^2}{4\pi a^2} \right] \sin(\omega t)$$ \quad (43)

$$\mathbf{B}(r, t) \approx \left[ \frac{\mu_0 I_0 \rho}{2\pi a^2} - \frac{\mu_0 I_0 \rho^3}{16\pi a^2 c^2} \right] \cos(\omega t)$$ \quad (44)

(b) Inside the plates we find the energy densities (keeping only terms up to $\omega^2$):

$$w_e = \frac{1}{4}(\mathbf{E} \cdot \mathbf{D}^\ast) = \frac{\epsilon_0}{4} \left[ -\frac{iI_0}{\epsilon_0}\frac{1}{\pi a^2} + \frac{i\mu_0 I_0 \rho^2}{4\pi a^2} \right] \left[ -\frac{I_0}{\epsilon_0}\frac{1}{\pi a^2} - \frac{\mu_0 I_0 \rho^2}{4\pi a^2} \right]$$

$$\approx \frac{I_0^2}{4\epsilon_0(\pi a^2 \omega)^2} \left[ 1 - \frac{1}{2\epsilon_0\mu_0 \rho^2} \right]$$ \quad (45)

$$w_m = \frac{1}{4\mu_0}(\mathbf{B} \cdot \mathbf{B}^\ast) = \frac{1}{4\mu_0} \left[ \frac{\mu_0 I_0 \rho}{2\pi a^2} - \frac{\mu_0 I_0 \rho^3}{16\pi a^2} \right] \left[ \frac{\mu_0 I_0 \rho}{2\pi a^2} - \frac{\mu_0 I_0 \rho^3}{16\pi a^2} \right]$$

$$\approx \frac{1}{4\mu_0} \left[ \left( \frac{\mu_0 I_0 \rho}{2\pi a^2} \right)^2 - \frac{\mu_0 I_0 \rho^3}{4\pi a^2 c^2} \right] \omega^2$$

$$= \left( \frac{\mu_0 I_0 \rho}{4\pi a^2} \right)^2 \left[ 1 - \frac{\rho^2 \omega^2}{4c^2} \right]$$ \quad (46)

We can now find the integrals:

$$\int w_e d^3 r = \frac{I_0^2}{4\epsilon_0(\pi a^2 \omega)^2} 2\pi \int_0^a \rho \left( 1 - \frac{1}{2c^2} \rho^2 \omega^2 \right) d\rho$$

$$= \frac{I_0^2}{4\epsilon_0 \pi a^2 \omega^2} \left[ 1 - \frac{a^2 \omega^2}{4c^2} \right]$$ \quad (47)

$$\int w_m d^3 r = \frac{\mu_0 I_0^2}{16(\pi a^2)^2} 2\pi \int_0^a \rho \left( \rho^2 - \frac{\rho^4 \omega^2}{4c^2} \right) d\rho$$

$$= \frac{\mu_0 I_0^2}{32\pi} \left[ 1 - \frac{a^2 \omega^2}{6c^2} \right]$$ \quad (48)

Notice that the corrections to the field between the plates indicates that the charge density found earlier is not accurate. We have:

$$\sigma = \epsilon_0 \left[ \frac{iI_0}{\pi \epsilon_0 a^2} \left( 1 - \frac{\rho^2 \omega^2}{4c^2} \right) \right]$$ \quad (49)

The total charge can now be found by integrating over the plate area:

$$Q = \frac{2iI_0}{a^2 \omega} \int_0^a \rho \left( 1 - \frac{\rho^2 \omega^2}{4c^2} \right) d\rho$$

$$= \frac{iI_0}{\omega} \left( 1 - \frac{\omega^2 a^2}{8c^2} \right)$$ \quad (50)
which gives the input current:

\[ I_i = -i\omega Q = I_0 \left( 1 - \frac{\omega^2 a^2}{8c^2} \right) \]  

(51)

and its square amplitude:

\[ |I_i|^2 = I_0^2 \left( 1 - \frac{\omega^2 a^2}{4c^2} \right) \]  

(52)

Substituting into the integrals, we find:

\[ \int w_e d^3r = \frac{|I_i|^2 d}{4\epsilon_0 \pi a^2 \omega^2} \]  

(53)

\[ \int w_m d^3r = \frac{\mu_0 |I_i|^2 d}{32\pi} \left( 1 + \frac{\omega^2 a^2}{12c^2} \right) \]  

(54)

(c) We now find the reactance:

\[ \chi \approx \frac{4\omega}{|I_i|^2} \int_V (w_m - w_e) d^3x \]

\[ = \frac{4\omega}{|I_i|^2} \left[ \frac{\epsilon_0 a^2 \omega^2 \mu_0 |I_i|^2 d \left( 1 + \frac{\omega^2 a^2}{12c^2} \right) - 8|I_i|^2 d}{32\epsilon_0 \pi a^2 \omega^2} \right] \]

\[ = \frac{\mu_0 da^2}{96\pi c^2 \omega^3} + \frac{\mu_0 d}{8\pi} - \frac{d}{\epsilon_0 \pi a^2 \omega} \]  

(55)

Since the capacitive and inductive reactances have the form \( \chi_C = \frac{1}{\omega C} \) and \( \chi_L = \omega L \), we can see by inspection:

\[ C = \frac{\pi \epsilon_0 a^2}{d} \quad L = \frac{\mu_0 d}{8\pi} \]  

(56)

The resonant frequency of an RLC circuit is given by:

\[ \omega_0 = \frac{1}{\sqrt{LC}} = \frac{\sqrt{8}}{\sqrt{\epsilon_0 \mu_0 a}} = \frac{2\sqrt{2}c}{a} \]  

(57)

4: Jackson 6.18

(a) We have the Dirac expression:

\[ \mathbf{A}(\mathbf{r}) = \frac{g}{4\pi} \int_L \frac{d\ell' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \]  

(58)

Since the string is along the \( \hat{z} \) axis, we have \( \mathbf{r}' = z' \hat{z} \) and \( d\ell' = \hat{z}dz' \):

\[ \mathbf{A}(\mathbf{r}) = \frac{g}{4\pi} \int_{-\infty}^{0} \frac{\hat{z} \times (\mathbf{r} - z' \hat{z})}{|\mathbf{r} - z' \hat{z}|^3} dz' \]  

(59)
This integral is easiest to perform in cylindrical coordinates:

\[ A(r) = \frac{g}{4\pi} \int_{-\infty}^{0} \frac{\hat{z} \times (r - z') \hat{z}}{|r - z'\hat{z}|^3} dz' \]

\[ = \frac{g}{4\pi} \int_{-\infty}^{0} \frac{\hat{z} \times r}{|r - z'\hat{z}|^3} dz' \]

\[ = \frac{g}{4\pi} (\hat{z} \times r) \int_{-\infty}^{0} \frac{1}{|\rho^2 + (z - z')^2|^2} dz' \]

\[ = \frac{g}{4\pi \rho^2} (\hat{z} \times r) \left(1 - \frac{z}{\sqrt{\rho^2 + z^2}}\right) \quad (60) \]

We can evaluate the cross product in Cartesian coordinates:

\[ A(r) = \frac{g}{4\pi \rho^2} \left( y\hat{i} - x\hat{j} \right) \left(1 - \frac{z}{r}\right) \quad (61) \]

Noting that \( y\hat{i} - x\hat{j} = \rho\hat{\phi} \), we now convert to cylindrical, and then spherical coordinates:

\[ A(r) = \frac{g}{4\pi \rho} (1 - \cos(\theta)) \hat{\phi} \]

\[ = \frac{g}{4\pi r \sin(\theta)} (1 - \cos(\theta)) \hat{\phi} = \frac{g}{4\pi r} \tan\left(\frac{\theta}{2}\right) \hat{\phi} \quad (62) \]

(b) The magnetic field is thus:

\[ B = \nabla \times A = \frac{1}{rsin(\theta)} \frac{\partial}{\partial \theta} (A_\phi \sin(\theta)) \hat{r} - \frac{1}{r} \frac{\partial}{\partial r} (r A_\phi) \hat{\theta} \]

\[ = \frac{g}{4\pi r^2 \sin(\theta)} \frac{\partial}{\partial \theta} \left(\tan\left(\frac{\theta}{2}\right) \sin(\theta)\right) \hat{r} \quad (63) \]

\[ = \frac{g}{4\pi r^2} \hat{r} \quad (64) \]

(c) Converting to cylindrical coordinates:

\[ B = \frac{g}{4\pi \rho^2 + z^2} \hat{z} + \frac{g}{4\pi \rho^2 + z^2} \rho \hat{\rho} \quad (66) \]

We now find the flux through the loop:

\[ \Phi = 2\pi \int_{0}^{R \sin(\theta)} g \frac{\cos(\theta)}{4\pi \rho^2 + z^2} \rho d\rho \]

\[ = g \frac{1}{2} \int_{0}^{R \sin(\theta)} \frac{z}{(\rho^2 + z^2)^{\frac{3}{2}}} \rho d\rho \]

\[ = \frac{g}{2} \left( \frac{z}{\sqrt{\rho^2 + z^2}} \right)_{R \sin(\theta)}^{0} \]

\[ = \frac{g}{2} \cos(\theta) \left(1 - \frac{1}{\cos(\theta)}\right) \quad (67) \]
Notice that there is a discontinuity at $\theta = \frac{\pi}{2}$. We can see this by writing the flux piecewise:

- $\theta < \frac{\pi}{2}$:
  \[
  \Phi = \frac{g}{2} |\cos(\theta)| \left( \frac{1}{|\cos(\theta)|} - 1 \right)
  \] (68)

- $\theta > \frac{\pi}{2}$:
  \[
  \Phi = \frac{g}{2} |\cos(\theta)| \left( \frac{1}{|\cos(\theta)|} + 1 \right)
  \] (69)

(d) The line integral can be found easily:

\[
\oint \mathbf{A} \cdot d\ell = 2\pi R \sin(\theta) A_\phi = \frac{g}{2} (1 - \cos(\theta))
\] (70)

By inspection, we see that the flux is identical for $\theta < \frac{\pi}{2}$, but there is a difference of $g\cos(\theta)$ for $\theta > \frac{\pi}{2}$. 