MATH 731: HODGE THEORY

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These are notes from Math 731 taught by Professor Mircea Mustata in Fall 2019, \LaTeX’ed by Aleksander Horawa (who is the only person responsible for any mistakes that may be found in them).

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http://www-personal.umich.edu/~ahorawa/index.html

If you find any typos or mistakes, please let me know at ahorawa@umich.edu.

The goal of the course is to give an introduction to the basic results in Hodge theory. The prerequisites are: familiarity with algebraic varieties and sheaf cohomology (no familiarity with scheme theory is required) and with smooth manifolds (the tangent bundle, differential forms, integration).

The course will not follow a single textbook but most of the material covered can be found in [Voi07]. Additional references will be given where appropriate throughout the notes.

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1. The classical topology on a complex algebraic variety

The set up is as follows. Let $X$ be an algebraic variety over $\mathbb{C}$, i.e. (the closed points of) a reduced scheme of finite type over $\mathbb{C}$.

Suppose $X$ is affine. Then there is a closed immersion $X \hookrightarrow \mathbb{A}_\mathbb{C}^N = \mathbb{C}^N$. Note that $\mathbb{C}^N$ carries the Euclidean topology. By definition, the classical topology on $X$ is the induced subspace topology. This is well-defined: given two closed embeddings

\[ X \hookrightarrow \mathbb{C}^N \quad \text{and} \quad X \hookrightarrow \mathbb{C}^{N'} \]
there are polynomial functions $\mathbb{C}^N \to \mathbb{C}^{N'}$ and $\mathbb{C}^{N'} \to \mathbb{C}^N$ that make the triangle commute; since polynomial functions $\mathbb{C}^m \to \mathbb{C}^n$ are continuous with respect to the Euclidean topology, the two embeddings induce the same topology on $X$.

To give some more details, the two closed immersions correspond to surjective maps

$$
\begin{array}{ccc}
\mathbb{C}[x_1, \ldots, x_N] & \to & \mathcal{O}(X) \\
\uparrow & & \uparrow \\
\mathbb{C}[y_1, \ldots, y_N]
\end{array}
$$

and we may defined the map $\mathbb{C}[x_1, \ldots, x_N] \to \mathbb{C}[y_1, \ldots, y_N]$ (and vice versa) by taking any lifts from $\mathcal{O}(X)$.

**Proposition 1.1.**

1. The classical topology on $X$ is **finer** than the Zariski topology.
2. If $X$ is affine and $Z \hookrightarrow X$ is a closed subvariety, then in the classical topology on $Z$ is the subspace topology with respect to the classical topology on $X$.
3. The same is holds for an open subvariety $U \hookrightarrow X$ (if $U$ is affine).

**Proof.** It is enough to check (1) for $\mathbb{C}^N$; there, we use the definition of the Zariski topology and the fact that polynomial functions are continuous with respect to the Euclidean topology. Part (2) follows directly from the definition.

To prove (3), we first note that by covering $U$ by principal affine open subsets, we may assume that $U$ is principal affine:

$$U = \{ x \in X \mid f(x) \neq 0 \}$$

for some $f \in \mathcal{O}(X)$. Indeed, if $U = U_1 \cup \cdots \cup U_r$ and $U_i$ is principal affine with respect to $X$ (hence also $U$), then the principal case implies that the topology on $U_i$ is the subspace topology on $U_i$ with respect to both the classical topologies on $U$ and on $X$, and hence the assertion holds for $U$.

Given a closed immersion $X \hookrightarrow \mathbb{C}^N$ and $f \in \mathcal{O}(X)$, choose $g \in \mathbb{C}[x_1, \ldots, x_N]$ such that $g|_X = f$ and embed $U$ in $\mathbb{C}^{N+1}$ as $\{(u, t) \mid u \in X, g(u)t = 1\}$:

$$
\begin{array}{ccc}
X & \to & \mathbb{C}^N \\
\downarrow & & \\
U & \hookrightarrow & \mathbb{C}^{N+1}
\end{array}
$$

The two maps:

- $\mathbb{C}^{N+1} \to \mathbb{C}^N$ given by projecting only the first component,
- $\mathbb{C}^N \setminus V(g) \to \mathbb{C}^{N+1}$ given by $u \mapsto (u, \frac{1}{g(u)})$
are both continuous with respect to the Euclidean topology. Thus the topology on $U$ as a subspace of $\mathbb{C}^N$ and $\mathbb{C}^{N+1}$ coincide. □

We now glue the above construction. Let $X$ be an algebraic variety over $\mathbb{C}$. Take an affine open cover $X = \bigcup_{i=1}^{r} U_i$. Each $U_i$ has the classical topology introduced above.

We know that:

1. $U_i \cap U_j$ is open in both $U_i$ and $U_j$ with respect to the classical topology,
2. by covering $U_i \cap U_j$ by affine open subsets, Proposition 1.1 (3) implies that the classical topology on $U_i$ and $U_j$ induce the same topology on $U_i \cap U_j$.

It is now easy to check that in this case there is a unique topology on $X$ such that the subspace topology on each $U_i$ is the classical topology.

By definition, a subset $U \subseteq X$ is classically open if $U \cap U_i$ is open for all $i$ in the classical topology. Note that each $U_i$ is open in $X$ in this topology.

It is easy to see that:

1. the definition is independent of the choice of cover,
2. Proposition 1.1 extends to an arbitrary complex variety.

**Definition 1.2.** We write $X^{\text{an}}$ for the topological space $X$ with the classical topology.

**Remarks 1.3.**

1. If $f: X \to Y$ is a morphism of algebraic varieties, then $f: X^{\text{an}} \to Y^{\text{an}}$ is continuous. Indeed, by covering $X$ and $Y$ by affines, we reduce to $X$ and $Y$ affine. Given closed immersions

$$
\begin{array}{ccc}
X & \xrightarrow{f} & \mathbb{C}^m \\
\downarrow & & \downarrow g \\
Y & \xrightarrow{g} & \mathbb{C}^n
\end{array}
$$

there is a polynomial map $g$ that makes the square above commute. Since polynomial maps are continuous, this proves the assertion.

In particular, regular functions on $X$ are continuous with respect to the classical topology.

2. Every points has a countable basis of open neighborhoods in the classical topology.

3. If $X$, $Y$ are complex algebraic variety, then $(X \times Y)^{\text{an}} = X^{\text{an}} \times Y^{\text{an}}$ in the category of topological spaces (i.e. the classical topology of $X \times Y$ is the product of the classical topologies on $X$ and $Y$).

Indeed, we reduce again to $X$, $Y$ affine, and using the definition of classical topology, we reduce to $X = \mathbb{C}^m$, $Y = \mathbb{C}^n$ — then the claim follows since the Euclidean topology on $\mathbb{C}^{m+n}$ is the product topology.

**Theorem 1.4.** If $X$ is an irreducible complex algebraic variety and $U \subseteq X$ is Zariski open, nonempty, then $U$ is dense in the classical topology.

We will prove this theorem next time.
Corollary 1.5. Let $X$ be a complex algebraic variety and $U \subseteq X$ be Zariski open and dense in the Zariski topology, then $U$ is dense in the classical topology.

Proof. If $X = X_1 \cup \cdots \cup X_r$ is an irreducible decomposition and $U \subseteq X$ is Zariski dense, then $U \cap X_i \neq \emptyset$ for all $i$. Theorem 1.4 then implies that $U \cap X_i$ is dense in $X_i$ in the classical topology, and hence $U$ is dense in $X$ in the classical topology. □

The proof of Theorem 1.4 follows [Mum99].

Proof of Theorem 1.4. Step 1. We may assume that $X$ is affine. Indeed, given an open affine cover $X = U_1 \cup \cdots \cup U_r$ then $U \cap U_i \neq 0$ for all $i$, and if we know $U \cap U_i$ is dense in $(U_i)^{\text{an}}$, then $U$ is dense in $X^{\text{an}}$.

Step 2. We apply Noether normalization to get a finite surjective map

$$\pi : X \to \mathbb{C}^n.$$ 

We need to show that given and $p \in Z = X \setminus U$, we can find a sequence $y_m \to p$ with $y_m \in U$.

Let $u = \pi(p)$. Since $\pi$ is finite, $\pi(Z)$ is a closed proper subset of $\mathbb{C}^n$, there exists $g \neq 0$ in $\mathbb{C}[x_1, \ldots, x_n]$ such that $\pi(Z) \subseteq V(g)$.

Consider

$$\mathbb{R} \xrightarrow{\varphi} \mathbb{C}$$

$$t \mapsto g(tu + (1 - t)w)$$

where $w \in \mathbb{C}^n$ is such that $g(w) \neq 0$. Since $\varphi(0) \neq 0$ and $\varphi$ is a polynomial, $\varphi$ only vanishes at finite many points. Thus there is a sequence $u_m \to u$, $g(u_m) \neq 0$. After passing to a subsequence, we need to find $y_m \in \pi^{-1}(u_m)$ such that $y_m \to y$. Since $u_m \not\in V(g)$, $y_m \in U$, so $p \in U$.

Step 3. Finding the $y_m$’s. Recall that we have a map

$$\pi : X \to \mathbb{C}^n \ni u_m \to u$$

and we want to find elements in fibers over $u_m$’s. Write

$$\pi^{-1}(u) = \{p = p_1, p_2, \ldots, p_r\}.$$ 

Choose $g \in \mathcal{O}(X)$ such that $g(p) = 0$ but $g(p_j) \neq 0$ for $j \geq 2$. Since $\pi$ is finite, there exists $F \in \mathcal{O}(\mathbb{C}^n)[t]$ monic such that $F(x, g) = 0$.

Write

$$F = t^d + a_1(x)t^{d-1} + \cdots + a_r(x).$$

Since $\mathcal{O}(X)$ is a domain, we may assume that $F$ is irreducible. The map $\pi$ factors as

$$\xymatrix{ X \ar[dr]_{\pi_2} \ar[rr]_{\pi_1} & & \mathbb{C}^{n+1} \\
 & \mathbb{C}^n \ar[ul]_{\pi} & \ar[u] V(F) }$$
where \( \pi_2(x) = (\pi(x), g(x)) \). Since \( \pi \) is finite, \( \pi_2 \) is also finite. Since \( \pi \) is surjective, this shows that \( \pi_2 \) is also surjective (otherwise, \( \pi_2(X) \) has dimension less than \( n \), and hence so does \( \pi(X) \)).

Recall that \( g(p) = 0 \) and hence \( a_r(u) = 0 \). Since \( |a_r(x)| \) is the absolute value of the product of the roots of \( F(x, -) \) and \( u_m \to u \), we can choose \( t_m \) such that \( F(u_m, t_m) = 0 \) for all \( m \) and \( t_m \to 0 \).

Now choose \( y_m \in \pi_2^{-1}(u_m, t_m) \) arbitrarily.

We claim that, after passing to a subsequence, we assume that \( y_m \) converges to some \( y \). Since \( \pi(y_m) = u_m \to u \), we see that \( y \in \pi^{-1}(U) \). Since \( g(y) = \lim g(y_m) = \lim t_m = 0 \), we have that \( y \neq p_j \) for \( j \geq 2 \). Hence \( y = p \).

We thus just need to prove this claim. Choose generators \( h_1, \ldots, h_s \) of \( \mathcal{O}(X) \) to get a closed immersion \( X \hookrightarrow \mathbb{C}^s \) given by \((h_1, \ldots, h_s)\).

We use the fact that each \( h_i \) satisfies a monic equation:

\[
t^{d_i} + a_{i,1}(x)t^{d_i-1} + \cdots = 0.
\]

We want to show that each \((h_i(y_m))_m \geq 1\) is bounded. Since \( h_i \) satisfies the monic equation above, we just need to show that the coefficients \((a_{i,j}(\pi(y_m)))_m \geq 1\) is bounded. Since \( \pi(y_m) = u_m \) is convergent, it is bounded, and hence \( a_{i,j}(\pi(y_m)) \) is bounded for all \( i, j \).

\[\square\]

Corollary 1.6. Let \( X \) be an algebraic variety over \( \mathbb{C} \) and \( W \subseteq X \) is a constructible \(^1\) subset. Then \( \overline{W}^{Zar} = \overline{W}^{an} \).

Proof. Since the classical topology is finer than the Zariski topology, \( \overline{W}^{an} \subseteq \overline{W}^{Zar} \).

Since \( W \) is constructible, there exists \( U \subseteq W \) such that \( U \) is open and dense in \( \overline{W}^{Zar} \).

By Corollary 1.5, \( U \) is dense in \( \overline{W}^{Zar} \) in the classical topology. This shows that \( \overline{W}^{Zar} \subseteq \overline{W}^{an} \).

\[\square\]

Remark 1.7 (Chevalley’s theorem). Let \( f : X \to Y \) be a finite morphism. The the image of a constructible set under \( f \) is constructible.

Theorem 1.8. Suppose \( X \) is a complex algebraic variety. Then

\[
\begin{align*}
(1) & \text{ } X \text{ is separated if and only if } X^{an} \text{ is Hausdorff,} \\
(2) & \text{ } X \text{ is complete if and only if } X^{an} \text{ is compact,} \\
(3) & \text{ if } f : X \to Y \text{ is a morphism of separated varieties, then } f \text{ is proper (in the algebraic sense) if and only if } f : X^{an} \to Y^{an} \text{ is proper (i.e. } K \subseteq Y^{an} \text{ is compact implies that } f^{-1}(K) \text{ is compact).}
\end{align*}
\]

Proof. We first show (1). Recall that, in general, the diagonal map

\[\Delta : X \to X \times X\]

is a locally closed immersion. By definition, \( X \) is separated if \( \Delta \) is a closed immersion, i.e. \( \Delta(X) \) is closed in \( X \times X \) (in the Zariski topology).

\[\square\]

\(^1\)Recall that a subset is constructible if it is a finite union of locally closed sets.

\(^2\)Recall that compact means quasicompact and Hausdorff.
By Corollary 1.6, $\Delta(X)$ is Zariski closed in $X \times X$ if and only if it is closed in the classical topology. But $\Delta(X)$ is closed in $X^{an} \times X^{an}$ if and only if $X^{an}$ is Hausdorff.

We will now prove (2). Note first that $(\mathbb{P}^n)^{an}$ is compact. Indeed, we have a continuous surjective map from the $n$-sphere which is compact:

$$\left\{ z = (z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1} \mid \sum_{i=0}^{n} |z_i|^2 = 1 \right\} \rightarrow (\mathbb{P}^n)^{an}.$$ 

Suppose now that $X$ is complete. In particular, it is separated, so we know that $X^{an}$ is Hausdorff by (1). By Chow’s lemma, there is a surjective (birational) morphism

$$\pi : \tilde{X} \rightarrow X$$

with $\tilde{X}$ projective, $\tilde{X} \hookrightarrow \mathbb{P}^n$ Zariski-closed. Then $(\tilde{X})^{an}$ is closed in $(\mathbb{P}^n)^{an}$, hence compact. Therefore, $\pi(\tilde{X}^{an}) = X^{an}$ is compact.

Conversely, suppose $X^{an}$ is compact. In particular, it is Hausdorff and hence $X$ is separated. We need to show that for every algebraic variety $Y$, the projection map $X \times Y \rightarrow Y$ is closed in the Zariski topology, i.e. if $Z \subseteq X \times Y$ is Zariski-closed, $f(Z)$ is Zariski-closed.

By Chevalley’s theorem 1.7, $f(Z)$ is constructible. By Corollary 1.6, $f(Z)$ is Zariski closed if and only if it is closed in the classical topology. Suppose $y_n \in f(Z)$ is such that $\lim_{n \to \infty} y_n = b \in Y$. Then there exists $x_n \in X$ such that $(x_n, y_n) \in Z$ for all $n$. Since $X^{an}$ is compact, after passing to a subsequence, we may assume that $x_n \rightarrow a \in X$. Since $Z$ is Zariski closed in $X \times Y$, it is closed in the classical topology. Since $(x_n, y_n) \in Z$, $(a, b) \in Z$, and hence $b \in f(Z)$.

**Remark 1.9.** A related result is that if $X$ is an irreducible variety over $\mathbb{C}$, then $X^{an}$ is connected. We will come back to this when we discuss holomorphic function. A challenge exercise is to prove this statement directly.

**Exercise.** Show that if $f : X \rightarrow Y$ is a morphism of separated algebraic varieties, then $f$ is proper if and only if $f^{an}$ is proper.

**Remark 1.10.** From now on, all varieties over $\mathbb{C}$ will be assumed to be separated.

## 2. Holomorphic Functions

The reference for this section is [GH94].

### 2.1. Holomorphic functions in one variable

First, we consider the case of 1-variable functions.

**Setup.** Consider an open set $U \subseteq \mathbb{C} = \mathbb{R}^2$. All functions considered will be smooth ($C^\infty$). Coordinate functions on $U$ will be denoted by $z = x + yi$, $\bar{z} = x - yi$. We have

$$dz = dx + idy, \quad d\bar{z} = dz - idy,$$

$$dz \wedge d\bar{z} = (-2i)dx \wedge dy.$$
Dually, we have

\[ \frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \]

(this is dual to the basis of differentials given by \( dz, d\bar{z} \)).

These operators acts on \( z^m \) by

\[ \frac{\partial}{\partial z} (z^m) = mz^{m-1}, \quad \frac{\partial}{\partial \bar{z}} z^m = 0 \]

(by product rule, it is enough to check these for \( m = 1 \)).

If \( f: U \to \mathbb{C} \) is smooth, we write

\[ df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}. \]

**Exercise.** Check that \( \frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial z} \).

**Proposition 2.1** (Cauchy’s formula). Let \( \Delta \) be an open disc in \( \mathbb{C} \). If \( f \) is a smooth function on an open neighborhood of \( \Delta \), then

\[ f(z) = \frac{1}{2\pi i} \int_{\partial \Delta} \frac{f(w)}{w - z} dw + \frac{1}{2\pi i} \int_{\Delta} \frac{\partial f}{\partial \bar{w}} \frac{d\bar{w}}{w - z} \]

for all \( z \in \Delta \).

**Remark 2.2.**

(1) The loop \( \partial \Delta \) is oriented counterclockwise.
(2) Part of the statement is that the second integral is well-defined.
(3) We will define a holomorphic function to be annihilated by \( \frac{\partial}{\partial \bar{z}} \). In particular, the second integral vanishes when \( f \) is holomorphic.

**Proof.** Let \( \Delta_\epsilon \) be a disc of radius \( 0 < \epsilon \ll 1 \) around \( z \).

We apply Stokes’ formula for

\[ \eta = \frac{f(w)}{w - z} dw \]
on $\Delta \setminus \Delta_{\epsilon}$. Note that
\[
d\eta = -\frac{\partial}{\partial w} \left( \frac{f(w)}{w - z} \right) \, dw \wedge d\overline{w}.
\]
By the quotient rule and
\[
\frac{\partial}{\partial w} \left( \frac{1}{w - z} \right) = 0,
\]
we have that
\[
d\eta = -\frac{\partial f}{\partial w} \, dw \wedge d\overline{w}.
\]
Stokes’ theorem then says that
\[
-\int_{\Delta \setminus \Delta_{\epsilon}} \frac{\partial f}{\partial w} \, dw \wedge d\overline{w} = \int_{\partial \Delta} f(w) \, w - z \, dw - \int_{\partial \Delta_{\epsilon}} f(w) \, w - z \, dw.
\]
We evaluate the last integral. We change variables to $w = z + \epsilon e^{i\theta}$ for $\theta \in [0, 2\pi]$ and $dw = \epsilon i e^{i\theta} d\theta$:
\[
\int_{\partial \Delta_{\epsilon}} f(w) \, w - z \, dw = \int_{0}^{2\pi} f(z + \epsilon e^{i\theta}) \epsilon i e^{i\theta} d\theta = i \int_{0}^{2\pi} f(z + \epsilon e^{i\theta}) d\theta.
\]
For $\epsilon \to 0$, this converges (by the dominated convergence theorem, for example) to
\[
i \int_{0}^{2\pi} f(z) d\theta = 2\pi i f(z).
\]
Finally, we deal with the integral on the left hand side of Stokes’ theorem. Again, we change variables to $w = re^{i\theta} + z$ for $r \geq 0$ and $\theta \in [0, 2\pi]$ and
\[
dw = e^{i\theta} dr + ire^{i\theta} d\theta,
\]
\[
d\overline{w} = e^{-i\theta} dr - ire^{-i\theta} d\theta,
\]
\[
dw \wedge d\overline{w} = -2i r dr \wedge d\theta.
\]
Then
\[
\frac{dw \wedge d\overline{w}}{w - z} = -2i e^{-i\theta} dr \wedge d\theta.
\]
This is integrable on any compact subset of $\mathbb{C}$ and
\[
\int_{\Delta} \frac{\partial f}{\partial \overline{w}} \, dw \wedge d\overline{w} = \lim_{\epsilon \to 0} \int_{\Delta \setminus \Delta_{\epsilon}} \frac{\partial f}{\partial \overline{w}} \, dw \wedge d\overline{w}.
\]
This completes the proof. \hfill \Box

**Definition 2.3.** A smooth function $f: U \to \mathbb{C}$ is

- **holomorphic** if $\frac{\partial f}{\partial \overline{w}} = 0$ (if $f = u + iv$, this is equivalent to the Cauchy–Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$),
- **analytic** if for any $a \in U$, there exists an open disc $\Delta_{r}(a)$ centered at $a$ inside $U$ such that
\[
f(z) = \sum_{n \geq 0} c_{n} (z - a)^{n}
\]
for some $c_{n} \in \mathbb{C}$ where the convergence is absolute and uniform for $z \in \Delta_{r}(a)$.  

Theorem 2.4. A function $f$ is holomorphic if and only if it is analytic.

Proof. We start with the ‘only if’ implication. Given $a \in U$, let $\Delta$ be a disc centered at $a$ such that $\overline{\Delta} \subseteq U$. Cauchy’s formula 2.1 then shows that

$$f(z) = \frac{1}{2\pi i} \int_{\partial \Delta} \frac{f(w)}{w-z} dw.$$

We write

$$\frac{f(w)}{w-z} = \frac{f(w)}{(w-a) - (z-a)} = \frac{f(w)}{(w-a) \left(1 - \frac{z-a}{w-a}\right)}.$$

If $R$ is the radius of $\Delta$, we fix a disc $\Delta'$ centered at $a$ of radius $R' < R$. Then

$$\left|\frac{z-a}{w-a}\right| \leq \frac{R'}{R} < 1$$

for $z \in \Delta'$. Then

$$\frac{f(w)}{w-z} = \sum_{n \geq 0} \frac{f(w)}{(w-a)^{n+1}} (z-a)^n$$

converges absolutely and uniformly for $z \in \Delta'$ and $w \in \partial \Delta$. Therefore,

$$f(z) = \sum_{n \geq 0} c_n (z-a)^n$$

is absolutely and uniformly convergence for $z \in \Delta'$, where

$$c_n = \frac{1}{2\pi i} \int_{\partial \Delta} \frac{f(w)}{(w-a)^{n+1}} dw.$$

Hence $f$ is analytic.

For the ‘if’ implication, suppose $f$ is analytic and choose around $a \in U$ a small disc $\Delta$ such that $\overline{\Delta} \subseteq U$ and

$$f(z) = \sum_{n \geq 0} c_n (z-a)^n$$

converges absolutely and uniformly for $z \in \Delta$.

Since $\frac{\partial P}{\partial z} = 0$ for any polynomial $P$, if $P_n$ is the $n$th partial sum, by Cauchy’s theorem 2.1,

$$P_n(z) = \frac{1}{2\pi i} \int_{\partial \Delta} \frac{P_n(w)}{w-z} dw,$$

and hence

$$f(z) = \frac{1}{2\pi i} \int_{\partial \Delta} \frac{f(w)}{w-z} dw.$$

Therefore:

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2\pi i} \int_{\partial \Delta} \frac{\partial}{\partial \overline{z}} \left( \frac{f(w)}{w-z} \right) dw = 0$$

because the integrand is 0. Hence $f$ is holomorphic. \qed
Theorem 2.5 (\(\bar{\partial}\)-lemma in 1 variable). Let \(U \subseteq \mathbb{C}\) and \(g: U \to \mathbb{C}\) be a smooth function. If \(\Delta\) is a disc such that \(\overline{\Delta} \subseteq U\) and we define

\[
f(z) = \frac{1}{2\pi i} \int_{\Delta} g(w) \frac{dw \wedge d\bar{w}}{w - z}, \quad \text{for } z \in \Delta,
\]

then \(f\) is a smooth function and

\[
\frac{\partial f}{\partial z} = g \quad \text{on } \Delta.
\]

Remark 2.6. This theorem will later be used to compute Dolbeaux cohomology. See Lemma

Proof. Given \(z_0 \in \Delta\), choose discs centered at \(z_0\) such that \(\Delta' \subseteq \Delta'' \subseteq \Delta\) (and the closure of the previous is contained in the next).

We can write \(g = g_1 + g_2\) with \(g_1, g_2\) smooth on \(U\) such that

\[
\begin{cases}
g_1 = 0 & \text{inside } \Delta', \\
g_2 = 0 & \text{outside } \Delta''.
\end{cases}
\]

Consider separately

\[
f_i(z) = \frac{1}{2\pi i} \int_{\Delta} g_i(w) \frac{dw \wedge d\bar{w}}{w - z}, \quad \text{for } i = 1, 2.
\]

For \(z \in \Delta'\), \(f_1\) is clearly smooth and

\[
\frac{\partial f_1}{\partial z} = \frac{1}{2\pi i} \int_{\Delta} \frac{\partial}{\partial \bar{z}} \left( \frac{g_1(w)}{w - z} \right) dw \wedge d\bar{w} = 0.
\]

Now note that

\[
f_2(z) = \frac{1}{2\pi i} \int_{\Delta} g_2(w) \frac{dw \wedge d\bar{w}}{w - z}
\]

because \(g_2 = 0\) outside \(\Delta''\)

\[
= \int_0^{2\pi} e^{-i\theta} \int_0^\infty g_2(z + re^{i\theta}) dr d\theta \quad \text{where } w = z + re^{i\theta} \text{ and } \frac{dw \wedge d\bar{w}}{w - z} = -2ie^{-i\theta} dr \wedge d\theta.
\]
This implies that \( f_2 \) is smooth on \( \Delta \). After going back via the change of variables, we see that

\[
\frac{\partial f_2}{\partial \bar{z}} = \frac{1}{2\pi i} \frac{\partial g_2}{\partial w} \cdot \frac{dw \wedge d\bar{w}}{w - z}.
\]

Cauchy’s formula 2.1 for \( g_2 \) then shows that

\[
g_2(z) = \frac{1}{2\pi i} \int_{\partial \Delta} \frac{g_2(w)}{w - z} dw \wedge d\bar{w} = \frac{\partial f_2}{\partial z},
\]

on \( \Delta \). Since \( \frac{\partial f_1}{\partial \bar{z}} = 0 = g_1 \) on \( \Delta' \), this shows that

\[
\frac{\partial f}{\partial \bar{z}} = g \quad \text{on} \quad \Delta'.
\]

This shows that \( \frac{\partial f}{\partial \bar{z}} = g \) for any \( z \in \Delta \). \qed

**Remark 2.7.** The proof also shows that if \( g \) is a smooth function of \( U_1 \times \cdots \times U_n \subseteq \mathbb{C}^n \), so is \( f \). Moreover, if \( g \) is holomorphic (separately) in each of \( z_2, \ldots, z_r \), so is \( f \).

### 2.2. Holomorphic functions in several variables.

Let \( U \subseteq \mathbb{C}^n \) be open with coordinate functions \( z_1, \ldots, z_n \), \( z_j = x_j + iy_j \).

**Definition 2.8.** A smooth function \( f : U \to \mathbb{C} \) is

- **holomorphic** if it is holomorphic in each variable, i.e.
  \[
  \frac{\partial f}{\partial \bar{z}_i} = 0 \quad \text{on} \quad U.
  \]

- **analytic** if for every \( a \in U \), there is a polydisc \( B = B_r(a) = \{ z \mid |z_j - a_j| < r \text{ for all } j \} \) such that
  \[
  f(z) = \sum_{a \in \mathbb{N}^n} c_a (z - a)^a,
  \]

  where we use the multiindex notation:

  \[
  (z - a)^a = \prod_{i=1}^n (z_i - a_i)^{a_i}.
  \]

**Theorem 2.9.** If \( f : U \to \mathbb{C} \) is a smooth function, the following are equivalent:

1. \( f \) is holomorphic,
2. \( f \) is analytic,
3. for every polydisc \( \Delta = \prod_{i=1}^n \{ z_i \mid |z_i - a_i| < \alpha_i \} \subseteq U \),

\[
f(z) = \left( \frac{1}{2\pi i} \right)^n \int_{|z_i - a_i| = \alpha_i} \frac{f(w)}{(w_1 - z_1) \cdots (w_n - z_n)} dw_1 \wedge \cdots \wedge dw_n,
\]

where the integral is over the product of circles with product orientation.
Proof. It is clear that if \( f \) is analytic, it is analytic in each variable, hence holomorphic in each variable, i.e. \( f \) is holomorphic. This proves that (2) implies (1). To prove (3) implies (2), we argue as in the proof of Theorem 2.4. We get

\[
    f(z) = \sum_{\beta \in \mathbb{N}^n} c_\beta (z - a)^\beta
\]

where

\[
    c_\beta = \left( \frac{1}{2\pi i} \right)^n \int_{|z_i - a_i| = \alpha_i} \frac{f(w)}{(w - 1 - z_1)^{\beta_1 + 1} \cdots (w_n - z_n)^{\beta_n + 1}} \, dw_1 \wedge \cdots \wedge dw_n.
\]

For (1) implies (3), use Cauchy’s formula 2.1 for holomorphic functions in each variable:

\[
    f(z) = \frac{1}{2\pi i} \int_{|z_n - a_n| = \alpha_n} \frac{f(z_1, \ldots, z_{n-1}, w_n)}{z_n - w_n} \, dw_n = \ldots
\]

and use that \( f \) is continuous and Fubini’s theorem.

For an open subset \( U \subseteq \mathbb{C}^n \), we write

\[
    \mathcal{O}(U) = \{ f : U \to \mathbb{C} | f \text{ holomorphic} \}.
\]

Then the following are true.

- The subset \( \mathcal{O}(U) \subseteq C^\infty(U) \) is a \( \mathbb{C} \)-subalgebra. To prove this, use the fact that \( \frac{\partial}{\partial z_j} \) are linear (so closed under + and scalar multiplication) and derivations (so closed under product).
- If \( f \in \mathcal{O}(U) \) and \( f(z) \neq 0 \) for all \( z \in U \), then \( \frac{1}{f} \in \mathcal{O}(U) \). Indeed, \( \frac{\partial}{\partial z_j} \) satisfies the quotient rule.

**Definition 2.10.** A function \( f = (f_1, \ldots, f_m) : U \to \mathbb{C}^m \) is *holomorphic* if all \( f_j \) are holomorphic.

We start by checking that the composition of holomorphic functions is holomorphic.

Identifying \( \mathbb{C}^n = \mathbb{R}^{2n} \) with coordinates \( z_1, \ldots, z_n \) and \( \mathbb{C}^m = \mathbb{R}^{2m} \) with coordinates \( z'_1, \ldots, z'_m \), and \( f_j = u_j + iv_j \), we have a map

\[
    T_p \mathbb{R}^{2n} \xrightarrow{df_p} T_{f(p)} \mathbb{R}^{2m}
\]

which can be written explicitly as

\[
    \frac{\partial}{\partial x_j}(p) \mapsto \sum_{k=1}^m \frac{\partial u_k}{\partial x_j}(p) \frac{\partial}{\partial x'_k}(f(p)) + \sum_{k=1}^m \frac{\partial v_k}{\partial x_j}(p) \frac{\partial}{\partial y'_k}(f(p))
\]

\[
    \frac{\partial}{\partial y_j}(p) \mapsto \ldots
\]
Exercise. Show that after we tensor with $\mathbb{C}$, we have the formulas
\[
\frac{\partial}{\partial z_j}(p) \mapsto \sum_{k=1}^{m} \frac{\partial f_k}{\partial z_j}(p) \frac{\partial}{\partial z'_k}(f(p)) + \sum_{k=1}^{m} \frac{\partial f_k}{\partial z_j}(p) \frac{\partial}{\partial z_k}(f(p)),
\]
\[
\frac{\partial}{\partial \overline{z}_j}(p) \mapsto \sum_{k=1}^{m} \frac{\partial f_k}{\partial \overline{z}_j}(p) \frac{\partial}{\partial z'_k}(f(p)) + \sum_{k=1}^{m} \frac{\partial f_k}{\partial \overline{z}_j}(p) \frac{\partial}{\partial z_k}(f(p)).
\]

The upshot is that if $f$ is holomorphic, then $\frac{\partial f}{\partial z_j} = \frac{\partial f}{\partial \overline{z}_j} = 0$. Therefore
\[
\text{span} \left( \frac{\partial}{\partial z_j} \middle| j \right) \rightarrow \text{span} \left( \frac{\partial}{\partial z'_k} \middle| k \right),
\]
\[
\text{span} \left( \frac{\partial}{\partial \overline{z}_j} \middle| j \right) \rightarrow \text{span} \left( \frac{\partial}{\partial z'_k} \middle| k \right).
\]

Consider maps $U \xrightarrow{f} V \xrightarrow{g} \mathbb{C}$. The upshot is that if $g$ is also holomorphic, then $g \circ f$ is also holomorphic. In fact, for any $g$, we have that
\[
\frac{\partial (g \circ f)}{\partial \overline{z}_j}(p) = \sum_{k=1}^{m} \frac{\partial f_k}{\partial \overline{z}_j} \left( \frac{\partial g}{\partial z'_k} \circ g \right).
\]

This implies that
\begin{itemize}
  \item if $f$, $g$ is holomorphic, then $g \circ f$ is also holomorphic,
  \item if $f$ is a holomorphic diffeomorphism $f: U \rightarrow V$ (for $U, V \subseteq \mathbb{C}^n$), $g \circ f$ is holomorphic, and the matrix \( \left( \frac{\partial f_i}{\partial z_j} \right)_{i,j} \) is invertible at every point, then $g$ is holomorphic.
\end{itemize}

Moreover, if both $f$ and $g: V \rightarrow \mathbb{C}$ are holomorphic, then
\[
\frac{\partial (g \circ f)}{\partial z_j} = \sum_{k} \frac{\partial f_k}{\partial z_j} \left( \frac{\partial g_k}{\partial z'_k} \circ g \right).
\]

Remark 2.11. We only assume that $g: V \rightarrow \mathbb{C}$ to simplify the notation. The above assertions also hold for $g: V \rightarrow \mathbb{C}^p$ in general.

Let $U \subseteq \mathbb{C}^n$ and $f: U \rightarrow \mathbb{C}^n$. The next goal is to prove the inverse function theorem. We want to compare the real Jacobian of $f$ with $\text{det} \left( \frac{\partial f_i}{\partial z_j} \right)$, and deduce it from the inverse function theorem for smooth functions.

Write $f = (f_1, \ldots, f_n)$ and $(z_1, \ldots, z_n)$ for the variables on $U$ and $\mathbb{C}^n$. One can compute that:
\[
f^*(dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n) = \left( \text{determinant of real Jacobian of } f \right) dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n,
\]
\[
dz_j \wedge dz_j = (dx_j + idy_j) \wedge (dx_j - idy_j) = (-2i)dx_j \wedge dy_j,
\]
and hence (after tensoring with \( \mathbb{C} \)):

\[
f^*(dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_n) = \left( \frac{\text{determinant of}}{\text{real Jacobian}} \right) dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_n.
\]

The left hand side is equal to

\[
df_1 \wedge df_1 \wedge \cdots \wedge df_n \wedge df_n.
\]

Recall that

\[
df = \sum_j \left( \frac{\partial f}{\partial x_j} \, dx_j + \frac{\partial f}{\partial y_j} \, dy_j \right) = \sum_j \left( \frac{\partial f}{\partial z_j} \, dz_j + \frac{\partial f}{\partial \overline{z}_j} \, d\overline{z}_j \right).
\]

In particular, if \( f \) is holomorphic, then each \( f_k \) is holomorphic, and hence

\[
df_k = \sum_{j=1}^n \frac{\partial f_k}{\partial z_j} dz_j,
\]

\[
df_k = \sum_{j=1}^n \frac{\partial f_k}{\partial \overline{z}_j} d\overline{z}_j.
\]

Finally, this shows that

\[
df_1 \wedge df_1 \wedge \cdots \wedge df_n \wedge df_n = \left( \det \frac{\partial f_j}{\partial z_k} \right) \left( \det \frac{\partial f_j}{\partial \overline{z}_k} \right) dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_n
\]

\[
= \left| \det \frac{\partial f_j}{\partial z_k} \right|^2 dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_n.
\]

The overall conclusion is that

\[
\left( \frac{\text{determinant of the}}{\text{real Jacobian}} \right) = \left| \frac{\text{determinant of the}}{\text{complex Jacobian}} \right|^2.
\]

In particular:

- the left hand size is \( \geq 0 \),
- the left hand size is \( = 0 \) if and only if the right hand side is.

**Theorem 2.12** (Holomorphic inverse function theorem). If \( U \subseteq \mathbb{C}^n \) is open and \( f : U \to \mathbb{C}^n \) is holomorphic. Then for \( p \in U \) such that \( \det \left( \frac{\partial f_j}{\partial z_k} \right)(p) \neq 0 \), there are open neighborhoods \( U' \subseteq U \) of \( p \) and \( V' \subseteq \mathbb{C}^n \) of \( f(p) \) such that \( f \) gives a bijective map \( U' \to V' \) and its inverse is holomorphic.

**Proof.** By the previous discussion, the hypothesis implies that the determinant of the real Jacobian of \( f \) is nonzero at \( p \). The inverse function theorem for smooth maps implies that there are open subsets \( U' \), \( V' \) as above such that \( U' \xrightarrow{f} V' \) is bijective and its inverse is smooth. We may assume that \( \det \left( \frac{\partial f_j}{\partial z_k} \right) \neq 0 \) on \( U' \), and hence \( g \) is holomorphic on \( V \) since \( f \) and \( g \circ f \) are. \( \square \)

**Remark 2.13.**
(1) If \( f: U \to \mathbb{C} \) is holomorphic, then \( \frac{\partial^{|\alpha|} f}{\partial z^\alpha} \) is holomorphic for all \( \alpha \) (since \( \frac{\partial}{\partial z_j} \) and \( \frac{\partial}{\partial z_k} \) commute).

(2) If \( a \in U \) is such that \( \frac{\partial^{|\alpha|}}{\partial z^\alpha}(a) = 0 \) for all \( \alpha \), then \( f \equiv 0 \) in a neighborhood of \( a \). Indeed, if \( B = \{ z \mid |z_i - a_i| < \epsilon \text{ for all } i \} \) is such that \( B \subseteq U \), then

\[
f(z) = \left(\frac{1}{2\pi i}\right)^n \int_{|z_i - a_i| = \alpha_i} \frac{f(w)}{(w_1 - z_1) \cdots (w_n - z_n)} dw_1 \wedge \cdots \wedge dw_n,
\]

and using this we get

\[
\sum_{\alpha \in \mathbb{N}^n} c_\alpha(z - a)^\alpha
\]

for \( z \in B \), where

\[
c_\alpha = \left(\frac{1}{2\pi i}\right)^n \int_{|z_i - a_i| = \alpha_i} \frac{f(w)}{(w - 1 - z_1)^{\alpha_1+1} \cdots (w_n - z_n)^{\alpha_n+1}} dw_1 \wedge \cdots \wedge dw_n
\]

\[
= \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial z^\alpha}(a)
\]

\[
= 0.
\]

**Proposition 2.14.** Suppose \( f: U \to \mathbb{C} \) is holomorphic and \( U \) is connected. If \( f = 0 \) on some \( V \subseteq U \) open, then \( f = 0 \).

**Proof.** Let \( U' = \{ z \in U \mid f = 0 \) on some open neighborhood of \( z \} \). This set is non-empty by hypothesis and clearly open. It is enough to show that it is closed. If \( z_n \in U' \) converges to \( a \), then for every \( \alpha \)

\[
\frac{\partial^{|\alpha|}}{\partial z^\alpha}(z_n) = 0
\]

and hence

\[
\frac{\partial^{|\alpha|}}{\partial z^\alpha}(a) = 0.
\]

This holds for all \( \alpha \), so \( f = 0 \) in a neighborhood of \( a \). Thus \( a \in U' \). \( \square \)

The next goal is to state and prove the maximum modulus principle.

**Theorem 2.15** (Maximum modulus principle). If \( U \subseteq \mathbb{C}^n \) is open and connected, and \( f: U \to \mathbb{C} \) is a holomorphic function such that \( |f| \) has a local max at \( a \in U \), then \( f \) is constant.

**Proof.** By Proposition 2.14, it is enough to show that there is an open neighborhood \( U_0 \) of \( a \) such that \( f \) is constant on \( U_0 \).

We first reduce to the case \( n = 1 \). Take \( U_0 \) to be an polydisc containing \( a \),

\[
U_0 = \{ z \mid |z_i - a_i| < \epsilon \text{ for all } i \}.
\]

For any \( z \in U_0 \), consider the 1-variable function

\[
\mathbb{C} \ni w \mapsto f(wa + (1 - w)z) \in \mathbb{C} \quad \text{for } |wa_i - (1 - w)z_i - a_i| < \epsilon.
\]
This function is defined on an open subset of $\mathbb{C}$ containing 0 and 1. It is a holomorphic function and its absolute value has a local maximum at $w = 1$. The 1-variable case then implies that this is constant, and hence $f(z) = f(a)$.

We now prove the theorem for $n = 1$. Let $\Delta = B_R(a)$ be a disc centered at $a$ such that $\overline{\Delta} \subseteq U$. Cauchy’s formula 2.1 implies that

$$f(a) = \frac{1}{2\pi i} \int_{\partial \Delta} \frac{f(w)}{w-a} dw = \frac{1}{2\pi i} \int_0^1 f\left(a + \operatorname{Re}^{2\pi i\theta}\right) \operatorname{Re}^{2\pi i\theta} \cdot 2\pi i d\theta = \int_0^1 f\left(a + \operatorname{Re}^{2\pi i\theta}\right) d\theta$$

where $w = a + \operatorname{Re}^{2\pi i\theta}$. Therefore,

$$|f(a)| \leq \int_0^1 |f\left(a + \operatorname{Re}^{2\pi i\theta}\right)| d\theta \leq |f(a)| \int_0^1 d\theta = |f(a)|,$$

assuming that $|f(z)| \leq |f(a)|$ in a neighborhood of $\overline{\Delta}$ (this is true for $R$ small enough). Therefore, the above inequalities are all equalities. Since $(\ast)$ is an equality and $f$ is continuous, we conclude that $|f(z)| = |f(a)|$ for all $z \in \partial \Delta$.

The same holds for any $0 < R' \leq R$, so $|f(z)|$ is constant in an open neighborhood of $a$. □

**Exercise.** Show that if $f = u + iv$ is holomorphic on some open connected subset and $u^2 + v^2$ is constant, then $f$ is constant. (Apply $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, and Cauchy–Riemann equations.)

### 3. Complex manifolds

If $U \subseteq \mathbb{C}^n$ is an open subset, consider the sheaf $\mathcal{O}_U$ of $\mathbb{C}$-algebras on $U$ defined by

$$\mathcal{O}_U(V) = \{ f : V \to \mathbb{C} \mid f \text{ holomorphic} \}.$$

We check that this is indeed a sheaf:

1. have restriction maps: if $V_1 \subseteq V_2$ and $f$ is holomorphic on $V_2$, then $f$ is holomorphic on $V_1$,
2. if $V = \bigcup V_i$ and $\varphi_i : V_i \to \mathbb{C}$ are holomorphic functions such that $\varphi_i|_{V_i \cap V_j} = \varphi_j|_{V_i \cap V_j}$, then there exists a unique $\varphi : V \to \mathbb{C}$ such that $\varphi|_{V_i} = \varphi_i$ for all $i$; indeed, if $\varphi$ is such that $\varphi|_{V_i}$ is holomorphic for all $i$, then $\varphi$ is holomorphic.

**Definition 3.1.** A complex manifold of dimension $n$ is a pair $(X, \mathcal{O}_X)$ where

1. $X$ is a topological space, assumed Hausdorff and having a countable basis of open subsets,
2. $\mathcal{O}_X \subseteq \mathcal{C}_X$ is a subsheaf of the sheaf of continuous $\mathbb{C}$-valued functions on $X$,

such that $X$ can be written as

$$X = \bigcup_i U_i, \quad U_i \subseteq X \text{ open}$$

such that each $(U_i, \mathcal{O}_{U_i}) \cong (V_i, \mathcal{O}_{V_i})$ for some $V_i \subseteq \mathbb{C}^n$ is open with $\mathcal{O}_{V_i}$ is the sheaf of holomorphic functions on $V_i$. 
Remark 3.2. Suppose \( V_1, V_2 \subseteq \mathbb{C}^n \) are open and \( f: (V_1, \mathcal{O}_{V_1}) \to (V_2, \mathcal{O}_{V_2}) \) is an isomorphism, i.e. a homeomorphism \( f: V_1 \to V_2 \) which induces an isomorphism of sheaves: for all \( U \subseteq V_2 \),
\[
\mathcal{O}(V_2) \xrightarrow{\cong} \mathcal{O}(f^{-1}(V_2)),
\]
\[
\varphi \mapsto \varphi \circ f.
\]
This forces \( f \) and \( f^{-1} \) to be holomorphic functions. The converse is also true.

Definition 3.3.

1. If \( (X, \mathcal{O}_X) \) is a complex manifold, the section of \( \mathcal{O}_X \) are the holomorphic functions on \( X \).
2. If \( (X, \mathcal{O}_X) \) and \( (Y, \mathcal{O}_Y) \) are complex manifolds, then a holomorphic map
\[
(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y)
\]
is a continuous map \( f: X \to Y \) that induces a map of sheaves, i.e. for any \( V \subseteq Y \) open and \( \varphi \in \mathcal{O}_Y(V) \), we have that \( \varphi \circ f \in \mathcal{O}(f^{-1}(V)) \).

Remark 3.4. If \( X \subseteq \mathbb{C}^n \) and \( Y \subseteq \mathbb{C}^m \) are open subsets, this coincides with the previous definition.

Remark 3.5. If \( U \subseteq \mathbb{C}^n, \ p \in U \),
\[
\mathcal{O}_{\mathbb{C}^n, p} = \lim_{V \ni p} \mathcal{O}_U(V).
\]
To check that this is a local ring, we note that we have a map
\[
\mathcal{O}_{U, p} = \lim_{V \ni p} \mathcal{O}_U(V) \to \mathbb{C}
\]
\[
(V, \varphi) \mapsto \varphi(p)
\]
whose kernel \( \{(V, \varphi) \mid \varphi(p) = 0\} = \mathfrak{m} \) is the unique maximal ideal. Indeed, if \( (V, \varphi) \not\in \mathfrak{m} \), we may assume that \( \varphi(z) \neq 0 \) for all \( z \in V \), and hence \( \frac{1}{\varphi} \in \mathcal{O}(V) \). Hence \( (\mathcal{O}_{U, p}, \mathfrak{m}) \) is a local ring.

Remark 3.6. All such local rings for manifolds of fixed dimension are isomorphic. This is very different from the algebraic case.

Remark 3.7. One can define complex manifolds using atlases: \( X \) is a topological space with suitable properties and \( X = \bigcup U_i \) is an open cover together with homeomorphisms \( \varphi_i: U_i \xrightarrow{\cong} V_i \subseteq \mathbb{C}^n \), where \( V_i \subseteq \mathbb{C}^n \) are open, such that for all \( i, j \) the map
\[
\varphi_i(U_i \cap U_j) \xrightarrow{\varphi_j \circ \varphi_i^{-1}} \varphi_j(U_i \cap U_j)
\]
is biholomorphic.

We identify two such objects \( (X, \mathcal{A}), (X, \mathcal{A}') \) if \( \mathcal{A} \) and \( \mathcal{A}' \) are compatible.

Remark 3.8. It is clear from the definition via atlases (Remark 3.7), using that holomorphic maps \( \mathbb{C}^n \supseteq U \to \mathbb{C}^n \) are smooth, that every complex manifold of dimension \( n \) has an underlying real smooth manifold structure of dimension \( 2n \). To avoid confusion, we will write \( X_\mathbb{R} \) for this real smooth manifold (if necessary). We have an inclusion of sheaves
\[
\mathcal{O}_X \subseteq \mathcal{C}^\infty_{X, \mathbb{C}}.
\]
Next, we will discuss:

- vector bundles in the smooth/holomorphic category,
- submanifolds,
- complex manifold associated to a smooth complex algebraic variety.

3.1. Vector bundles. If $M$ is a smooth real manifold, a real (or complex) vector bundle of rank $r$ on $M$ is a smooth manifold $E$ with a smooth map $E \to M$ such that for any $x \in M$, $\pi^{-1}(x)$ has the structure of a vector space over $\mathbb{R}$ (respectively $\mathbb{C}$) of dimension $r$ such that there is an open cover $M = \bigcup U_i$ such that we have isomorphisms

$$\pi^{-1}(U_i) \xrightarrow{\cong} U_i \times \mathbb{R}^r \quad \text{(resp. } U_i \times \mathbb{C}^r)$$

(respectively, $U_i \times \mathbb{C}^r$), inducing linear maps on the fibers.

Given such $E$, we get a sheaf $E$ on $M$ such that $E(U) = \{ s : U \to E \text{ smooth} \mid \pi \circ s = 1_U \}$.

This gives an equivalence of categories

$$\begin{align*}
\{ \text{real (complex) vector bundles} & \quad \text{on } M \text{ (of rank } r) \} \\
\text{of } \mathcal{C}^\infty_{M,\mathbb{R}} \text{-modules (} \mathcal{O}^\infty_{M,\mathbb{C}} \text{-modules) } \end{align*}$$

We will consider the corresponding notion in the category of complex manifolds. For complex vector bundles, we assume that $E$ is a complex manifold, $\pi$ is holomorphic.

These correspond to locally free sheaves of $\mathcal{O}_M$-modules. Note that associated to such $E$, we will have: sheaves of smooth sections and sheaves of holomorphic sections.

Definition 3.9. Let $X$ be a complex manifold of dimension $n$. A closed submanifold of $X$ of codimension $r$ is a closed subset $Y \subseteq X$ such that for all $p \in Y$, there is a chart $p \in U \xrightarrow{\varphi} V \subseteq \mathbb{C}^n$ such that

$$\varphi(U \cap Y) = \{ z \in V \mid z_1 = \cdots = z_r = 0 \}.$$ 

It is easy to see that by restricting such charts to $Y$, we get a holomorphic atlas on $Y$, making it a complex manifold of dimension $n - r$.

The universal property of submanifolds is: given a holomorphic $g : Z \to X$ such that $g(Z) \subseteq Y$, there is a unique holomorphic map $g' : Z \to Y$ such that $\text{incl} \circ g' = g$.

Proposition 3.10. If $U \subseteq \mathbb{C}^n$ is open and $f_1, \ldots, f_r \in \mathcal{O}(U)$ are such that

$$\text{rank} \left( \frac{\partial f_i}{\partial z_j}(p) \right) = r \leq n$$

for all $p \in U$, then

$$Y = \{ z \in U \mid f_1(z) = f_2(z) = \cdots = f_r(z) = 0 \}$$
is a closed submanifold of $U$ of codimension $r$.

**Proof.** Given $p \in Y$, we may assume that $\text{rank} \left( \frac{\partial f_i}{\partial z_j}(p) \right)_{1 \leq i, j \leq n} \neq 0$. Define

$$\varphi : U \to \mathbb{C}^n,$$

$$z \mapsto (f_1(z), \ldots, f_r(z), z_{r+1}, \ldots, z_n).$$

Then

$$\det \left( \frac{\partial \varphi_i}{\partial z_j}(p) \right) \neq 0$$

and we apply the Inverse Function Theorem 2.12 to see that $\varphi$ is biholomorphic in some neighborhood of $p$. In the neighborhood, $\varphi$ is the desired chart. \qed

Basic properties of holomorphic functions we discussed extend to this setting. We recall a few of them for completeness. Let $X$ be a complex manifold.

1. If $f \in \mathcal{O}(X)$ is such that $f|_U = 0$ for some $U \subseteq X$ open, and $X$ is connected, then $f = 0$.

2. (Maximum modulus principle) If $f \in \mathcal{O}(X)$ is such that $|f|$ has a local max, $X$ is connected, then $f$ is constant.

**Corollary 3.11.** If $X$ is a compact connected complex manifold, then $\Gamma(X, \mathcal{O}_X) = \mathbb{C}$.

**Proof.** Since $X$ is compact, for any $f \in \mathcal{O}(X)$, $|f|$ has a maximum. Then the maximum modulus principle implies that $f$ is constant. \qed

### 3.2. The complex manifold associated to a smooth complex algebraic variety.

Let $X$ be a smooth complex algebraic variety of pure dimension $n$. Choose an affine open subset $U \subseteq X$ and let $U \hookrightarrow \mathbb{C}^N$ be a closed immersion, $r = N - n$. Since $U$ and $\mathbb{C}^N$ are smooth, can cover $\mathbb{C}^N$ by open subsets $V_i$ (in the Zariski topology) such that if $V_i \cap U \neq \emptyset$ then $V_i \cap U \hookrightarrow V_i$ is cut out by $r$ equations $f_1, \ldots, f_r \in \mathcal{O}(V_i)$ with

$$\text{rank} \left( \frac{\partial f_i}{\partial z_j}(p) \right) = r$$

for all $p \in V_i \cap U$.

Applying Proposition 3.10, each $V_i \cap U \hookrightarrow V_i$ is a closed complex submanifold of codimension $r$.

**Exercise.** Check that the resulting transition maps are holomorphic, using the fact that rational maps are holomorphic.

**Exercise.** Show that if $f : X \to Y$ is a morphism between smooth complex algebraic varieties, then the induced map $X^{an} \to Y^{an}$ is holomorphic.

We now discuss an application.

**Theorem 3.12.** If $X$ is a connected complex algebraic variety, then $X^{an}$ is connected.

We first prove this theorem when $X$ is a smooth connected projective curve over $\mathbb{C}$. 

Proof when $X$ is a smooth, projective curve. We first prove this when $X$ is a smooth connected projective curve over $\mathbb{C}$. We know that $X^{an}$ is a 1-dimensional complex manifold, which is compact since $X$ is complete (by Theorem 1.8).

Suppose that $X^{an} = U \cup V$ is a disjoint union with $U, V$ open in $X^{an}$ and nonempty. Take $P \in U$. If $n \gg 0$ ($n \geq 2 \cdot \text{genus}(X)$), $\mathcal{O}_X(nP)$ is globally generated. Then there exists $s \in \Gamma(X, \mathcal{O}_X(nP))$ which does not vanish at $P$. Then $nP \sim Q_1 + \cdots + Q_n$ for $Q_i \neq P$ so there exists $\varphi \in \mathbb{C}(X)$ such that $\text{div}(\varphi) = (Q_1 + \cdots + Q_n) - nP$, so $\varphi$ gives a regular function $X \setminus \{P\} \to \mathbb{C}$. Note that it is holomorphic. By restricting to $V$, we get a holomorphic map $V \xrightarrow{g} \mathbb{C}$. Since $V$ is a compact complex manifold, $g$ is constant by Corollary 3.11. In particular, $\varphi$ takes the same value infinitely many times, so $\varphi$ is constant, and hence $\div \varphi = 0$. This is a contradiction. □

To reduce the general case to $\dim X = 1$, we use the following result.

**Proposition 3.13.** Let $X$ be an algebraic variety over $k = \overline{k}$. For any $x_1, x_2 \in X$, there is an irreducible curve $C \subseteq X$ such that $x_1, x_2 \in C$.

**Proof.** We may assume that $n = \dim X \geq 2$.

(1) By Chow’s lemma, there is a surjective morphism $\pi: \tilde{X} \to X$ where $\tilde{X}$ is irreducible and quasi-projective. If $\tilde{x}_1, \tilde{x}_2$ lie above $x_1, x_2$, it is enough to find a curve $\tilde{C}$ on $\tilde{X}$ through $\tilde{x}_1, \tilde{x}_2$ and take $C = \pi(\tilde{C})$. We may hence assume $X$ is quasi-projective.

(2) Choose a locally closed immersion $X \hookrightarrow \mathbb{P}^N$. It is enough to prove the statement for $X$. We may hence assume that $X$ is projective.

Consider the blow up of $X$ at $\{x_1, x_2\}$:

$$ Y = \text{Bl}_{\{x_1, x_2\}} X \leftarrow E_i = p^{-1}(x_i) $$

$$ \downarrow p $$

$$ X $$

where $\dim E_i = n - 1$.

The variety $Y$ is projective since $X$ is, so we may choose an embedding $Y \hookrightarrow \mathbb{P}^N$. Cut $Y$ with $n - 1$ general hyperplane $H_1, \ldots, H_{n-1}$. Since $\dim E_i = n - 1$,

$$ E_i \cap H_1 \cap \cdots \cap H_{n-1} \neq \emptyset \quad \text{for } i = 1, 2. $$

If $Z = Y \cap H_1 \cap \cdots \cap H_{n-1}$, the curve $C = p(Z)$ satisfies the requirements. Using Bertini’s Theorem: a general hyperplane section of an irreducible projective variety of dimension $\geq 2$ is irreducible, and hence $Z$ is irreducible.

We need to assume that $\dim(Z \cap E_i) = 0$ for $i = 1, 2$. This is okay since the $H_i$s are general. □

We can now finish the proof of Theorem 3.12.
Proof of Theorem 3.12. We just have to reduce to the smooth, projective curve case from the general case.

First, we may assume that $X$ is irreducible (since by hypothesis we can go from any irreducible component to any other one via points of intersection).

For an irreducible algebraic variety $X$ over $k = \overline{k}$, for any $x, y \in X$, there is an irreducible curve $C$ such that $x, y \in C$ by Proposition 3.13. We may hence assume that $X$ is an irreducible curve.

If $\tilde{X} \to X$ is the normalization, it is enough to show that $\tilde{X}^{\text{an}}$ is connected. We may hence assume that $X$ is smooth.

Finally, let $X \subseteq \overline{X}$ where $\overline{X}$ is a smooth, projective, connected curve. We showed last time that $\overline{X}^{\text{an}}$ is connected.

We now use that if $M$ is smooth real manifold of dimension $\geq 2$, $p \in M$, and $M$ is connected, then $M \setminus \{p\}$ is also connected.

Indeed, if $M \setminus \{p\} = U \cup V$ is a disjoint union of open non-empty sets, then $p \in U \cap V$ because $M$ is connected. Choose a neighborhood $W$ of $p$ such that $W$ is isomorphic to a ball. Then $W \setminus \{p\}$ is disconnected. This is a contradiction, since it is clearly path-connected. □

3.3. More examples of complex manifolds. Suppose $X$ is a complex manifold and $G$ is a group acting on $X$ via holomorphic maps. Suppose

(1) for all $x \in X$, there exists an open neighborhood $U \ni x$ such that $U \cap gU \neq \emptyset$ implies $g = e$ (this is sometimes called a properly discontinuous action),

(2) for all $x, y \in X$ such that $x, y$ are not in the same orbit, there exist open neighborhoods $U \ni x, V \ni y$ such that $gU \cap V = \emptyset$ for all $g$.

Note that (1) implies that the quotient map $\pi : X \to X/G$ is a covering space. Moreover, since the transition maps are holomorphic, there is a unique complex manifold structure on $X/G$ such that $\pi$ is holomorphic. Condition (2) implies that $X/G$ is Hausdorff.

Example 3.14 (Complex tori). Let $V$ be an $n$-dimensional complex vector space and $\Lambda \subseteq V$ be a lattice (i.e. a free abelian group of rank $2n$ such that $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} \cong V$). The natural action of $\Lambda$ by translations satisfies (1) and (2) above, and hence

$$V \xrightarrow{\mathbb{Z}} Z = V/\Lambda$$

gives a complex manifold $Z$. Note that

$$V/\Lambda \cong \mathbb{R}^{2n}/\mathbb{Z}^{2n} \cong (S^1)^{2n},$$

and hence topologically, $Z$ is a $2n$-dimensional torus.

When $n = 1$, the resulting $Z$ is an analytic construction of elliptic curves, which are algebraic. We will see that for $n \geq 2$, most of these do not come from algebraic varieties. However, they are still Kähler manifold.

Example 3.15 (Hopf surface). Consider the action of $\mathbb{Z}$ on $\mathbb{C}^2 \setminus \{(0, 0)\}$, where the generator $\gamma$ of $\mathbb{Z}$ acts by $(z_1, z_2) \mapsto (2z_1, 2z_2)$. 
This clearly satisfies conditions (1) and (2), so we get a complex manifold structure on the quotient.

We have a diffeomorphism:

\[ \mathbb{C}^2 \setminus \{(0,0)\} \xrightarrow{\cong} S^3 \times \mathbb{R} \]

\[ (z_1, z_2) \mapsto \left( \frac{1}{\sqrt{|z_1|^2 + |z_2|^2}} (z_1, z_2), \log \sqrt{|z_1|^2 + |z_2|^2} \right) \]

under which the action of \( \gamma \) translates to

\[ (u, t) \mapsto (u, t + \log 2). \]

Therefore, the Hopf surface is topologically

\[ \mathbb{C}^2 \setminus \{(0,0)\}/\mathbb{Z} \cong S^3 \times S^1. \]

We will later see these manifolds are not even Kähler, and hence do not come from algebraic surfaces.

3.4. Orientation. If \( V \) is a 1-dimensional real vector space, an orientation on \( V \) is a choice of element in \( V/\mathbb{R}^*_0 \). Note that an orientation of \( V \) is the same as an orientation of \( V^* \).

If \( V \) is an \( n \)-dimensional vector space, an orientation on \( V \) is an orientation on \( \Lambda^n V \).

If \( X \) is a smooth real manifold and \( E \) is a real vector bundle on \( X \), an orientation on \( E \) is a compatible system of orientations on \( E(x) \) for all \( x \in X \), i.e. locally have trivializations \( \pi^{-1}(U) \cong U \times \mathbb{R}^r \) where \( \pi : E \to X \), preserving the orientations on the fibers.

Note that an orientation on \( E \) corresponds to an orientation on \( E^* \).

Definition 3.16. An orientation on a smooth real manifold \( X \) is an orientation on the tangent bundle \( TX \) (or equivalently on the cotangent bundle \( T^*X \)).

Giving an orientation is equivalent to giving a system of charts such that for all transition maps

\[ f = (f_1, \ldots, f_n) : U \to \mathbb{R}^n, \quad \det \frac{\partial f_i}{\partial x_j} > 0. \]

Note that if \( X \) is a complex manifold and we consider the smooth manifold structure, we saw that if we take a system of holomorphic charts, then for the transition maps \( f = (f_1, \ldots, f_n) : U \to \mathbb{C}^n \) where \( U \subseteq \mathbb{C}^n \),

\[ \det(\text{real Jacobian}) = \det \left| \frac{\partial f_i}{\partial z_j} \right|^2 > 0. \]

Therefore, we have a canonical orientation on \( X \).

By convention: given chart \( f : U \to \mathbb{C}^n \), the orientation on \( U \) corresponds to the orientation on \( \mathbb{C}^n \) given by the top form \( dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n \).
3.5. The analytic space associated to an algebraic variety. We first discuss the local model. For $U \subseteq \mathbb{C}^n$ open in the classical topology, $f_1, \ldots, f_r \in \mathcal{O}(U)$, consider

$$Z = \{ u \in U \mid f_1(u) = \cdots = f_r(u) = 0 \}.$$ 

Consider on $Z$ the sheaf given by

$$\mathcal{O}_Z(V) = \{ f: V \to \mathbb{C} \mid \text{locally } f \text{ extends to a holomorphic function on an open subset in } \mathbb{C}^n \}.$$ 

If $Z \hookrightarrow U$ is the inclusion, we get a map

$$\mathcal{O}_U \to j_*\mathcal{O}_Z$$

and the kernel is $\mathcal{I}_{Z/U}$ given by

$$\Gamma(V, \mathcal{I}_{Z/U}) = \{ f: V \to \mathbb{C} \mid f|_{V \cap Z} = 0 \}.$$ 

Note that $(Z, \mathcal{O}_Z)$ is a locally ringed space.

**Definition 3.17.** A (reduced) analytic space is a locally ringed space $(X, \mathcal{O}_X)$ such that

1. $X$ is a Hausdorff topological space with a countable basis for the topology,
2. there is an open cover $X = \bigcup_i W_i$ such that each $(W_i, \mathcal{O}_{W_i})$ is isomorphic as a locally ringed space to a local model as above.

The sections of $\mathcal{O}_X$ are called holomorphic functions on $X$.

**Definition 3.18.** A holomorphic map between analytic spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is a continuous map $f: X \to Y$ such that for any $V \subseteq Y$ open and $\varphi \in \mathcal{O}_Y(V)$, we have $\varphi \circ f \in \mathcal{O}_X(\varphi^{-1}(V))$.

**Examples 3.19.**

1. Every complex manifold is canonically an analytic space.
2. If $X$ is a separated algebraic variety, we have a sheaf $\mathcal{O}_{X^{an}}$ on $X^{an}$ that makes it an analytic space. We do it locally. Choose affine open subsets covering $X$; each such open subspace covering $U$ has a closed immersion $U \hookrightarrow \mathbb{C}^N$ (cut out by finitely many polynomials), so we have a sheaf $\mathcal{O}_{U^{an}}$ on $U^{an}$ making it an analytic space. It is easy to check that these sheaves are compatible on intersections, so we get a sheaf $\mathcal{O}_{X^{an}}$ on $X^{an}$.

We get in this way a functor

$$\{ \text{complex algebraic varieties} \} \to \{ \text{analytic spaces} \}.$$ 

However, in this class, we deal with smooth varieties, and hence we only have to work with complex manifolds.

3.6. Comparison results. Let $X$ be a complex algebraic variety. As we saw above, it has an associated analytic space $X^{an}$.

We have a morphism of locally ringed spaces:

$$(\varphi, \varphi^\#): (X^{an}, \mathcal{O}_{X^{an}}) \to (X, \mathcal{O}_X).$$
defined by \( \varphi(x) = x \) and

\[
\varphi^\#: \mathcal{O}_X \to \varphi_* \mathcal{O}_{X^{an}} \\
\mathcal{O}_X(U) \to \mathcal{O}_{X^{an}}(U) \\
f \mapsto f
\]

(since every regular function on \( U \) is holomorphic). The corresponding ring homomorphism \( \mathcal{O}_{X,x} \to \mathcal{O}_{X^{an},x} \) is a local homomorphism.

Given an \( \mathcal{O}_X \)-module \( F \), let

\[
F^{an} = \varphi^*(F) = \varphi^{-1}(F) \otimes_{\varphi^{-1}(\mathcal{O}_X)} \mathcal{O}_{X^{an}},
\]

which is an \( \mathcal{O}_{X^{an}} \)-module.

In particular, for every \( x \in X \), we have a canonical isomorphism

\[
(F^{an})_x \cong F_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X^{an},x}.
\]

We will see later that \( \mathcal{O}_{X^{an},x} \) is a Noetherian ring and the morphism \( \mathcal{O}_{X,x} \to \mathcal{O}_{X^{an},x} \) is flat. In particular, this will imply that the functor \( F \mapsto F^{an} \) is exact.

Note that we have canonical maps:

- \( \text{Hom}_{\mathcal{O}_X}(F, G) \to \text{Hom}_{\mathcal{O}_{X^{an}}}(F^{an}, G^{an}) \),
- \( H^i(X, F) \to H^i(X^{an}, F^{an}) \),
- more generally, \( \text{Ext}_I^{\mathcal{O}_X}(F, G) \to \text{Ext}_I^{\mathcal{O}_{X^{an}}}(F^{an}, G^{an}) \).

**Theorem 3.20** (GAGA, part 1). If \( X \) is a complete variety, then the functor \( F \mapsto F^{an} \) is fully faithful on coherent sheaves. Moreover, for all \( F, G \) coherent, the map

\[
\text{Ext}_I^{\mathcal{O}_X}(F, G) \to \text{Ext}_I^{\mathcal{O}_{X^{an}}}(F^{an}, G^{an})
\]

is an isomorphism.

The theorem is due to Serre when \( X \) is projective and due to Grothendieck when \( X \) is complete. There is also a relative version for proper morphisms. We will prove this theorem only when \( X \) is projective.

The category which is the target of this functor (i.e. which \( F^{an} \) belongs to) still have to be defined.

**Definition 3.21.** In general, if \( (X, \mathcal{O}_X) \) is a locally ringed space, an \( \mathcal{O}_X \)-module, \( F \) is **locally finitely generated** if for any \( x \in X \), there is an open neighborhood \( U \ni x \) and \( s_1, \ldots, s_n \in F(U) \) such that

\[
s_{1,y}, \ldots, s_{n,y} \in F_y
\]

generate \( F_y \) over \( \mathcal{O}_{X,y} \) for all \( y \in U \).

**Definition 3.22.** An \( \mathcal{O}_X \)-module \( F \) is **coherent** if

- it is locally finitely generated,
- for every open subset \( U \subseteq X \), \( s_1, \ldots, s_r \in F(U) \), the kernel of the induced map

\[
\ker(\mathcal{O}_U^{\text{pr}} \to F)
\]

is locally finitely generated.
Exercise. Check that on algebraic varieties, this coincides with the definition in Hartshorne [Har77].

**Theorem 3.23** (Oka). *If X is an analytic space, then \( O_X \) is coherent. (In particular, also locally free \( O_X \)-modules of finite rank are coherent)*

If \( X \) is an algebraic variety over \( \mathbb{C} \), then any coherent sheaf on \( F \) on \( X \) has a finite presentation, so \( F^{an} \) is coherent.

**Theorem 3.24** (GAGA, part 2). *If \( X \) is complete, the functor
\[
\{\text{coherent } O_X \text{-modules}\} \to \{\text{coherent } O_X^{an} \text{-modules}\}
\]
\[
F \mapsto F^{an}
\]
is an equivalence of categories.*

**Remark 3.25.** In particular, in this case we have an equivalence of categories
\[
\{\text{locally free } O_X \text{-modules}\} \to \{\text{locally free } O_X^{an} \text{-modules}\}.
\]
To show that \( F \) is locally free if \( F^{an} \) is, use the fact that \( O_{X,x} \to O_{X^{an},x} \) is faithfully flat.

**Remark 3.26.**

1. In general, \((O_X)^{an} = O_{X^{an}}\).
2. If \( X \) is a complex algebraic variety and \( E \xrightarrow{\pi} X \) is an algebraic vector bundle with sheaf of sections \( E \), the holomorphic vector bundle \( E^{an} \xrightarrow{\pi^{an}} X^{an} \) has the sheaf of sections \( \mathcal{E}^{an} \).
3. Applying the theorem for coherent ideal sheaves, in the setting of the theorem, every closed analytic subspace of \( X^{an} \) is equal to \( Y^{an} \) for some closed subvariety \( Y \subseteq X \). (When \( X = \mathbb{P}^N \), this was known as Chow’s Theorem.)
4. Using (3) and the graph, any morphism \( f: X^{an} \to Y^{an} \) comes from a morphism \( X \to Y \). Therefore, the functor
\[
\{\text{complete algebraic varieties}\} \to \{\text{compact analytic spaces}\}
\]
\[
X \mapsto X^{an}
\]

3.7. The ring \( O_{\mathbb{C}^n,0} \).

**Definition 3.27.** An element \( f \in \mathbb{C}[z_1, \ldots, z_n] \) is *convergent* if there is an \( R \) such that
\[
f = \sum_{\alpha \in \mathbb{N}^n} a_{\alpha} z^\alpha
\]
converges uniformly and absolutely for \( |z_i| < R \) for all \( i \). We write
\[
\mathbb{C}\{z_1, \ldots, z_n\} = \{f \in \mathbb{C}[z_1, \ldots, z_n] \mid f \text{ is convergent}\} \subseteq \mathbb{C}[z_1, \ldots, z_n].
\]
It is easy to check that \( f = \sum a_{\alpha} z^\alpha \) is convergent if and only if there exists \( R > 0 \) such that \( \{|a_{\alpha}| R^\alpha\}_{\alpha} \) is bounded. This is also equivalent to
\[
\limsup_{|\alpha| \to \infty} |a_{\alpha}|^{1/|\alpha|} < \infty
\]
(by the Cauchy-Hadamard Theorem).
We have a map
\[ \mathcal{O}_{\mathbb{C}^n,0} \to \mathbb{C}[z_1, \ldots, z_n] \]
\[ f \mapsto \sum_{\alpha \in \mathbb{N}^n} a_{\alpha} z^{\alpha} \]
where \( a_{\alpha} = \frac{1}{\alpha!} \partial^{\|\alpha\|} f(0) \).

Here \( \mathcal{O}_{\mathbb{C}^n,0} \) is the ring of germs of holomorphic functions at 0. Recall that if \( f \) is holomorphic at 0, then
\[ f(z) = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \partial^{\|\alpha\|} f(0) z^{\alpha} \]
converges absolutely and uniformly in a neighborhood of 0. By definition, the image of the above map is hence
\[ \mathbb{C}\{z_1, \ldots, z_n\} \]
and it is clear it is injective. Moreover, it is clearly a ring homomorphism.

**Conclusion.** If \( p \in M \) where \( M \) is a complex manifold, then
\[ \mathcal{O}_{M,p} \cong \mathbb{C}\{z_1, \ldots, z_n\} \]
where \( n = \dim X \).

The next goal is to show that \( \mathbb{C}\{z_1, \ldots, z_n\} \) is Noetherian. The idea is to proceed by induction and the key ingredient is the Weierstrass Preparation Theorem.

**Definition 3.28.** A *Weierstrass polynomial with respect to \( z_n \) is an element of \( \mathbb{C}\{z_1, \ldots, z_n\} \) of the form*
\[ z_n^d + a_1(z_1, \ldots, z_{n-1})z_n^{d-1} + \cdots + a_d(z_1, \ldots, z_{n-1}) \]
*such that \( a_0(0) = 0 \) for \( 1 \leq i \leq d \).*

**Theorem 3.29 (Weierstrass Preparation Theorem).** *Given \( f \in \mathbb{C}\{z_1, \ldots, z_n\} \) such that \( f(0, \ldots, 0, z_n) \neq 0 \), there exist unique \( g, h \in \mathbb{C}\{z_1, \ldots, z_n\} \) such that \( h(0) \neq 0 \), \( g \) is a Weierstrass polynomial, and*
\[ f = g \cdot h. \]

**Remark 3.30.**

1. If \( n = 1 \) and \( f \in \mathbb{C}\{z\}, f \neq 0 \), then
\[ f = z^d h \]
where \( h(0) \neq 0 \). Weierstrass Preparation Theorem 3.29 is a generalization of this statement to more variables.

2. Note that (1) implies that (still for \( n = 1 \)) if \( f \in \mathcal{O}(U) \), the zeroes of \( f \) do not accumulate in \( U \).

3. The condition that \( f(0, \ldots, 0, z_n) \neq 0 \) can always be achieved (if \( f \neq 0 \)) by a linear change of variables.

Recall (a special case of) the Residue Theorem. Suppose \( f \in \mathcal{O}(U \setminus \{a_1, \ldots, a_r\}) \) and there is a disc \( \Delta \subseteq U \) such that \( a_i \in \Delta \).
Then

\[ \frac{1}{2\pi i} \int_{\partial \Delta} \varphi(z) dz = \sum_{i=1}^{r} \text{Res}_{a_i}(\varphi_i) \]

In fact, we will only need this when \( \varphi \) is meromorphic at \( a_i \) with pole of order \( \leq 1 \). Using \( d(\varphi(z)dz) = 0 \) and Stokes' Theorem, we can reduce the computation of the integral to the case \( r = 1 \) by cutting out small discs around \( a_1, \ldots, a_r \).

In this case, we may write \( \varphi = \frac{\psi}{z-a} \) and then \( \text{Res}_{a}(\varphi) = \psi(a) \). Then

\[ \frac{1}{2\pi i} \int_{|w-a|=r} \frac{\psi(w)}{w-a} dw = \psi(a) \]

by Cauchy's formula 2.1.

In our case, we will take \( f \in \mathcal{O}(U) \), \( \Delta \subseteq U \), and consider

\[ \frac{1}{2\pi i} \int_{\partial \Delta} z^j \frac{f'(z)}{f(z)} dz \]

where \( f \) has no zeroes on \( \partial \Delta \). Suppose \( a \) is a zero of \( f \) and write \( f = (z-a)^m h \), \( h(a) \neq 0 \). Then

\[ z^j \frac{f'(z)}{f(z)} = z^j \left( \frac{m}{z-a} + \frac{h'}{h} \right) \]
which implies that 
\[ \text{Res}_a z^j \frac{f'(z)}{f(z)} = ma^j. \]

Overall, the conclusion is that:

\[(1) \frac{1}{2\pi i} \int_{\partial \Delta} z^j \frac{f'(z)}{f(z)} \, dz = \lambda_1^j + \cdots + \lambda_m^j. \]

if \( \lambda_1, \ldots, \lambda_m \) are the roots of \( f \) in \( \Delta \), listed with multiplicity.

**Proof of Weierstrass Preparation Theorem 3.29.** Let \( z' = (z_1, \ldots, z_{n-1}) \) and write
\[ f_{z'}(z_n) = f(z', z_n) \]

where \( f \) is a holomorphic function on \( \mathbb{C}^n \supseteq U \ni 0 \). Let \( \epsilon_n > 0 \) be such that
\[ f(0, \ldots, 0, z_n) \neq 0 \text{ for } 0 < |z_n| \leq \epsilon_n. \]

Choose \( \epsilon' > 0 \) such that if \( z' \) satisfies that if \( |z_i| < \epsilon' \) for \( 1 \leq i \leq n - 1 \) and \( |z_n| = \epsilon_n \), then \( f(z', z_n) \neq 0 \), and
\[ \{ z \mid |z_i| < \epsilon' \text{ for } i \leq n - 1, \ |z_n| < \epsilon_n \} \subseteq U. \]

Otherwise, one can choose \( z_i \to 0 \) such that \( f(z', z_n) = 0 \) and continuity of \( f \) will contradict the way we chose \( \epsilon_n \).

Given \( z' \) such that \( |z_i| < \epsilon' \) for \( i \leq n - 1 \), let
\[ \lambda_1(z'), \ldots, \lambda_m(z') \]

be the zeroes of \( f_{z'} \) in
\[ \{ z_n \mid |z_n| < \epsilon_n \}, \]
listed with multiplicities. By equation (1),
\[ \sum_{i=1}^{m} \lambda_i(z')^j = \frac{1}{2\pi i} \int_{|w| = \epsilon_n} w^j \cdot \frac{\partial f}{\partial z_n}(z', w) \cdot \frac{f(z', w)}{f(z', w)} \, dw. \]

Note that the right hand side is a holomorphic function as a function of \( z' \). For \( j = 0 \), the left hand side is an integer, and hence constant. This shows that
\[ m = \text{ord}_{z_n} f(0, \ldots, 0, z_n) \]

by taking \( z' = 0 \).

If \( \sigma_1(z'), \ldots, \sigma_m(z') \) are the symmetric functions of \( \lambda_1(z'), \ldots, \lambda_m(z') \), then each \( \sigma_i \) is holomorphic for \( |z_j| < \epsilon' \), \( j \leq n - 1 \) and \( \sigma_i(0) = 0 \) for \( 1 \leq i \leq m \). Let
\[ g = z_m^m - \sigma_1(z') z_m^{m-1} + \cdots + (-1)^m \sigma_m(z') \]

which is a Weierstrass polynomial.

It is clear that the function \( \frac{f}{g} \) is well-defined and holomorphic in
\[ \{ z \mid |z_j| < \epsilon' \text{ for } j \leq n - 1, \ |z_n| < \epsilon_n \} \setminus \{ g = 0 \}. \]

For every \( z' \), \( \frac{f(z', -)}{g(z', -)} \) extends to a holomorphic function of \( z_n \) for \( |z_n| < \epsilon_n \).
Exercise. Check that therefore \( h = \frac{f}{g} \) is in fact holomorphic in a neighborhood of 0 and \( h(0) \neq 0 \).

This proves existence.

Uniqueness is straightforward. If \( f = g' \cdot h' \) as in the theorem and \( g' = z_n^{d'} + \cdots \), we see that \( f(0, \ldots, 0, z_n) = z_n^{d'} \cdot h(0, \ldots, 0, z_n) \) which implies that \( d' = m \). For every \( z' \), \( f(z', -) \) has \( d \) roots in \( |z_n| < \epsilon_n \), so \( g'(z', -) \) vanishes on these with the right multiplicities. For degree reasons, this implies that \( g' = g \). \( \square \)

Corollary 3.31. For any \( n, \mathbb{C}\{z_1, \ldots, z_n\} \) is Noetherian.

Proof. We proceed by induction on \( n \geq 0 \). When \( n = 0 \), this ring is a field, which is Noetherian. Suppose \( I \subseteq \mathbb{C}\{z_1, \ldots, z_n\}, I \neq 0 \) is an ideal. Let \((f_\lambda)_{\lambda \in \Lambda}\) be a set of generators for \( I \). Fix \( \lambda_0 \in \Lambda \) such that \( f_{\lambda_0} \neq 0 \). Do a linear change of variables to assume that \( f_{\lambda_0}(0, \ldots, 0, z_n) \neq 0 \).

For \( \lambda \neq \lambda_0 \), if \( f_\lambda(0, \ldots, 0, z_n) = 0 \), replace \( f_\lambda \) by \( f_\lambda + f_{\lambda_0} \). We may hence assume that \( f_\lambda(0, \ldots, 0, z_n) \neq 0 \) for all \( \lambda \).

Now, by Weierstrass Preparation Theorem 3.29, we may write for each \( \lambda \)
\[
f_\lambda = (\text{invertible element}) \cdot (\text{element of } \mathbb{C}\{z_1, \ldots, z_{n-1}\}[z_n]).
\]
This shows that \( I \) is generated (as an ideal) by
\[
I \cap \mathbb{C}\{z_1, \ldots, z_{n-1}\}[z_n]
\]
which is finitely generated by inductive hypothesis and Hilbert’s basis theorem (if \( R \) is Noetherian, \( R[x] \) is also Noetherian). \( \square \)

Remark 3.32. The same proof shows that \( \mathbb{C}[z_1, \ldots, z_n] \) is Noetherian. However, it is easier to see that it is the completion of \( \mathbb{C}\{z_1, \ldots, z_n\} \) (as shown in the proof of Proposition 3.33), and hence Noetherian.

Proposition 3.33. If \( X \) is a smooth algebraic variety, then the ring homomorphism
\[
\mathcal{O}_{X,x} \rightarrow \mathcal{O}_{X,a,n,x}
\]
is (faithfully) flat for every \( x \in X \).

Proof. Step 1. Suppose \( X = \mathbb{A}^n \). Let \( R = \mathbb{C}\{z_1, \ldots, z_n\} \supseteq m = \{ f \mid f(0) = 0 \} \). It is easy to check that \( m = (z_1, \ldots, z_n) \). Moreover,
\[
R/m^N \cong \mathbb{C}[z_1, \ldots, z_n]/(z_1, \ldots, z_n)^N.
\]
Therefore,
\[
\hat{R} = \varprojlim R/m^N \cong \varprojlim \mathbb{C}[z_1, \ldots, z_n]/(z_1, \ldots, z_n)^N \cong \mathbb{C}[z_1, \ldots, z_n].
\]
We have the following commuting square:
Recall that if \((S,n)\) is a local Noetherian ring, the map \(S \to \hat{S}\) is (faithfully) flat. Since the vertical maps are faithfully flat (as \(R\) is Noetherian by 3.31), the top horizontal map is faithfully flat.

(Since \(\hat{R}\) is a regular ring of dimension \(n\), \(R\) is a regular ring of dimension \(n\).)

**Step 2.** Prove the following fact.

**Exercise.** If \(X \subseteq Y\) are smooth algebraic varieties where \(X\) is defined by the coherent ideal \(I\) and we consider \(X^{\text{an}} \subseteq Y^{\text{an}}\), the ideal of \(O_{Y^{\text{an}}}\) vanishing on \(X^{\text{an}}\) is \(I^{\text{an}}\). (Hint: reduce to the case \((x_1, \ldots, x_r = 0) = X \subseteq \mathbb{C}^n = Y\).

In general, if \(X\) is a smooth algebraic variety, \(X \subseteq \mathbb{C}^N\) defined by the ideal \(I\), then the exercise shows that

\[
\mathcal{O}_{\mathbb{C}^N,x}/I\mathcal{O}_{\mathbb{C}^N,x} = \mathcal{O}_{X,x} \xrightarrow{\varphi} \mathcal{O}_{X^{\text{an}},x} = \mathcal{O}_{(\mathbb{C}^N)^{\text{an}},x}/I\mathcal{O}_{(\mathbb{C}^N)^{\text{an}},x}
\]

Since we have shown that \(\psi\) is flat, this shows that \(\varphi\) is also flat. Because this is a local statement, it was enough to consider the case \(X \subseteq \mathbb{C}^N\).

The exercise in the proof of Proposition 3.33 has another consequence. If \(i: X \hookrightarrow Y\) is a closed immersion of smooth algebraic varieties and \(F\) is a sheaf on \(X\), then the canonical map

\[
(i_*(\mathcal{F}))^{\text{an}} \to (i^{\text{an}})_*(\mathcal{F}^{\text{an}})
\]

is an isomorphism.

**Exercise.** Show that we have such a morphism which gives an isomorphism on stalks. (Hint: use the other consequence of the exercise)

We can finally prove a part of GAGA, part 1, 3.20.

**Theorem 3.34.** If \(X\) is a smooth projective complex algebraic variety. Then:

1. the functor \(\mathcal{F} \mapsto \mathcal{F}^{\text{an}}\) is exact,
2. if \(\mathcal{F}, \mathcal{G}\) are coherent on \(X\), we have an isomorphism

\[
\text{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \xrightarrow{\cong} \text{Ext}^i_{\mathcal{O}_{X^{\text{an}}}}(\mathcal{F}^{\text{an}}, \mathcal{G}^{\text{an}})
\]
Assume that:

\[ H^i((\mathbb{P}^n)^{an}, \mathcal{O}_{(\mathbb{P}^n)^{an}}) = \begin{cases} \mathbb{C}, & i = 0, \\ 0, & i > 0. \end{cases} \]

We will see this later via Hodge theory when we will compute \( H^*(\mathbb{P}^n)^{an}, \mathbb{C} \).

**Proof of Theorem 3.34.** We have a closed immersion \( X \hookrightarrow \mathbb{P}^n \). We first treat the case \( \mathcal{F} = \mathcal{O}_X \). To show that

\[ H^i(X, \mathcal{G}) \to H^i(X^{an}, \mathcal{G}^{an}), \]

by pushing forward to \( \mathbb{P}^N \), we may assume that \( X = \mathbb{P}^n \).

Next, suppose \( \mathcal{G} = \mathcal{O}_{\mathbb{P}^n}(m) \) and argue by induction on \( n \). The case \( n = 0 \) is trivial. The key exact sequence to use is

\[ 0 \to \mathcal{O}_{\mathbb{P}^n}(-1) \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n-1} \to 0 \]
tensored with \( \mathcal{O}(m) \). Since we know the assertion for \( \mathcal{O}_{\mathbb{P}^n-1}(m) \), then 5-Lemma implies that the assertion holds for \( \mathcal{O}_{\mathbb{P}^n}(m) \) if and only if it holds for \( \mathcal{O}_{\mathbb{P}^n}(m - 1) \). Since we assume we know this for \( m = 0 \), we know it for all \( m \).

Now, work with general \( \mathcal{G} \). We argue by decreasing induction on \( i \) that to show that

\[ H^i(X, \mathcal{G}) \to H^i(X^{an}, \mathcal{G}^{an}) \]
is an isomorphism. For \( i \gg 0 \), both are 0, so the assertion is true. For the induction step, given \( \mathcal{G} \) have a short exact sequence

\[ 0 \to \mathcal{G}' \to \mathcal{E} \to \mathcal{G} \to 0 \]

where \( \mathcal{E} \) is a direct sum of \( \mathcal{O}(m) \), so we know the assertion for \( \mathcal{E} \). The long exact sequence in cohomology then gives

\[ H^i(X, \mathcal{G}') \to H^i(X, \mathcal{E}) \to H^i(X, \mathcal{G}) \to H^{i+1}(X, \mathcal{G}') \to H^{i+1}(X, \mathcal{E}) \]

\[ \downarrow \beta \quad \downarrow \cong \quad \downarrow \alpha \quad \downarrow \cong \]

\[ H^i(X^{an}, (\mathcal{G}')^{an}) \to H^i(X^{an}, \mathcal{E}^{an}) \to H^i(X^{an}, \mathcal{G}^{an}) \to H^{i+1}(X^{an}, (\mathcal{G}')^{an}) \to H^{i+1}(X^{an}, \mathcal{E}^{an}) \]

By the 5-Lemma, \( \alpha \) is surjective for every \( \mathcal{G} \). Hence \( \beta \) is surjective as well (applying this to \( \mathcal{G}' \) instead of \( \mathcal{G} \)), which implies by the 5-Lemma that \( \alpha \) is injective. This shows that \( \alpha \) is an isomorphism, as required.

Finally, we know that for every \( X \) smooth projective, any \( \mathcal{F}, \mathcal{G} \) where \( \mathcal{F} \) locally free,

\[ H^i(X, \mathcal{G} \otimes \mathcal{F}^\vee) \cong \text{Ext}_{\mathcal{O}_X}^i(\mathcal{F}, \mathcal{G}) \to \text{Ext}_{\mathcal{O}_X}^i(\mathcal{F}^{an}, \mathcal{G}^{an}) \]
is an isomorphism. For general \( \mathcal{F} \), use increasing induction using

\[ 0 \to \mathcal{F}' \to \mathcal{E} \to \mathcal{F} \to 0 \]
4. Dolbeaux cohomology

4.1. The tangent bundle of a complex manifold.

4.1.1. The complexification of a real vector space. Let \( V_\mathbb{R} \) be a finite-dimensional vector space over \( \mathbb{R} \). To give a complex vector space structure on \( V_\mathbb{R} \) is equivalent to giving a linear map \( J: V_\mathbb{R} \to V_\mathbb{R} \) such that \( J^2 = -\text{Id} \) (where \( J \) is multiplication by \( i \)).

Given such \( J \), we write \( V \) for the corresponding \( \mathbb{C} \)-vector space. This is called a complexification of \( V \). We write

\[
V_\mathbb{C} = V \otimes_\mathbb{R} \mathbb{C}
\]

for the extension of scalars. Then \( J \) induces

\[
J_\mathbb{C}: V_\mathbb{C} \to V_\mathbb{C}
\]

\[
v \otimes \lambda \mapsto J(v) \otimes \lambda.
\]

Then \( J^2_\mathbb{C} = -\text{Id} \) and we have a decomposition

\[
V_\mathbb{C} = V' \oplus V''
\]

where

\[
V' = \{ v \in V_\mathbb{C} \mid J_\mathbb{C}(v) = iv \},
\]

\[
V'' = \{ v \in V_\mathbb{C} \mid J_\mathbb{C}(v) = -iv \}
\]

are \( \mathbb{C} \)-subspaces of \( V_\mathbb{C} \).

Denote by \( u \mapsto \overline{u} \) the conjugate-linear map

\[
V_\mathbb{C} \to V_\mathbb{C}
\]

\[
v \otimes \lambda \mapsto v \otimes \overline{\lambda}.
\]

We have an embedding

\[
V_\mathbb{R} \xrightarrow{j} V_\mathbb{C}
\]

\[
v \mapsto v \otimes 1
\]

such that \( V_\mathbb{R} = \text{Fix}(u \mapsto \overline{u}) \).

We claim that the composition

\[
V \xrightarrow{j} V_\mathbb{C} \xrightarrow{\text{pr}_1} V'
\]

is a complex isomorphism,

\[
V \xrightarrow{j} V_\mathbb{C} \xrightarrow{\text{pr}_2} V''
\]

is a conjugate-linear isomorphism.

If \( v \in V \), write \( v \otimes 1 = v' + v'' \) for \( v' \in V' \), \( v'' \in V'' \). Then

\[
Jv \otimes 1 = iv' - iv''.
\]
Hence
\[ v' = \frac{1}{2}(v - iJ_C(v)), \]
\[ v'' = \frac{1}{2}(v + iJ_C(v)). \]

This implies that \( v'' = \overline{v'} \).

**Exercise.** Check that \( v \mapsto v' \) is \( \mathbb{C} \)-linear and \( v \mapsto v'' \) is conjugate-linear.

By the above formulas, the two maps are injective. Hence they are bijective by dimension considerations. This proves the above claim. Moreover, we note that
\[ V'' = \overline{V'}. \]

Let us now describe the decomposition \( V_C = V' \oplus V'' \) in terms of bases. Suppose \( x_1, \ldots, x_n \) give a basis of \( V \) over \( \mathbb{C} \). This implies that if \( y_j = J(x_j) \), then \( x_1, \ldots, x_n, y_1, \ldots, y_n \) give a basis of \( V_\mathbb{R} \). Consider these in \( V_C \) via \( j : V \hookrightarrow V_C \). Let \( e_j \) be the \( V' \)-component of \( x_j \):
\[ e_j = \frac{1}{2}(x_j - iy_j), \]
\[ \overline{e_j} = \frac{1}{2}(x_j + iy_j). \]

Then \( e_1, \ldots, e_n \) is a basis of \( V' \) and \( \overline{e_1}, \ldots, \overline{e_n} \) of \( V'' \).

Consider now \( U = \text{Hom}_\mathbb{R}(V, \mathbb{R}) \). This has a complex structure given by
\[ (\lambda \varphi)(v) = \varphi(\lambda v) \quad \text{for} \quad \lambda \in \mathbb{C}. \]

By the above, we have a decomposition \( U_C = U' \oplus U'' \). On the other hand
\[ (V_C)^* = \text{Hom}_\mathbb{C}(V \otimes_\mathbb{R} \mathbb{C}, \mathbb{C}) \cong \text{Hom}_\mathbb{R}(V, \mathbb{C}) \cong U \otimes \mathbb{C} \]
are isomorphisms of complex vector spaces.

**Exercise.** Check that via these isomorphisms \( (J_{V,C})^* \) corresponds to \( J_{U,C} \).

This implies that \( U' = (V')^* \) and \( U'' = (V'')^* \).

In the above description of \( V_C = V' \oplus V'' \) using bases, we see that \( x_1^*, \ldots, x_n^*, y_1^*, \ldots, y_n^* \) is a basis of \( U = \text{Hom}_\mathbb{R}(V, \mathbb{R}) \). Moreover, it is a simplex exercise to check that \( y_j^* = -J_{U,C}(x_j^*) \).

The bases are then
\[ \text{basis of } U' : \quad x_j^* + iy_j^* \quad 1 \leq j \leq n, \]
\[ \text{basis of } U : \quad x_j^* - iy_j^* \quad 1 \leq j \leq n. \]

The decomposition \( U_C = U' \oplus U'' \) induces a decomposition
\[ \bigwedge^p U \otimes_\mathbb{C} U_C = \bigwedge^p (U_C) = \bigwedge^p (U' \oplus U'') = \bigoplus_{i+j=p} \left( \bigwedge^i U' \otimes \bigwedge^j U'' \right). \]

The conjugation on \( U_C \), which maps \( U' \) to \( U'' \) and \( U'' \) to \( U' \), induces a conjugation on \( (\bigwedge^p U) \otimes_\mathbb{C} U_C \) which maps
\[ \bigwedge^i U' \otimes \bigwedge^j U'' \to \bigwedge^j U' \otimes \bigwedge^i U''. \]
4.1.2. The complexification of real line bundles. This globalizes as follows. Suppose $M$ is a smooth real manifold and $E$ is a smooth real vector bundle on $M$.

Giving a complex structure on $E$ is equivalent to giving a morphism of vector bundles $J: E \to E$ such that $J^2 = -\text{Id}$.

In this case, the previous discussion globalizes to a decomposition

$$E_{\mathbb{C}} = E \otimes_{\mathbb{R}} \mathbb{C} = E' \oplus E''$$

and we have an isomorphism of complex vector bundles $E \to E'$ and a complex conjugate-linear isomorphism $E \to E''$. We also have a conjugation operator on $E_{\mathbb{C}}$ mapping $E'$ to $E''$.

The dual $E^*$ also has a complex structure. We get a corresponding decomposition of $(\bigwedge^p E^*)_{\mathbb{C}}$ etc.

4.1.3. The tangent bundle.

**Definition 4.1.** Let $M$ be a smooth real manifold. An almost complex structure on $M$ is a complex structure on the tangent bundle $T M$, i.e. a morphism $J: T M \to T M$ of vector bundles such that $J^2 = -\text{Id}$.

**Proposition 4.2.** If $M$ is a complex manifold, then $M$ carries a canonical almost complex structure. Moreover, if the corresponding decomposition is

$$T M_{\mathbb{C}} = T^{1,0} M \oplus T^{0,1} M$$

then $T^{1,0} M$ is a holomorphic vector bundle.

**Proof.** It is enough to treat the case of open subsets of $\mathbb{C}^n$ and then show that biholomorphic maps preserve this structure.

If $U \subseteq \mathbb{C}^n$ is an open subset with complex coordinates $z_1, \ldots, z_n$ and $z_j = x_j + iy_j$, then $TU$ is trivialized by

$$\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}.$$

Define a complex structure by

$$J \left( \frac{\partial}{\partial x_j} \right) = \frac{\partial}{\partial y_j}, \quad J \left( \frac{\partial}{\partial y_j} \right) = -\frac{\partial}{\partial x_j}.$$

We have a decomposition $TU_{\mathbb{C}} = T^{1,0} U \oplus T^{0,1} U$ where

- $T^{1,0} U$ is trivialized by $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$,
- $T^{0,1} U$ is trivialized by $\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}$.

The key point is that we showed that if $f: U \to V$ is holomorphic, then the canonical map

$$TU_{\mathbb{C}} \to f^* TV_{\mathbb{C}}$$

maps

- $T^{1,0} U$ to $f^* T^{1,0} V$,
- $T^{0,1} U$ to $f^* T^{0,1} V$. 

In particular, if \( f \) is biholomorphic, the isomorphism between \( TU_C \) and \( f^*TV_C \) preserves the decomposition. Hence any biholomorphic map respects the two complex structure. Here, we use the following fact: if \( \varphi: V \to W \) is an \( \mathbb{R} \)-linear isomorphism between complex vector spaces such that \( \varphi \otimes 1: V_C \to W_C \) maps to \( V' \) to \( W' \) and \( V'' \) to \( W'' \), then \( \varphi \) is a \( \mathbb{C} \)-linear isomorphism. Indeed, we have

\[
\begin{array}{ccc}
V & \xrightarrow{\varphi} & W \\
\cong & & \cong \\
V' & \xrightarrow{\varphi \otimes 1} & W'
\end{array}
\]

and since the map \( V' \to W' \) is \( \mathbb{C} \)-linear, so is \( \varphi \). This proves the first statement.

The second follows, since we saw that if \( f: V \to W \) is a holomorphic map between open subsets \( V \subseteq \mathbb{C}^n \), \( W \subseteq \mathbb{C}^m \), then \( f^*T^{1,0}W \to T^{1,0}V \) is given by the matrix \( \left( \frac{\partial f_i}{\partial z_j} \right)_{i,j} \). In particular, the transition maps of \( T^{1,0}M \) are holomorphic. \( \square \)

4.2. The Dolbeault complex. Last time, we saw that if \( M \) is a complex manifold, then \( TM \) has a canonical complex structure such that

\[
TM_C \cong T^{1,0}M \oplus T^{0,1}M
\]

where \( T^{1,0}M \) is a holomorphic vector bundle.

**Definition 4.3.** The bundle \( T^{1,0} \) is the holomorphic tangent bundle of \( M \).

**Remark 4.4.**

1. As in the case of the tangent bundle to a smooth manifold, \( T^{1,0}M \) can be described as the derivations on the local rings \( O_{M,x} \) for \( x \in M \).
2. If \( X \) is a smooth algebraic variety and \( M = X^{an} \), then

\[
T^{1,0}M \cong (TX)^{an}.
\]

If \( f: M \to M' \) is a holomorphic map, then we have

\[
\begin{array}{ccc}
TM_C & \longrightarrow & f^*TM'_C \\
\uparrow & & \uparrow \\
T^{1,0}M & \longrightarrow & f^*T^{1,0}M'
\end{array}
\]

Dually, we have a decomposition

\[
T^*M_C = A^{1,0}_M \oplus A^{0,1}_M,
\]

where \( A^{i,j}_M \) is the dual of \( T^{i,j}M \). Moreover, we also have a corresponding decomposition for

\[
\bigwedge^p (T^*M_C).
\]

Let \( A^m_M \) be the sheaf of real smooth \( m \)-forms on \( M \). Moreover, let \( A^m_{M,C} = A^m_M \otimes \mathbb{R} \mathbb{C} \). We have a decomposition

\[
A^m_{M,C} = \bigoplus_{p+q=m} A^{p,q}_M
\]
where $\mathcal{A}^{p,q}_M$ is the sheaf of $(p, q)$-forms on $M$. Note that
\[ \mathcal{A}^{p,q}_M = \mathcal{A}^{q,p}_M. \]

Note that $\mathcal{A}^{p,0}_M$ is the sheaf of smooth sections of a holomorphic vector bundle. We have a subsheaf $\Omega^p \subseteq \mathcal{A}^{p,0}_M$ of holomorphic sections of a holomorphic vector bundle.

For example,
\[ \Omega^1 = \text{sheaf of holomorphic sections of } (T^{1,0}M)^*. \]

Locally, in a chart with coordinates $z_1, \ldots, z_n$,
\[ \mathcal{A}^{p,q} = \text{free } C^\infty_{M,\mathbb{C}}\text{-module, with basis } dz_I \wedge d\overline{z}_J \text{ for } |I| = p, |J| = q, \]
where for $I$ given by $i_1 < \ldots < i_p$ we write
\[ dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}, \quad d\overline{z}_I = d\overline{z}_{i_1} \wedge \cdots \wedge d\overline{z}_{i_p}. \]

Recall that we have a de Rham differential $d: \mathcal{A}^m_M \to \mathcal{A}^{m+1}_M$. Given $p, q$, consider

\[ d: \mathcal{A}^{p,q}_M \to \mathcal{A}^{p+q+1}_M, \]

\[ \overline{\partial} \quad \text{proj} \quad d \quad \text{proj} \]

Proposition 4.5. We have that $d = \partial + \overline{\partial}$.

Proof. Let us compute $\partial$ and $\overline{\partial}$ locally. Consider a chart with coordinates $z_1, \ldots, z_n$. Consider
\[ \omega = f \ dz_I \wedge d\overline{z}_J \text{ for } |I| = p, |J| = q. \]

Then
\[ d\omega = df \wedge dz_I \wedge d\overline{z}_J \]
where
\[ df = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j + \sum_{j=1}^{n} \frac{\partial f}{\partial y_j} dy_j = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} dz_j + \sum_{j=1}^{n} \frac{\partial f}{\partial \overline{z}_j} d\overline{z}_j. \]

This shows that $\partial \omega = df \wedge dz_I \wedge d\overline{z}_J$ and $\overline{\partial} \omega = \overline{df} \wedge dz_I \wedge d\overline{z}_J$. This shows that $d\omega = \partial \omega + \overline{\partial} \omega$. \[\square\]

Corollary 4.6. We have that $\partial^2 = 0$, $\overline{\partial}^2 = 0$, and $\partial \overline{\partial} + \overline{\partial} \partial = 0$. 

**Proof.** Use the fact that $d^2 = 0$ and look in the corresponding graded pieces.

**Corollary 4.7.** The maps $\partial$ and $\overline{\partial}$ are derivations, i.e.

$$\partial(\omega_1 \wedge \omega_2) = \partial \omega_1 \wedge \omega_2 + (-1)^{\deg \omega_1} \omega_1 \wedge \partial \omega_2$$

and similarly for $\overline{\partial}$.

**Proof.** Use that $d$ is a derivation and look in the corresponding graded pieces.

For every $p$, we have the following complex:

$$0 \to A^{p,0}_M \overset{\overline{\partial}}{\to} A^{p,1}_M \overset{\overline{\partial}}{\to} \cdots \overset{\overline{\partial}}{\to} A^{p,n}_M \to 0$$

where $n$ is the dimension of $M$ as a complex manifold.\(^3\)

**Definition 4.8.** The global sections $\Gamma(M, A^p_M \cdot)$ form the $p$th Dolbeault complex of $M$. The $(p, q)$ Dolbeault cohomology is

$$H^{p,q}(M) = H^q(\Gamma(M, A^p_M \cdot)).$$

**Remark 4.9.** Note that

$$\ker(A^{p,0}_M \overset{\overline{\partial}}{\to} A^{p,1}_M) = \Omega^p_M,$$

i.e. the sheaf of holomorphic sections of the sheaf of $(p, 0)$ forms. Indeed, locally, $\omega = \sum_I f_iz_I$ is in the kernel if and only if $\frac{\partial f_I}{\partial z_j} = 0$ for all $j$ and $I$, which means that $f_I$ is holomorphic for all $I$.

More generally, suppose $E$ is a holomorphic vector bundle with sheaf of smooth sections $\mathcal{E}$. We claim that we can define a canonical $\overline{\partial}_E$:

$$\overline{\partial}_E: A^{p,q}_M \otimes \mathcal{E} \to A^{p,q+1}_M \otimes \mathcal{E}$$

where the tensor product is over $C^\infty_{M, \mathcal{E}}$ such that $\overline{\partial}_E^2 = 0$ and $\overline{\partial}_E$ is a derivation.

We prove this claim now. We first work on a chart $U$ such that $\mathcal{E}|_U$ has a trivialization by holomorphic sections $s_1, \ldots, s_r$. Let $\omega$ be a section of $A^{p,q}_M \otimes \mathcal{E}$ and write it as

$$\omega = \sum_{i=1}^r \omega_i s_i.$$ 

Define

$$\overline{\partial}_E(\omega) = \sum_{i=1}^r \overline{\partial}(\omega_i) s_i.$$ 

This is independent of trivialization. Indeed,

$$\overline{\partial}(f \omega_i) = f \overline{\partial} \omega_i$$

if $f$ is holomorphic (so $\overline{\partial}f = 0$). Therefore, these local maps $\overline{\partial}_E$ glue to give $\overline{\partial}_E$ on $M$. It is clear from the local description that $\overline{\partial}_E^2 = 0$ and $\overline{\partial}_E$ is a derivation.

Altogether, this gives a *twisted Dolbeault complex*:

\(^3\)Note that $A^{p,q}_M = 0$ whenever $p$ or $q$ is bigger than $n$.\]
\[ 0 \rightarrow A^p_M \otimes \mathcal{E} \xrightarrow{\overline{\partial}_E} A^{p+1}_M \otimes \mathcal{E} \xrightarrow{\overline{\partial}_E} \cdots \xrightarrow{\overline{\partial}_E} A^{p+n}_M \otimes \mathcal{E} \rightarrow 0. \]

**Definition 4.10.** The \((p, q)\) Dolbeault cohomology of \(\mathcal{E}\) is
\[ H^{p,q}(M; \mathcal{E}) = H^q(\Gamma(M, A^{p, q}_M \otimes \mathcal{E})). \]

There are two things to do:
- \(A^{p,q}_M\) is acyclic: \(H^q(A^{p, q}_M) = 0\) for \(q \geq 1\),
- can use this complex to compute \(H^q(M, \Omega^p_M)\).

**Theorem 2.5** generalizes to functions of several variables.

**Proposition 4.11** (\(\overline{\partial}\)-lemma). If \(\omega\) is a \((p, q)\) form on \(U \subseteq M\) such that \(\overline{\partial}\omega = 0\), \(q \geq 1\), then locally, we can find \(\beta \in A^{p, q-1}_M\) such that \(\overline{\partial}\beta = \omega\). (Then \(A^{p, q}_M\) is acyclic.)

**Proof.** We work locally, so we may assume that we have a chart with coordinates \(z_1, \ldots, z_n\).

**Step 1.** Reduce to the case \(p = 0\). Write \(\omega = \sum_{I, J} d_{I,J} d z_I \wedge d \overline{z}_J\) and assume that \(\overline{\partial}\omega = 0\). For every \(I\), consider
\[ \omega_I = \sum_J f_{I,J} d z_J. \]

Since \(\overline{\partial}\omega = 0\), \(\overline{\partial}\omega_I = 0\) for all \(I\). If we know the \(p = 0\) case, then locally \(\omega_I = \overline{\partial}\beta_I\) for some \(\beta_I\) which are \((0, q-1)\) forms. If we take \(\beta = \sum_I (-1)^p d z_I \wedge \beta_I\), then \(\overline{\partial}\beta = \omega\).

**Step 2.** Assume \(p = 0\). Working locally, in a chart with coordinates \(z_1, \ldots, z_n\), we may write
\[ \omega = \sum_{|J|=q} f_J d \overline{z}_J. \]

Let \(k\) be the largest index so that \(d \overline{z}_k\) shows up in some \(d \overline{z}_J\) with non-zero coefficient. We proceed by increasing induction on \(k\).

First, suppose \(\omega \neq 0\) and the smallest \(k\) is \(q\). Then
\[ \omega = f d \overline{z}_1 \wedge \cdots \wedge d \overline{z}_q. \]

Note that \(\overline{\partial}\omega = 0\) if and only if \(\frac{\partial f}{\partial \overline{z}_i} = 0\) for \(i > q\), i.e. \(f\) is holomorphic in the variables \(z_{q+1}, \ldots, z_n\). By **Theorem 2.5** (the one variable \(\overline{\partial}\)-lemma), locally there is a function \(g\) which is smooth and holomorphic with respect to the variables \(z_{q+1}, \ldots, z_n\), and
\[ \frac{\partial g}{\partial \overline{z}_1} = f. \]

Then
\[ \overline{\partial}(g d \overline{z}_1 \wedge \cdots \wedge \overline{z}_q) = \omega. \]

For the induction step, write \(\omega = \omega_1 + \omega_2 \wedge d \overline{z}_k\) where \(\omega_1\) is a \((0, q)\)-form and \(\omega_2\) is a \((0, q-1)\)-form such that \(\omega_1\) and \(\omega_2\) only involve \(d \overline{z}_1, \ldots, d \overline{z}_{k-1}\). Since \(\overline{\partial}\omega = 0\), the coefficients of \(\omega_1\), \(\omega_2\)
are holomorphic in $z_1, \ldots, z_{k+1}$. Write

$$\omega_2 = \sum_{|J|=q-1} a_J d\bar{z}_J.$$ 

Applying Theorem 2.5, we can find locally smooth functions $b_J$, holomorphic in $z_{k+1}, \ldots, z_n$ such that

$$\frac{\partial b_J}{\partial \bar{z}_k} = a_J.$$ 

Then

$$\bar{\partial} \left( \sum_{|J|=q-1} b_J d\bar{z}_J \right) = \sum_{|J|=q-1} (-1)^{q-1} a_J d\bar{z}_J \wedge d\bar{z}_k + (\text{terms involving only } d\bar{z}_1, \ldots, d\bar{z}_{k-1}).$$

Therefore,

$$\omega - (-1)^{q-1} \bar{\partial}(\beta')$$

only involves $d\bar{z}_1, \ldots, d\bar{z}_{k-1}$. By inductive hypothesis, this must be equal to $\bar{\partial}(\gamma)$ for some $\gamma$, and hence

$$\omega = \bar{\partial}(\gamma + (-1)^q \beta').$$

This completes the proof. \[\Box\]

Exercise. Repeat the whole argument when $M$ is a smooth manifold to show that if $\omega$ is a $p$-form for $p \geq 1$ which is closed ($d\omega = 0$), then $\omega$ is locally exact.

**Corollary 4.12.** For every $p \geq 0$, we have an exact complex of sheaves on $M$:

$$0 \longrightarrow \Omega^p_M \longrightarrow A^{p,0} \xrightarrow{\bar{\partial}} A^{p,1} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} A^{p,n} \longrightarrow 0.$$ 

More generally, if $E$ is any holomorphic vector bundles, with sheaf of smooth sections $\mathcal{E}$ and sheaf of holomorphic sections $\mathcal{E}^{\text{hol}}$, then we have an exact complex

$$0 \longrightarrow \Omega^p_M \otimes_{\mathcal{O}_M} \mathcal{E}^{\text{hol}} \longrightarrow A^{p,0} \otimes_{\mathcal{C}^\infty_{M,C}} \mathcal{E} \xrightarrow{\bar{\partial}_E} A^{p,1} \otimes_{\mathcal{C}^\infty_{M,C}} \mathcal{E} \longrightarrow \cdots$$

Exactness follows since locally the complex is isomorphic to a direct sum of $r = \text{rk} E$ copies of the Dolbeault complex.

In the case of smooth manifolds, we have similar exact complexes. We have the de Rham complex:

$$0 \longrightarrow \mathbb{R} \longrightarrow A^0_M \xrightarrow{d} \cdots \xrightarrow{d} A^n_M \longrightarrow 0.$$ 

4.3. **Soft sheaves on paracompact spaces.** Let $X$ be a topological space and $\mathcal{F}$ be a sheaf of abelian groups on $X$. If $Z$ is any subset of $X$ and $i: Z \hookrightarrow X$ is the inclusion map, then we define

$$\mathcal{F}(Z) = \Gamma(Z, \mathcal{F}) = \Gamma(Z, i_! \mathcal{F}).$$
to be the set of sections $s: Z \to \prod_{x \in Z} F_x$ such that $s(x) \in F_x$ for all $x \in Z$ and for all $x \in Z$, there is an open neighborhood of $x$ in $X$ and $t \in F(U)$ such that

$$s(y) = t_y \text{ for all } y \in U \cap Z.$$  

**Remark 4.13.** If $Z' \subseteq Z$, we have natural restriction maps $F(Z) \to F(Z')$ which are functorial.

**Proposition 4.14.** If $F$ is as above and $Z_1, \ldots, Z_r$ are closed subsets of $X$, we have an exact sequence:

$$0 \to F\left(\bigcup_i Z_i\right) \to \prod_i F(Z_i) \to \prod_{i,j} F(Z_i \cap Z_j)$$

induced by restriction maps.

**Proof.** Suppose $(s_i)_{1 \leq i \leq r}$ are sections

$$s_i: Z_i \to \prod_{x \in Z_i} F_x$$

such that $s_i(x) \in F_x$ which are compatible, i.e. $s_i(x) = s_j(x)$ for all $x \in Z_i \cap Z_j$. We want to show there is a unique $s: \bigcup_i Z_i \to \prod_{x \in \bigcup_i Z_i} F_x$ such that $s(x) \in F_x$ for all $x$, and $s|_{Z_i} = s_i$.

Fix $x \in X$. We may replace $X$ be an open neighborhood of $x$. Since $Z_i$'s are closed, we may assume that $x \in Z_1 \cap \cdots \cap Z_r$ by making this open neighborhood smaller if necessary. Moreover, we may assume that for all $i$, there exists $t_i \in F(X)$ such that $(t_i)_y = s_i(y)$ for all $y \in Z_i$. In particular,

$$(t_1)_x = \cdots = (t_r)_x.$$  

Further replacing $X$, we may assume $t_1 = \cdots = t_r = t$. Clearly, $(t)_y = s(y)$ for all $y$. 

**Definition 4.15.** A topological space $X$ is **paracompact** if

- $X$ is Hausdorff,
- every open cover admits a refinement which is locally finite.

We suppose throughout this section that $X$ is paracompact.

**Examples 4.16.**

1. Topological manifolds are paracompact.
2. CW complexes are paracompact.

**Remark 4.17.** If $Z$ is closed in $X$ and $X$ is paracompact, then $Z$ is paracompact.

**Remark 4.18.** If $X = \bigcup_{i \in I} U_i$ is a locally finite open cover and $X$ is paracompact, then there is an open cover $X = \bigcup V_i$ such that $\overline{V_i} \subseteq U_i$.

**Example 4.19.** If $A, B \subseteq X$ are closed, $A \cap B = \emptyset$, there exist $U, V$ open such that $A \subseteq U$, $B \subseteq V$, $U \cap V = \emptyset$. (In other words, $X$ is a **normal space**).
Definition 4.20. A sheaf $F$ of abelian groups on $X$ is soft if for any $Z \subseteq X$ closed, the restriction map $F(X) \to F(Z)$ is surjective.

Compare this to the definition of flasque sheaves. A sheaf $F$ is flasque if for any $U \subseteq X$ open, the map $F(X) \to F(U)$ is surjective.

Fact 4.21. Flasque sheaves are acyclic, i.e. their higher cohomology vanishes. Therefore, one can compute cohomology via flasque resolutions.

We will see next time that if $X$ is paracompact, then the same holds for soft sheaves. Moreover, we will see that there is a large supply of soft sheaves on complex manifolds.

There are some supplementary notes on the course website


covering

- soft sheaves,
- comparison between singular cohomology and sheaf cohomology with constant coefficients.

Proposition 4.22. Let $F$ be a sheaf of abelian groups on $X$. If $Z \subseteq X$ is closed, for any section $s \in F(Z)$, there exists $U \supseteq Z$ open and $t \in F(U)$ such that $t|_Z = s$.

Proof. See the notes on soft sheaves on the course website (Lemma 2.3). \qed

Corollary 4.23. If $F$ is flasque, then $F$ is soft.

Proposition 4.24. If

$$0 \longrightarrow F' \xrightarrow{\varphi} F \xrightarrow{\psi} F'' \longrightarrow 0$$

is exact and $F'$ is soft, then

$$0 \longrightarrow F'(X) \longrightarrow F(X) \longrightarrow F''(X) \longrightarrow 0$$

is exact.

Proof. We only need to show that if $s \in F''(X)$, there exists $\tilde{s} \in F(X)$ such that $\psi(\tilde{s}) = s$. Since $\psi$ is surjective, there is an open cover $X = \bigcup_i U_i$, $\tilde{s}_i \in F(U_i)$ such that $\psi(\tilde{s}_i) = s|_{U_i}$.

By paracompactness of $X$, after passing to some refinement, we may assume this is a locally finite cover. Hence there is an open cover $X = \bigcup_i V_i$ such that $\overline{V}_i \subseteq U_i$.

For $J \subseteq I$, $Z_J = \bigcup_{i \in J} \overline{V}_i$ is closed in $X$ by local finiteness.

Consider pairs $(J, t)$ with $J \subseteq I$ and $t \in F(Z_J)$ such that $\psi(t) = s|_{Z_J}$. We order the pairs by declaring $(J, t) \leq (J', t')$ if $J \subseteq J'$ and $t'|_{Z_J} = t$. By Zorn’s Lemma, we may choose a maximal $(J, t)$. 

If \( I = J \), we are done. Otherwise, there exists \( i \in I \setminus J \), and we will produce a contradiction with maximality of \( J \). We have \( t \in \mathcal{F}(Z_J) \) and \( \tilde{s}_i|_{\overline{V}_i} \in \mathcal{F}(\overline{V}_i) \). Note that

\[
\psi(t|_{Z_J \cap \overline{V}_i}) = \psi(\tilde{s}_i|_{Z_J \cap \overline{V}_i})
\]

and hence

\[
t|_{Z_J \cap \overline{V}_i} - \tilde{s}_i|_{Z_J \cap \overline{V}_i} = \varphi(w)
\]

for some \( w \in \mathcal{F}'(Z_J \cap \overline{V}_i) \). Since \( \mathcal{F}' \) is soft, there exists \( \tilde{w} \in \mathcal{F}'(X) \) such that \( \tilde{w}|_{Z_J \cap \overline{V}_i} = w \). Replace

\[
\tilde{s}_i|_{U_i} \text{ by } \tilde{s}_i|_{U_i} + \varphi(\tilde{w}|_{U_i})
\]

to assume that \( t|_{Z_J \cap \overline{V}_i} = \tilde{s}_i|_{Z_J \cap \overline{V}_i} \).

By Proposition 4.14, there exists \( t' \in \mathcal{F}(Z_{J \cup \{i\}}) \) such that \( t'|_{Z_J} = t \) and \( t'|_{\overline{V}_i} = \tilde{s}_i|_{\overline{V}_i} \). Then

\[
\psi(t') = s|_{Z_{J \cup \{i\}}} \text{, contradicting maximality of the pair } (J, t).
\]

\[\square\]

**Corollary 4.25.** If \( 0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0 \) is exact, and \( \mathcal{F}' \) and \( \mathcal{F} \) are soft, then \( \mathcal{F}'' \) is soft.

**Proof.** Consider \( Z \subseteq X \) closed. Then \( Z \) is paracompact and \( \mathcal{F}''|_{Z} \) is soft. By Proposition 4.24, the diagram

\[
\begin{array}{cccccc}
0 & \to & \mathcal{F}'(X) & \to & \mathcal{F}(X) & \to & \mathcal{F}''(X) & \to & 0 \\
& \downarrow & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{F}'(Z) & \to & \mathcal{F}(Z) & \to & \mathcal{F}''(Z) & \to & 0
\end{array}
\]

has exact rows, and hence \( \mathcal{F}''(X) \to \mathcal{F}''(Z) \) is surjective. \[\square\]

We finally show that we can compute cohomology using soft sheaves.

**Theorem 4.26.**

1. If \( \mathcal{E} \) is a soft sheaf on \( X \), \( H^i(X, \mathcal{E}) = 0 \) for all \( i > 0 \).
2. If \( \mathcal{F} \) has a resolution

\[
0 \to \mathcal{F} \to \mathcal{E}^0 \to \mathcal{E}^1 \to \cdots
\]

with all \( \mathcal{E}^i \) soft, then

\[
H^i(X, \mathcal{F}) = \mathcal{H}^i(\Gamma(X, \mathcal{E}^\bullet)).
\]

**Proof.** Part (2) follows from part (1) by general reasons. For (1), we argue by induction on \( i \). Consider the short exact sequence

\[
0 \to \mathcal{E} \to \mathcal{I} \to \mathcal{G} \to 0
\]

with \( \mathcal{I} \) flasque. In particular, \( \mathcal{I} \) is soft by Corollary 4.23. Then, by Corollary 4.25, \( \mathcal{G} \) is also soft. The long exact sequence in cohomology then shows that:

\[
\Gamma(X, \mathcal{I}) \to \Gamma(X, \mathcal{G}) \to H^1(X, \mathcal{E}) \to H^1(X, \mathcal{I}), \text{ so } H^1(X, \mathcal{E}) = 0,
\]

surjective by Prop. 4.24

\[\square\]
\[ H^{i+1}(X, \mathcal{E}) = H^i(X, \mathcal{G}) = 0 \text{ for } i \geq 1 \text{ by inductive hypothesis.} \]

This completes the proof. \( \square \)

**Exercise.** Suppose \( f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y) \) is a morphism of ringed spaces with \( X, Y \) paracompact. Let \( \mathcal{F} \) be a sheaf on \( Y \) and \( \mathcal{G} \) be a sheaf on \( X \) so that we have a morphism \( f^* \mathcal{F} \to \mathcal{G} \). This induces maps 
\[ H^i(Y, \mathcal{F}) \to H^i(X, f^* \mathcal{F}) \to H^i(X, \mathcal{G}). \]

Show that if \( \mathcal{F} \to \mathcal{E}^\bullet \), \( \mathcal{G} \to \mathcal{M}^\bullet \) are soft resolutions and we have induced morphisms
\[
\begin{align*}
  f^* \mathcal{F} &\to f^* \mathcal{E}^\bullet \\
  \mathcal{G} &\to \mathcal{M}^\bullet
\end{align*}
\]
then we have a commutative diagram
\[
\begin{array}{ccc}
  H^i(Y, \mathcal{F}) & \xrightarrow{\cong} & H^i(\Gamma(Y, \mathcal{E}^\bullet)) \\
  \downarrow & & \downarrow \\
  H^i(X, \mathcal{G}) & \xrightarrow{\cong} & H^i(\Gamma(X, \mathcal{M}^\bullet))
\end{array}
\]

**Proposition 4.27.** If \( M \) is a smooth real manifold, then any \( \mathcal{C}_M^\infty \)-module is soft.

**Proof.** Let \( \mathcal{F} \) be a \( \mathcal{C}_M^\infty \)-module and \( Z \subseteq X \) be a closed subset. Consider \( s \in \mathcal{F}(Z) \). We want to extend it to a section on \( X \).

By Proposition 4.22, there is an open subset \( U \ni Z \) and a section \( \bar{s} \in \mathcal{F}(U) \) such that \( \bar{s}|_Z = s \).

Considering \( Z \subseteq U \), there is an open subset \( U_1 \) such that
\[ Z \subseteq U_1 \subseteq \overline{U_1} \subseteq U \]
and an open subset \( U_2 \) such that
\[ \overline{U_1} \subseteq U_2 \subseteq \overline{U_2} \subseteq U. \]

Then smooth version of Urysohn’s Lemma say that there exists a smooth function \( \varphi \) such that
\[ \begin{cases} 
  \varphi = 1 & \text{on } \overline{U_1}, \\
  \varphi = 0 & \text{on } X \setminus U_2.
\end{cases} \]

Consider \( \varphi|_U \cdot \bar{s} \) which is 0 on \( U \setminus U_2 \). Then there exists a section \( s' \in \mathcal{F}(X) \) such that \( s'|_{X \setminus \overline{U_2}} = 0 \) and \( s'|_U = \varphi|_U \bar{s}|_U \) since \( \varphi = 0 \) on \( X \setminus \overline{U_2} \). Note that
\[ s'|_{U_1} = \bar{s}|_{U_1} \]
since \( \varphi = 1 \) on \( U_1 \). Hence \( s'|_Z = s \). \( \square \)

**Applications.**

(1) If \( M \) is a smooth manifold of dimension \( n \), we have a resolution of \( \mathbb{R} \) given by
0 \longrightarrow \mathbb{R} \longrightarrow A^0_M \xrightarrow{d} A^1_M \xrightarrow{d} \cdots \xrightarrow{d} A^n_M \longrightarrow 0,

so we recover the de Rham Theorem:

\[ H^p(X, \mathbb{R}) \cong H^p_{dR}(X). \]

This gives a simple interpretation of the cup product on cohomology (which is messy to define otherwise) via \( \wedge \) of differential forms.

These are also isomorphic to singular cohomology. This is proven in the notes on the course website.

**Fact 4.28.** Since \( M \) is paracompact and locally contractible,

\[ H^p(M, \mathbb{R}) \cong H^p(M, \mathbb{R}). \]

(2) If \( M \) is a complex manifold of dimension \( n \), for all \( p \), we have an exact complex

\[ 0 \longrightarrow \Omega^p_M \longrightarrow A^{p,0}_M \xrightarrow{\delta} A^{p,1}_M \xrightarrow{\delta} \cdots \xrightarrow{\delta} A^{p,n}_M \longrightarrow 0, \]

which shows that

\[ H^{p,q}(X) = H^q(\Gamma(M, A^{p,*}_M)) \cong H^q(M, \Omega^p_M) \]

(\text{where the first equality is the definition of Dolbeaux cohomology}).

More generally, if \( E \) is a holomorphic vector bundle with sheaf of holomorphic sections \( \mathcal{E} \), the sheaf of sections is \( C^\infty_M \otimes_{\mathcal{O}_M} \mathcal{E} = \mathcal{E}_{\text{sm}} \). By taking \( (\ast) \otimes_{c^\infty_M} \mathcal{E}_{\text{sm}} = (\ast) \otimes_{\mathcal{O}_M} \mathcal{E} \), we get a complex

\[ 0 \longrightarrow \Omega^p_M \otimes_{\mathcal{O}_M} \mathcal{E} \longrightarrow A^{p,0}_E \xrightarrow{\delta_E} A^{p,1}_E \xrightarrow{\delta_E} \cdots \xrightarrow{\delta_E} A^{p,n}_{M,E} \longrightarrow 0. \]

Then

\[ H^{p,q}(X, \mathcal{E}) = H^q(\Gamma(M, A^{p,*}_{M,E})) \cong H^q(M, \Omega^p_M \otimes \mathcal{E}) \]

(\text{where the first equality is the definition of Dolbeaux cohomology}).

5. **Hodge theory on compact, oriented, Riemannian manifolds**

We are now done with introductory material to the class and we start Hodge theory. During the next few lectures, we discuss Hodge theory for Riemannian manifolds.

5.1. **Linear algebra background: the \( \ast \) operator.** Let \( V \) be a finite-dimensional vector space over \( \mathbb{R} \).

**Definition 5.1.** A *scalar product* on \( V \) is a symmetric bilinear form \( \langle \cdot, \cdot \rangle \) on \( V \times V \) valued in \( \mathbb{R} \) which is positive definite, i.e.

\[ \langle v, v \rangle > 0 \text{ for all } v \neq 0. \]

Given a scalar product on \( V \), we get an isomorphism

\[ V \xrightarrow{\cong} V^* \]

\[ v \mapsto \varphi_v = \langle v, \cdot \rangle. \]

We put a scalar product on \( V^* \) such that

\[ \langle \varphi_v, \varphi_w \rangle = \langle v, w \rangle \text{ for all } v, w. \]
Example 5.2. If $e_1, \ldots, e_n$ is an orthonormal basis of $V$ (i.e. $\langle e_i, e_j \rangle = \delta_{i,j}$ for all $i, j$) and $e_1^*, \ldots, e_n^*$ is the dual basis on $V^*$, then $\varphi_{e_i} = e_i^*$ and hence $e_1^*, \ldots, e_n^*$ is an orthonormal basis for $V^*$.

Exercise. Given a scalar product $\langle \cdot, \cdot \rangle$ on $V$, we have an induced scalar product on each $\bigwedge^p V$ such that
$$\langle v_1 \wedge \cdots \wedge v_p, w_1 \wedge \cdots \wedge w_p \rangle = \det(\langle v_i, w_j \rangle).$$
(Hint: use this as the definition and show that if $e_1, \ldots, e_n$ is an orthonormal basis, then we get an orthonormal basis for $\bigwedge^p V$ given by $\{e_I = e_{i_1} \wedge \cdots \wedge e_{i_p} \mid I = \{i_1 < \ldots < i_p\} \subseteq \{1, \ldots, n\}\}$.)

Suppose now that, in addition, that on $V$ we also have an orientation (i.e. an orientation on the 1-dimensional top exterior power of $V$).

In this case, we get a canonical volume element $\text{vol} \in \bigwedge^n V$ for $n = \dim V$ by choosing an orthonormal basis $e_1, \ldots, e_n$ such that $e_1 \wedge \cdots \wedge e_n$ is positive and letting
$$\text{vol} = e_1 \wedge \cdots \wedge e_n.$$  
This is independent on the choice of basis. If $e'_1, \ldots, e'_n$ is another such basis and we write
$$e'_i = \sum a_j a_{i,j} e_j,$$  
then for $A = (a_{i,j})$ we have $A \cdot A^t = I_n$, so $\det(A)^2 = 1$. Hence $\det(A) = \pm 1$ and we have
$$e'_1 \wedge \cdots \wedge e'_n = (\det A)e_1 \wedge \cdots \wedge e_n$$  
so $\det A = 1$ since $e'_1 \wedge \cdots \wedge e'_n$ and $e_1 \wedge \cdots \wedge e_n$ are both positive.

We now define the $\ast$ operator for $(V, \langle \cdot, \cdot \rangle, \text{orientation})$ where $n = \dim V$.

Proposition 5.3. For every $p$, $0 \leq p \leq n$, there is a unique isomorphism
$$\bigwedge^p V \xrightarrow{\ast} \bigwedge^{n-p} V$$  
such that
$$v \wedge (\ast w) = \langle v, w \rangle \text{vol} \quad \text{in} \quad \bigwedge^n V$$  
for all $v, w \in \bigwedge^p V$.

Proof. Recall that there is a canonical nondegenerate bilinear map
$$\bigwedge^p V \times \bigwedge^{n-p} V \to \bigwedge^n V \cong \mathbb{R}.$$  
given by $\wedge$. Given $w \in \bigwedge^p V$, we may consider the map
$$\langle -, w \rangle \text{vol}: \bigwedge^p V \to \bigwedge^n V \cong \mathbb{R}.$$  
Using the pairing (2), there is an element $\ast w \in \bigwedge^n V$ such that
$$\langle -, w \rangle \text{vol} = - \wedge \ast w.$$
We get a linear map \( *: \bigwedge^p V \to \bigwedge^{n-p} V \). It is clear this is injective: if \( *w = 0 \), then for all \( v \), \( \langle v, w \rangle = 0 \), so \( w = 0 \) since \( \langle -, - \rangle \) is non-degenerate.

By dimension considerations, \( * \) is an isomorphism. \( \square \)

We now describe \( * \) via an orthonormal basis. Recall that

\[
e_J \wedge (*e_J) = \langle e_J, e_I \rangle \text{vol}.
\]

This shows that, by taking \( I = J \),

\[
*e_I = \epsilon(I, \mathcal{T}) \cdot e_\mathcal{T} \quad \text{where} \quad \mathcal{T} = \{1, \ldots, n\} \setminus I
\]

where \( \epsilon(I, \mathcal{T}) \) is the signature of the permutation \( (I, \mathcal{T}) \). In other words,

\[
e_I \wedge e_\mathcal{T} = \epsilon(I, \mathcal{T}) e_1 \wedge \cdots \wedge e_n.
\]

**Properties of \( * \):**

1. \( *(\text{vol}) = 1 \),
2. \( **: \bigwedge^p V \to \bigwedge^p V \) is equal to \( (-1)^{p(n-p)} \) because

\[
(a_1 \wedge \cdots \wedge a_p) \wedge (b_1 \wedge \cdots \wedge b_{n-p}) = (-1)^{p(n-p)} (b_1 \wedge \cdots \wedge b_{n-p}) \wedge (a_1 \wedge \cdots \wedge a_p).
\]

5.1.1. The global situation. Let \( M \) be a smooth manifold and \( E \) be a smooth real vector bundle on \( M \) of rank \( n \). Write \( \mathcal{E} \) for the sheaf of sections.

**Definition 5.4.** A metric (or scalar product) on \( E \) is a smoothly varying family of scalar products on the fibers of \( E \). Concretely, for all \( p \in M \), we have a scalar product \( \langle \cdot, \cdot \rangle \) on \( \mathcal{E}_p \) such that for sections \( s, t \in \mathcal{E}(U) \), the map

\[
U \ni p \mapsto \langle s(p), t(p) \rangle \in \mathbb{R}
\]

is a smooth function. (Note that it is enough to check this for \( s_1, \ldots, s_n \) which trivialize \( \mathcal{E} \) over open subsets.)

**Example 5.5.** If \( E = M \times \mathbb{R}^n \), then the standard scalar product on \( \mathbb{R}^n \) gives a scalar product on each fiber, which is a matrix on \( E \).

In particular, we always have such metrics locally on any \( E \) locally on \( M \). By using partitions of unity, get metrics on \( E \). Hence on every \( E \), we have such a metric.

If, in addition, we have an orientation of \( E \) (i.e. a compatible system of orientations of all fibers), we get an element \( \text{vol} \in \Gamma \left( M, \bigwedge^n \mathcal{E} \right) \) which is everywhere nonzero, belonging to the (positive) orientation.

We get a global \( * \) operator

\[
*: \bigwedge E \to \bigwedge E
\]

globalizing the one on each fiber.
5.1.2. The tangent bundle.

**Definition 5.6.** A Riemannian metric on $M$ is a metric on $TM$.

A Riemannian metric induces a metric on $T^*M$ and on $\bigwedge^p T^*M$.

If $M$ is oriented, i.e. we have an orientation on $TM$ (or equivalently on $T^*M$), we can apply the previous considerations.

In particular, we have an $n$-form (where $n$ is the dimension of $M$) $dV$ called the volume element which is everywhere nonzero and positively oriented.

We get $*: \mathcal{A}^p_M \cong \mathcal{A}^{p-1}_M$ such that $\omega \wedge (*\eta) = \langle \omega, \eta \rangle dV$.

From now on, let $M$ be a compact manifold with orientation and a Riemannian structure. Compactness allows us to define a scalar product on $\mathcal{A}^p(M)$ by

$$\langle\langle \omega, \eta \rangle\rangle = \int_M \langle \omega, \eta \rangle dV.$$

It is clearly bilinear, symmetric and positive-definite: if $\omega \neq 0$, then $\langle\langle \omega, \omega \rangle\rangle > 0$.

One caveat is that $\mathcal{A}^p(M)$ is not finite-dimensional. Actually, it is not even complete with respect to the metric induced by $\langle\langle \cdot, \cdot \rangle\rangle$.

**Definition 5.7.** Let $d^*: \mathcal{A}^p_M \to \mathcal{A}^{p-1}_M$ be given by

$$d^* = (-1)^{n(p-1)+1} d *$$

$$= (-1)^p * d *$$

**Proposition 5.8.** For every $p$, $d^*: \mathcal{A}^{p+1}(M) \to \mathcal{A}^p(M)$ is the formal adjoint of $d: \mathcal{A}^p(M) \to \mathcal{A}^{p+1}(M)$ with respect to $\langle\langle \cdot, \cdot \rangle\rangle$. Explicitly,

$$\langle\langle d\omega, \eta \rangle\rangle = \langle\langle \omega, d^*\eta \rangle\rangle$$

for all $\eta \in \mathcal{A}^{p+1}(M)$, $\omega \in \mathcal{A}^p(M)$.

This is only a formal adjoint because these are not really Hilbert spaces (they are not complete).

**Remark 5.9.** Such a formal adjoint is unique if it exists: if $\overline{d}^*$ is another such operator, then

$$\langle\langle \omega, d^*\eta \rangle\rangle = \langle\langle \omega, \overline{d}^*\eta \rangle\rangle,$$

then for $\omega = d^*\eta - \overline{d}^*\eta$ we have $\langle\langle \omega, \omega \rangle\rangle = 0$, so $\omega = 0$. 
Proof. We compute \( \langle \langle \omega, d^* \eta \rangle \rangle \) using the definition:

\[
\langle \langle \omega, d^* \eta \rangle \rangle = \int_M \langle \omega, (-1)^{p+1} d * \eta \rangle dV
\]

\[
= (-1)^{p+1} \int_M \omega \wedge d^* \eta
\]

\[
= - \int_M d(\omega \wedge \eta) + \int_M d\omega \wedge \eta
\]

\[
= \langle \langle d\omega, \eta \rangle \rangle dV
\]

\[
= \langle \langle d\omega, \eta \rangle \rangle,
\]

where we have used that \( d \) is a derivation. \( \square \)

We define the Laplace–Beltrami operator:

\[
\Delta = dd^* + d^* d : \mathcal{A}^p(M) \to \mathcal{A}^p(M).
\]

Proposition 5.10. The operator \( \Delta \) is formally self-adjoint.

Proof. We have that

\[
\langle \langle \Delta \omega, \eta \rangle \rangle = \langle \langle dd^* \omega + d^* d\omega, \eta \rangle \rangle
\]

\[
= \langle \langle d^* \omega, \eta \rangle \rangle + \langle \langle d\omega, d\eta \rangle \rangle
\]

by Proposition 5.8.

By symmetry, this is also equal to \( \langle \langle \omega, \Delta \eta \rangle \rangle \). \( \square \)

Definition 5.11. A form \( \omega \in \mathcal{A}^p(M) \) is harmonic if \( \Delta \omega = 0 \).

Proposition 5.12. A form \( \omega \) is harmonic if and only if \( d\omega = 0 \) and \( d^* \omega = 0 \).

Proof. The ‘if’ implication is clear from the definition of \( \Delta \). For the ‘only if’ implication, we use the formula \( \langle \langle \Delta \omega, \eta \rangle \rangle = \langle \langle d^* \omega, d^* \eta \rangle \rangle + \langle \langle d\omega, d\eta \rangle \rangle \) from the proof of Proposition 5.10 for \( \eta = \omega \) such that \( d\omega = 0 \) to conclude that

\[
0 = \| d^* \omega \|^2 + \| d\omega \|^2,
\]

so \( d\omega = 0 \) and \( d^* \omega = 0 \). \( \square \)

The goal is to prove that every de Rham cohomology class is represented by a unique harmonic representative. This is the famous Hodge Theorem 5.21. We start with the following lemma.

Lemma 5.13. For a given de Rham cohomology class, a representative \( \omega \) is harmonic if and only if \( \| \omega \| \) is minimal.

Proof. Given a \( p \)-form \( \omega \) such that \( d\omega = 0 \), consider \( \omega + d\eta \) for all \( (p - 1) \)-forms \( \eta \). Then

\[
\| \omega + d\eta \|^2 = \| \omega \|^2 + \| d\eta \|^2 + 2 \langle \langle \omega, d\eta \rangle \rangle \frac{\langle \langle d^* \omega, \eta \rangle \rangle}{\langle \langle d^* \omega, \eta \rangle \rangle}
\]

(3)

The goal is to prove that every de Rham cohomology class is represented by a unique harmonic representative. This is the famous Hodge Theorem 5.21. We start with the following lemma.

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Proof. Given a \( p \)-form \( \omega \) such that \( d\omega = 0 \), consider \( \omega + d\eta \) for all \( (p - 1) \)-forms \( \eta \). Then

\[
\| \omega + d\eta \|^2 = \| \omega \|^2 + \| d\eta \|^2 + 2 \langle \langle \omega, d\eta \rangle \rangle \frac{\langle \langle d^* \omega, \eta \rangle \rangle}{\langle \langle d^* \omega, \eta \rangle \rangle}
\]

(3)
Since \( d\omega = 0 \), \( \omega \) is harmonic if and only if \( d^*\omega = 0 \). If this holds, then 
\[
\|\omega + d\eta\|^2 \geq \|\omega\|^2
\]
for all \( \eta \) by formula (3).

Conversely, if \( \|\omega\|^2 \) is minimal among all \( \|\omega + d\eta\|^2 \), then 
\[
\frac{d}{dt}\|\omega + t\eta\|^2_{t=0} = 0.
\]
According to formula (3) derivative is equal to \( 2\langle\langle \omega, d\eta \rangle\rangle \). For \( \eta = d^*\omega \), we get that 
\[
0 = \langle\langle \omega, dd^*\omega \rangle\rangle = \langle\langle d^*\omega, d^*\omega \rangle\rangle,
\]
so \( d^*\omega = 0 \).

\[\square\]

Note that if \( \omega \) and \( \omega' \) are harmonic in the same cohomology class, then \( \|\omega\|^2 = \|\omega'\|^2 \). If we write \( \omega = \omega' + d\eta \), then \( \|d\eta\|^2 = 0 \) by formula (3), so \( \omega = \omega' \). This shows uniqueness.

Proving existence will be much more difficult. The problem is that the space \( \mathcal{A}^p(M) \) is not only infinite-dimensional but also not complete. Therefore, there is no abstract way to conclude that the desired minimum exists.

**Proposition 5.14.** The operators \( \ast \) and \( \Delta \) commute: \( \ast\Delta = \Delta\ast \).

**Corollary 5.15.** The form \( \omega \) is harmonic if and only if \( \ast\omega \) is harmonic.

**Proof of Proposition 5.14.** Computing on \( p \)-forms:
\[
\ast\Delta = \ast(dd^* + d^*d) = (-1)^{n(p-1)+1} \ast d \ast d \ast + (-1)^{p+1} d \ast d
\]
and
\[
\Delta\ast = (dd^* + d^*d)\ast = (-1)^{n(n-p-1)+1} d \ast d \ast + \underbrace{(-1)^n d \ast d \ast}_{(-1)^{n-p}} + (-1)^{n(p-1)+1} d \ast d \ast
\]
\[
= (-1)^{p+1} d \ast d + (-1)^{n(p-1)+1} d \ast d \ast,
\]
which agrees with \( \ast\Delta \) above.

\[\square\]

Note that if \( n \) is even (for example, \( M \) is a complex manifold), then the sign becomes simply \( d^* = -\ast d \ast \), which is easier to keep track of.

We have a formally self-adjoint operator
\[
\Delta: \mathcal{A}^p(M) \to \mathcal{A}^p(M).
\]

If we have a self-adjoint linear map \( T: V \to V \), where \( V \) is a finite-dimensional vector space with a scalar product \( \langle \cdot, \cdot \rangle \), then \( \ker(T) \) is perpendicular to \( \text{im}(T) \), because for \( Tu = 0 \), we have
\[
\langle u, Tv \rangle = \langle Tu, v \rangle = 0.
\]
Moreover, these have complementary dimension and hence $\langle \cdot, \cdot \rangle$ gives an isomorphism

$$V = \ker(T) \overset{\perp}{\oplus} \operatorname{im}(T).$$

The same holds if $T$ is an operator on a Hilbert space.

However, the spaces $\mathcal{A}^p(M)$ are not Hilbert spaces. We hence have to do more work in other to prove such a statement for $T = \Delta, V = \mathcal{A}^p(M)$.

### 5.2. Differential operators.

Let $M$ be a smooth manifold and $\mathcal{C}_M^\infty$ be the sheaf of real-valued smooth functions on $M$. Consider

$$\mathcal{D}_M \subseteq \mathfrak{End}_\mathbb{R}(\mathcal{C}_M^\infty)$$

generated as a sheaf of rings by $\mathcal{C}_M^\infty$ (acting by homotheties) and $\mathcal{D}\mathfrak{er}_\mathbb{R}(\mathcal{C}_M^\infty)$. Note that this is a sheaf of non-commutative rings.

If $U \subseteq M$ is a chart with coordinates $x_1, \ldots, x_n$, then $\mathcal{D}\mathfrak{er}(\mathcal{C}_M^\infty)$ is generated over $\mathcal{C}_M^\infty$ by $\partial_1, \ldots, \partial_n$, where $\partial_i = \frac{\partial}{\partial x_i}$.

Hence $\mathcal{D}_U$ is free over $\mathcal{C}_M^\infty$ (both as a left and as a right modulo), with basis given by

$$\partial^\alpha = \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n} \text{ for } \alpha = (\alpha_1, \ldots, \alpha_n).$$

Let $\mathcal{F}_k \mathcal{D}_M \subseteq \mathcal{D}_M$ for $k \geq 0$ be the subsheaf of locally generated (in charts as above) by $\partial^\alpha$ with $|\alpha| = \alpha_1 + \cdots + \alpha_n \leq k$. We call these differential operators of order $\leq k$. For example:

- $\mathcal{F}_0 \mathcal{D}_M = \mathcal{C}_M^\infty$,
- $\mathcal{F}_1 \mathcal{D}_M = \mathcal{C}_M^\infty + \mathcal{D}\mathfrak{er}(\mathcal{C}_M^\infty)$.

They satisfy the following obvious properties.

1. We see that

$$\mathcal{F}_k \mathcal{D}_M \cdot \mathcal{F}_\ell \mathcal{D}_M \subseteq \mathcal{F}_{k+\ell} \mathcal{D}_M$$

using $[\partial_k, g] = \frac{\partial g}{\partial x_k}$. This implies that

$$\operatorname{gr}_F \mathcal{D}_M = \bigoplus_{k \geq 0} \mathcal{F}_k \mathcal{D}_M / \mathcal{F}_{k-1} \mathcal{D}_M$$

has an induced graded ring structure.

2. We have that $[\mathcal{F}_k \mathcal{D}_M, \mathcal{F}_\ell \mathcal{D}_M] \subseteq \mathcal{F}_{k+\ell-1} \mathcal{D}_M$, so $\operatorname{gr}_F \mathcal{D}_M$ is a sheaf of commutative rings.

Note that

$$\operatorname{gr}_F \mathcal{D}_M = \mathcal{C}_M^\infty \oplus \mathcal{T}_M \oplus \cdots,$$

where we write $\mathcal{T}_M$ for the sheaf of sections of the tangent bundle $TM$ and identify it with $\mathcal{D}\mathfrak{er}_\mathbb{R}(\mathcal{C}_M^\infty)$.

By the universal property of the symmetric algebra, we get a morphism of sheaves of graded commutative $\mathcal{C}_M^\infty$-algebras:

$$\mathcal{S}\mathfrak{ym}_{\mathcal{C}_M^\infty}(\mathcal{T}_M) \to \operatorname{gr}_F \mathcal{D}_M.$$
Using the local description of $\mathcal{D}_X$ in a chart, we see that this is an isomorphism. Given an operator, $P \in \Gamma(M, \mathcal{D}_X)$ of order $k$ (i.e. order $\leq k$ but not $\leq k - 1$), the symbol of $P$ is the corresponding section

$$\sigma_k(P) \in \Gamma(M, \text{Sym}^k(\mathcal{T}_M)).$$

More generally, suppose $E, F$ are smooth (real) vector bundles on $M$, with corresponding sheaves $\mathcal{E}, \mathcal{F}$. Then

$$\text{Diff}_k(\mathcal{E}, \mathcal{F}) = \left\{ P \in \text{End}(\mathcal{E}, \mathcal{F}) \middle| \begin{array}{l}
\text{locally on open subsets } U \subseteq X \\
such that } \mathcal{E}|_U \cong \mathcal{O}_U^{\oplus r}, \mathcal{F}|_U \cong \mathcal{O}_U^{\oplus s}
\end{array} \\
P \text{ is given by } (P_{i,j}) \text{ with each } P_{i,j} \text{ a differential operator of order } \leq k \right\}.$$ 

**Example 5.16.** The map $d: \mathcal{A}^p_M \to \mathcal{A}^{p+1}_M$ is a differential operator of order 1.

If $P$ is a differential operator of order $\leq k$, we want to define $\sigma_k(P)$, as above. Locally on an open subset $U \subseteq X$ such that $\mathcal{E}|_U \cong \mathcal{O}_U^{\oplus r}, \mathcal{F}|_U \cong \mathcal{O}_U^{\oplus s}$, if $P$ is given by $(P_{i,j})_{i,j}$, we consider $(\sigma_k(P_{i,j}))_{i,j}$ where

$$\sigma_k(P_{i,j}) \in \Gamma(U, \text{Sym}(\mathcal{T}_M)).$$

These glue together to give

$$\sigma_k(P) \in \Gamma(M, \text{Sym}(\mathcal{T}_M) \otimes \text{Hom}(\mathcal{E}, \mathcal{F})).$$

Given $x \in M$, we get a map

$$T_x^* M \to \text{Hom}_\mathbb{R}(E(x), F(x))$$

which is a homogeneous polynomial of degree $k$.

**Definition 5.17.** A differential operator $P \in \text{Diff}_k(E, F)$, with $\text{rank}(E) = \text{rank}(F)$, is elliptic if for all $x \in M$ and any non-zero $v \in T_x^* M$, $\sigma_k(P)_x(v)$ is an isomorphism.

If $P = \sum_{|\alpha| \leq k} a_\alpha(x) \partial^\alpha$, then

$$\sigma_k(P) = \sum_{|\alpha| = k} a_\alpha(x) y^\alpha.$$ 

Therefore, $P$ is elliptic if and only if for all $(y_1, \ldots, y_n) \neq 0$ and all $x$,

$$\sum_{|\alpha| = k} a_\alpha(x) y^\alpha \neq 0.$$ 

**Example 5.18** (Main example). The *Laplace-Beltrami operator* $\Delta$, where

$$\Delta = d^* d + dd^* \quad d^* = \pm * d *.$$ 

Recall that $*$ has order 0 and $d$ has order 1, so $\Delta$ is a differential operator of order $\leq 2$.

**Goals.**

- Compute $\Delta$ on $\mathbb{R}^n$ with the usual metric and orientation.
- Compute $\sigma_2(\Delta)$ in general and show that $\Delta$ is elliptic.
Recall that if $M$ is a smooth manifold, $X$ is a vector field, and $\omega$ is a $p$-form, $i_X \omega$ is the contraction of $\omega$ with respect to $X$ given by: if $X_1, \ldots, X_{p-1}$ are vector fields, then
\[(i_X \omega)(X_1, \ldots, X_{p-1}) = \omega(X, X_1, \ldots, X_{p-1}).\]

For example, $i_X(df) = X(f)$.

**Exercise.** The contraction along $X$, $i_X$, behaves well with respect to $\wedge$:
\[i_X (\alpha \wedge \beta) = i_X(\alpha) \wedge (-1)^{\deg \alpha} \alpha \wedge i_X(\beta).\]

We write down $i_X$ explicitly in local coordinates. If $\xi_1, \ldots, \xi_n$ trivialize $T^*M$ on $U$ and $\xi_1^*, \ldots, \xi_n^*$ is the dual basis of $T^*M$, we set
\[\xi_I^* = \xi_{i_1}^* \wedge \cdots \wedge \xi_{i_p}^* \quad \text{for} \ I : i_1 < \cdots < i_p.\]

Then
\[i_{\xi_j} (\xi_I^*) = \begin{cases} 0 & \text{if } j \not\in I \\ (-1)^{k-1} \xi_{I \setminus \{j\}}^* & \text{if } j = i_k \end{cases}\]

**Lemma 5.19.** Let $M$ be an oriented Riemannian manifold and $\xi_1, \ldots, \xi_n$ be an orthonormal positively oriented local basis for $T^*M$. Then for all $I$ with $|I| = p$,
\[* (\xi_j^* \wedge * \xi_I^*) = (-1)^{n(p-1)} i_{\xi_j} (\xi_I^*).\]

**Proof.** We first make sure that this equality is correct when we ignore the signs. For $j \not\in I$, both sides are clearly 0. Otherwise, if $j \in I$, the left hand side is
\[\pm * (\xi_j^* \wedge \pm \xi_I^*) = \pm * \xi_{I \setminus \{j\}}^* = \pm \xi_I^* .\]

Checking that the signs agree is left as an exercise.

We now compute $\sigma_2(\Delta)$ in general and $\Delta$ if $M = \mathbb{R}^n$ with standard metric and orientation. Consider a $p$-form $\omega$ which may be written in the local coordinates as $\omega = \sum_{|I| = p} f_I \xi_I^*$. Then
\[d^* \omega = (-1)^{n(p-1)+1} * d * (\omega) \]
\[= (-1)^{n(p-1)+1} \sum_{|I| = p} f_I * \xi_I^* \]
\[= (-1)^{n(p-1)+1} \sum_{|I| = p} \sum_{k=1}^n \xi_k(f_I)(\xi_k^* \wedge * \xi_I^*) + (-1)^{n(p-1)+1} \sum_{|I| = p} f_I d(* \xi_I^*).\]

Term B is
- $= 0$ in $\mathbb{R}^n$ with $\xi_i = \frac{\partial}{\partial x_i}$,
- in general, can be ignored in the computation of $\sigma_2(\Delta)$. 

We are hence left with computing Term A. We have that
\[
\text{Term A} = (-1)^{n(p-1)+1} \sum_{|I|=p} \sum_{k=1}^{n} \xi_k(f_I) \left( (\xi^*_k)^* \wedge \xi^*_I \right)
\]

\[
= - \sum_{I} \sum_{k} \xi_k(f_I)i_{\xi_k}(\xi^*_I).
\]
So far, we have shown that
\[
d^*\omega = - \sum_{I} \sum_{k} \xi_k(f_I)i_{\xi_k}(\xi^*_I) + (\text{linear operator in } \omega)
\]
and the linear operator in \(\omega\) is 0 if \(U = \mathbb{R}^n\) with the standard metric and basis. Moreover,
\[
d\omega = \sum_{|I|=p} \sum_{j} \xi_j(f_I)\xi^*_j \wedge \xi^*_I.
\]
Putting these together, we see that
\[
dd^* = - \sum_{I} \sum_{j,k} \xi_j \xi_k(f_I)\xi^*_j \wedge i_{\xi_k}(\xi^*_I) + (\text{operator of order } \leq 1 \text{ in } \omega),
\]
\[
d^*d = - \sum_{I} \sum_{j,k} \xi_k \xi_j(f_I)i_{\xi_k}(\xi^*_j \wedge \xi^*_I) + (\text{operator of order } \leq 1 \text{ in } \omega).
\]

and hence
\[
\Delta \omega = (dd^* + d^*d)\omega = - \sum_{I} \sum_{j,k} \xi_j \xi_k(f_I)(\xi^*_j \wedge i_{\xi_k}(\xi^*_I) + i_{\xi_k}(\xi^*_j \wedge \xi^*_I)) + (\text{operator of order } \leq 1 \text{ in } \omega)
\]
where we use the fact that \([\xi_k, \xi_j]\) is an operator of order \(\leq 1\). Using the formula (4) for \(i_X(\alpha \wedge \beta)\), we see that
\[
\Delta \omega = - \sum_{I} \sum_{j,k} \xi_k \xi_j(f_I)i_{\xi_k}(\xi^*_j) \wedge \xi^*_I - \sum_{k} \xi^2_k(f_I)\xi^*_k + (\text{operator of order } \leq 1 \text{ in } \omega).
\]

Conclusion.

1. If \(M = \mathbb{R}^n\), we get the formula
\[
\Delta \left( \sum_{I} f_I dx_I \right) = - \sum_{I} \left( \sum_{k=1}^{n} \frac{\partial^2 f}{\partial x_k^2} \right) dx_I
\]
which is the standard Laplace operator in \(\mathbb{R}^n\).

2. In general, \(\sigma_2(\Delta)\) ignores the operators of order \(\leq 1\), so we get the expression
\[
\sigma_2(\Delta)_x \left( \sum_{k=1}^{n} v_k \xi^*_k(x) \right) = - \left( \sum_{k=1}^{n} v_k^2 \right) \cdot \text{Id}
\]
so
\[
\sigma_2(\Delta)_x(v) = -\|v\|^2 \cdot \text{Id}
\]
which is an isomorphism if \(v \neq 0\). In particular, \(\Delta\) is an elliptic operator.
Suppose \( P \in \text{Diff}_k(\mathcal{E}, \mathcal{F}) \) where \( M \) is compact and oriented. If we have a metric on \( \mathcal{E} \) and a volume element \( dV \), we can define a scalar product on \( \mathcal{E}(M) \) by 
\[
\langle \langle s, t \rangle \rangle = \int_M \langle s, t \rangle \, dV. 
\]
Given \( P \in \text{Diff}_k(\mathcal{E}, \mathcal{F}) \) such that both \( \mathcal{E}, \mathcal{F} \) carry metrics, there is a formal adjoint 
\[
P^* \in \text{Diff}_k(\mathcal{F}, \mathcal{E})
\]
such that 
\[
\langle \langle Ps, t \rangle \rangle = \langle \langle s, P^* r \rangle \rangle 
\]
for all \( s \in \mathcal{E}(M), t \in \mathcal{F}(M) \).
Moreover, for all \( x \in M, v \in T^*_x(M), \)
\[
\sigma_k(P^*)_x(v) = (\sigma_k(P)_x(v))^* 
\]
where the right hand side is the adjoint with respect to the scalar product on the fibers.
In particular, if \( \text{rank}(E) = \text{rank}(F) \), then \( P \) is elliptic if and only if \( P^* \) is elliptic.

**Theorem 5.20 (Fundamental theorem).** Suppose \( M \) is compact and oriented, and \( E, F \) are smooth vector bundles of the same rank, with metrics, and we have a volume element \( dV \). For an elliptic differential operator \( P \in \text{Diff}_k(E, F) \), we have

1. \( \dim \ker(P) < \infty \) (so \( \text{codim}_\mathbb{R} \text{im}(P) < \infty \)),
2. \( \mathcal{E}(M) = \ker(P) \oplus \text{im}(P^*) \).

In particular, if \( P \) is self-adjoint, then 
\[
\mathcal{E}(M) = \ker(P) \oplus \text{im}(P). 
\]

Note that \( \ker(P) \perp \text{im}(P^*) \) by adjointness. The assertion in the theorem is that \( \mathcal{E}(M) \) is a direct sum of the two.

The subtle issue is that \( \mathcal{E}(M) \) and \( \mathcal{F}(M) \) are not complete with respect to \( \langle \langle \cdot, \cdot \rangle \rangle \), so we cannot apply the usual theory of Hilbert spaces. We have first to enlarge the spaces to suitable spaces of *distributions*. The hard part is then showing that, given an elliptic operator \( P^* \) and a section \( s \) of \( \mathcal{E} \) with coefficients in a distribution, if \( P^* s \) is smooth, then \( s \) is smooth.

What does this say about \( \Delta \)? Since \( \Delta \) is elliptic and self-adjoint, Theorem 5.20 implies that 

\[
\mathcal{A}^p(M) = \ker(\Delta) \oplus \text{im}(\Delta). 
\]

We already know that the kernel is the space of harmonic forms: 
\[
\ker(\Delta) = \mathcal{H}^p(M, \mathbb{R}) = \{ \omega \in \mathcal{A}^p(M) \mid \Delta \omega = 0 \}. 
\]

We need to compute the image. We have that 
\[
\Delta(\mathcal{A}^p(M)) \subseteq d(\mathcal{A}^{p-1}(M)) + d^*(\mathcal{A}^{p+1}(M))
\]
because \( \Delta = dd^* + d^*d \). Note that 

1. \( d(\mathcal{A}^{p-1}(M)) \perp d^*(\mathcal{A}^{p+1}(M)) \), because 
\[
\langle \langle d\omega, d^* \eta \rangle \rangle = \langle \langle d^2 \omega, \eta \rangle \rangle = 0, 
\]

(5) \[ \mathcal{A}^p(M) = \ker(\Delta) \oplus \text{im}(\Delta). \]
(2) $\mathcal{H}^p(M, \mathbb{R})$ is orthogonal to 
\[ d(A^{p-1}) + d^*(A^{p+1}(M)), \]
because for harmonic $\omega$, $d\omega = 0$ and $d^*\omega = 0$, so 
\[ \langle \omega, d\eta \rangle = \langle d^*\omega, \eta \rangle = 0 \]
and similarly $\langle \omega, d^*\theta \rangle = 0$.

The orthogonal decomposition (5) together with the inclusion (6) shows that 
\[ \Delta(A^p(M)) = d(A^{p-1}(M)) \perp d^*(A^{p+1}(M)). \]

What is the kernel $\ker(d: A^p(M) \to A^{p+1}(M))$? It contains $\mathcal{H}^p(M, \mathbb{R})$ and $d(A^{p-1}(M))$. We claim that 
\[ \ker(d: A^p(M) \to A^{p+1}(M)) = \mathcal{H}^p(M, \mathbb{R}) \perp d(A^{p-1}(M)). \]

For this, it is enough to show that if $d^*\eta \in \ker(d)$, then $d^*\eta = 0$ (using the decomposition (5)).
This is clear since $dd^*\eta = 0$ implies that 
\[ \langle d^*\eta, d^*\eta \rangle = \langle \eta, dd^*\eta \rangle = 0, \]
so $d^*\eta = 0$.

Finally, we deduce from this discussion the fundamental theorem of Hodge theory.

**Corollary 5.21** (Hodge theorem). Suppose $M$ is a compact oriented Riemannian manifold. We have a canonical isomorphism 
\[ H^p_{dR}(M, \mathbb{R}) \cong \mathcal{H}^p(M, \mathbb{R}). \]

**Corollary 5.22.** We have that $\dim_{\mathbb{R}} H^p_{dR}(M, \mathbb{R}) < \infty$.

**Proof.** This follows from part (1) of Theorem 5.20. \qed

One can also prove this theorem using triangulation of manifolds. This is, however, not much easier than the method we employed.

**Elementary application.** Let $M$ be a compact, orientable manifold. We then have a Poincaré duality. Put a metric on $M$ and choose an orientation. Then $*$ gives an isomorphism 
\[ \mathcal{H}^p(M, \mathbb{R}) \cong \mathcal{H}^{n-p}(M, \mathbb{R}). \]

depending on the choice of metric.

Here is a better statement: the pairing 
\[ H^p_{dR}(M, \mathbb{R}) \times H^{n-p}_{dR}(M, \mathbb{R}) \to \mathbb{R} \]
\[ (\omega, \eta) \mapsto \int_M \omega \wedge \eta \]
is non-degenerate. To see this, it is enough to show that for every $p$ and every $\alpha \in H^p_{dR}(M, \mathbb{R})$, there is a $\beta \in H^{n-p}_{dR}(M, \mathbb{R})$ such that 
\[ \int_M \alpha \wedge \beta \neq 0. \]
For this, choose a metric, and choose \( \omega \in \mathcal{H}^p(M, \mathbb{R}) \) such that \([\omega] = \alpha\). Then take \( \beta = [\ast \omega] \) where \( \ast \omega \) is also harmonic to obtain

\[
\int_M \omega \wedge \ast \omega = \int_M \langle \omega, \omega \rangle dV = \langle \langle \omega, \omega \rangle \rangle > 0
\]

if \( \omega \neq 0 \). This implies the non-degeneracy of the Poincaré duality.

6. Hodge theory of complex manifolds

We now discuss Hodge theory for complex manifolds.

6.1. Linear algebra background. Let \( V \) be a finite-dimensional complex vector space. Then \( V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C} = V' \oplus V'' \) with each \( V', V'' \) isomorphic to \( V \).

**Definition 6.1.** A Hermitian form on \( V \) is a bilinear map

\[
V \times V \rightarrow \mathbb{C}
\]

such that \( h(v, w) = \overline{h(w, v)} \).

Given such \( h \), we write it as \( h = S + iA \) where \( S, A : V \times V \rightarrow \mathbb{R} \) are bilinear over \( \mathbb{R} \) and \( S \) is symmetric, \( A \) is skew-symmetric. Note that

\[
S(iv, w) + iA(iv, w) = i(S(v, w) + iA(v, w))
\]

so

\[
S(iv, w) = -A(v, w),
A(iv, w) = S(v, w).
\]

It is easy to see that, giving a Hermitian form \( h \) on \( V \) is equivalent to giving a symmetric bilinear form \( S : V_{\mathbb{R}} \times V_{\mathbb{R}} \rightarrow \mathbb{R} \) such that \( S(iv, iw) = S(v, w) \) for all \( v, w \). In this case, \( A \) is defined by \( A(v, w) = -S(iv, w) \). It is also equivalent to giving a skew-symmetric bilinear form \( A : V_{\mathbb{R}} \rightarrow V_{\mathbb{R}} \rightarrow \mathbb{R} \) such that \( A(iv, iw) = A(v, w) \). In this case, \( S \) is given by \( S(v, w) = A(iv, w) \).

**Definition 6.2.** The Hermitian form \( h \) is a Hermitian metric such that \( h(v, v) > 0 \) for all \( v \in V \) non-zero.

In this case, \( S = S_h \) given a scalar product on \( V_{\mathbb{R}} \). This can be uniquely extended to a Hermitian form \( S_h \) on \( V_{\mathbb{C}} \). This is again a metric:

\[
S_h(v + iw, v + iw) = S_h(v, v) + S_h(w, w) + i(S_h(w, v) - S_h(v, w)) = S_h(v, v) + S_h(w, w) > 0
\]

if \( v + iw \neq 0 \).

**Lemma 6.3.** The canonical isomorphism \( V \cong V' \) given by

\[
\begin{array}{ccc}
V & \cong & V' \\
\downarrow & & \downarrow \\
V \otimes_{\mathbb{R}} \mathbb{C} & \overset{\text{proj}}{\rightarrow} & V'
\end{array}
\]
is compatible with the Hermitian forms (up to a constant scalar factor).

**Proof.** Recall that if \( J : V \to V \) is multiplication by \( i \), then

\[
V \to V',
\]

\[
v \mapsto \frac{1}{2}(v - iJv).
\]

We then have

\[
S_h \left( \frac{1}{2}(v - iJv), \frac{1}{2}(w - iJw) \right) = \frac{1}{4} \left( (S_h(v, w) + S_h(Jv, Jw)) - i(S_h(Jv, w) - S_h(v, Jw)) \right)
\]

\[
= \frac{1}{2} (S_h(v, w) - iS_h(Jv, w))
\]

\[
= \frac{1}{2} h(v, w)
\]

This is what we wanted to prove. \( \square \)

**Remark 6.4.**

1. If \( h : V \times V \to \mathbb{C} \) is a Hermitian metric, we get an isomorphism \( V \cong V^* \) given by

\[
v \mapsto h_v = h(-, v).
\]

Also, \( \bar{h} : \bar{V} \times V \to \mathbb{C} \), which gives a Hermitian metric on \( \bar{V} \). Combining these, we get a Hermitian metric \( h^* \) on \( V^* \) such that

\[
h^*(h_v, h_w) = \overline{h(v, w)}.
\]

2. Given a Hermitian metric \( h \) on \( V \), we get Hermitian metric on all \( \bigwedge V \) given by

\[
\langle v_1 \wedge \cdots \wedge v_p, w_1 \wedge \cdots \wedge w_p \rangle = \det(h(v_i, w_j)).
\]

Given a Hermitian metric \( h \) on \( V \), consider the Hermitian metric \( S_h \) on \( V_C \) and use it to put a Hermitian metric on all \( \bigwedge V_C \).

**Exercise.** This is the same as using the scalar product \( S_h \) on \( V \) to get a scalar product on \( \bigwedge V^* \) and then extending this by linearity to a Hermitian linear form on \( (\bigwedge \bigwedge)^* \).

**Lemma 6.5.** The decomoposition

\[
\bigwedge^p V_C^* \cong \bigoplus_{i+j=p} \left( \bigwedge^i (V')^* \oplus \bigwedge^j (V'')^* \right)
\]

is orthogonal with respect to the Hermitian metric.

**Proof.** To show this, it is enough to check that the decomposition \( V_C = V' \oplus V'' \) is orthogonal with respect to \( S_h \). But:

\[
S_h \left( \frac{1}{2}(v - iJv), \frac{1}{2}(w + iJw) \right) = \frac{1}{4} \left( (S_h(v, w) + S_h(Jv, Jw)) + i(S_h(v, Jw) + S_h(Jv, w)) \right)
\]

\[
= 0
\]
as required.

Suppose \((V, h)\) is as above. We have a scalar product \(S_h\) on \(V\). Since \(V\) is a complex vector space, we have a canonical orientation on \(V\). Let \(n = \dim_C V\). We get a canonical volume element \(dV \in \bigwedge^{2n} V^* \subseteq \bigwedge^{2n} V_C^*\).

We hence have the Hodge operator

\[ * : \bigwedge^p V_C^* \overset{\cong}{\to} \bigwedge^{2n-p} V^* \]

which is an isomorphism of \(\mathbb{R}\)-vector spaces. We extend scalars to \(\mathbb{C}\) to get

\[ * : \bigwedge^p V_C^* \overset{\cong}{\to} \bigwedge^{2n-p} V^*_C \]

We have the following analog of Proposition 5.3 which defined \(*\).

**Lemma 6.6.** For every \(\omega, \eta \in \bigwedge^p V_C^*, \) we have

\[ \omega \wedge * \eta = \langle \omega, \eta \rangle dV. \]

**Exercise.** Check this, using the fact that we know this if \(\omega, \eta \in \bigwedge^p V^*\).

We write \(\bigwedge^{p,q} V_C^*\) for \(\bigwedge^{p'} V'^* \otimes \bigwedge^q V'^*\).

**Corollary 6.7.** The map \(*\) maps \(\bigwedge^{p,q} V_C^*\) to \(\bigwedge^{n-q, n-p} V_C^*\).

Recall that if \(w : \bigwedge^m V_C^* \to \bigwedge^m V_C^*\) is the de Rham operator acting by multiplication with \((-1)^m\) on \(\bigwedge^m V^* C\), then

\[ ** = w. \]

**Note** that \(*\) is defined by scalar extension from \(\mathbb{R}\), so it is a real operator, i.e. \(\overline{\omega} = \overline{*\omega}\).

### 6.2. Globalization

Let \(M\) be an \(n\)-dimensional complex manifold. We can always choose on \(M\) a Hermitian metric. The key point is that a real positive function times a Hermitian metric is still a Hermitian metric, and a finite sum of Hermitian metrics is still a Hermitian metric, so we can construct such metrics locally and glue using partitions of unity.

Fix such a metric \(h\). Then \(S = \text{Re}(h)\) is a Riemannian metric on \(M\) with the standard orientation and we get a volume element \(dV\) which is a real \((n, n)\)-form on \(M\). The \(*\) operator

\[ * : \mathcal{A}^{p,q}_M \cong \mathcal{A}^{n-q, n-p}_M \]

is the unique map satisfying

\[ \langle \omega, \eta \rangle dV = \omega \wedge * \eta \]

(as in Lemma 6.6). Note that \(S\) also induces Hermitian metrics on all \(\mathcal{A}^m_{M, C}\) such that the \((p, q)\)-components are orthogonal.
From now on, suppose $M$ is compact. We get Hermitian metrics on each $\mathcal{A}^{p,q}(M)$ by
\[
\langle\langle \omega, \eta \rangle\rangle = \int_M \langle\omega, \eta\rangle \, dV = \int_M \omega \wedge \overline{\eta}.
\]
Note that $\langle\langle \omega, \omega \rangle\rangle > 0$ unless $\omega = 0$.

It is easy to say that the induced Hermitian metric on $\mathcal{A}^m_{M,C}(M) = \mathcal{A}^m_M(M) \otimes_R C$ is the one induced by extending to complexifications of the one we associated before to the Riemannian structure.

6.3. The operators $\partial^*$ and $\overline{\partial}^*$. We have operators
\[
\partial: \mathcal{A}_M^{p,q} \rightarrow \mathcal{A}_M^{p+1,q},
\overline{\partial}: \mathcal{A}_M^{p,q} \rightarrow \mathcal{A}_M^{p,q+1}
\]
such that $d = \partial + \overline{\partial}$. Recall that $d^* = - * d *$.

**Definition 6.8.** Define
\[
\partial^* = - * \overline{\partial}*: \mathcal{A}_M^{p+1,q} \rightarrow \mathcal{A}_M^{p,q},
\overline{\partial}^* = - * \partial*: \mathcal{A}_M^{p,q+1} \rightarrow \mathcal{A}_M^{p,q}.
\]
Clearly, $d^* = \partial^* + \overline{\partial}^*$.

**Proposition 6.9.** The partial $(\partial, \partial^*)$ and $(\overline{\partial}, \overline{\partial}^*)$ are formal adjoint pairs.

**Proof.** Let $u \in \mathcal{A}^{p,q}(M)$, $v \in \mathcal{A}^{p+1,q}(M)$. Then
\[
\langle\langle u, \partial^* v \rangle\rangle = \int_M u \wedge \overline{\partial^* v} = \int_M u \wedge \overline{** v}
\]
\[
= (-1)^{p+q} \int_M (u \wedge \partial^* v) = (-1)^{p+q} \int_M \overline{u \wedge \partial^* v}
\]
\[
= (-1)^{p+q} \int_M \partial(u \wedge \overline{v}) + \int_M \partial u \wedge \overline{v}
\]
where the first term is 0 by Stokes theorem. The proof of the second adjointness is similar. □

**Definition 6.10.** Let $\Delta' = \partial \partial^* + \partial^* \partial$, $\Delta'' = \overline{\partial} \overline{\partial}^* + \overline{\partial}^* \overline{\partial}$.

By Proposition 6.9, both $\Delta'$ and $\Delta''$ are formally self-adjoint.

**Definition 6.11.** A form $\omega$ is
\[
\bullet \, \partial\text{-harmonic if } \Delta'\omega = 0,
\]
• $\bar{\partial}$-harmonic if $\Delta''\omega = 0$.

We write

$$H^{(p,q)}_{\Delta'}(M) = \{\partial$-harmonic $(p,q)$-forms\}$$

$$H^{(p,q)}_{\Delta''}(M) = \{\bar{\partial}$-harmonic $(p,q)$-forms\}$$

As in the case of usual harmonic forms, one shows the following simple properties.

- A form $\omega$ is $\partial$-harmonic if and only if $\partial \omega = 0$ and $\partial^* \omega = 0$,
- A form $\omega$ is $\bar{\partial}$-harmonic if and only if $\bar{\partial} \omega = 0$ and $\bar{\partial}^* \omega = 0$,
- We have that $\bar{\partial}^* \omega = -\ast \bar{\partial} \ast \omega = -\ast \partial \ast \bar{\omega} = \bar{\partial}^* \bar{\omega}$, so $\Delta'\omega = \bar{\Delta}'\bar{\omega}$, and hence

$$H^{(p,q)}_{\Delta',x}(M) \cong H^{p,q}_{\Delta''}(M) \quad \omega \mapsto \bar{\omega},$$

- Both $\Delta'$ and $\Delta''$ commute with $\ast$: $\ast \Delta' = \Delta'' \ast, \ast \Delta'' = \Delta' \ast$. We check the first equality on $(p,q)$-forms:

$$\ast(\partial \partial^* + \partial^* \partial) = -\ast(\partial \ast \partial^* + \partial^* \ast \partial)$$

$$= -\ast \partial \ast \partial^* + (-1)^{p+q+1} \partial^* \ast \partial$$

and

$$\Delta'' \ast = (\bar{\partial} \bar{\partial}^* + \bar{\partial}^* \bar{\partial}) \ast$$

$$= -\ast \partial \ast \partial^* + \partial^* \ast \partial$$

$$= (-1)^{p+q+1} \partial^* \ast \partial - \partial^* \ast \partial,$$

so the two agree. Altogether, we get isomorphisms

$$H^{(p,q)}_{\Delta',x}(M) \xrightarrow{\ast} H^{n-q,n-p}_{\Delta''}(M) \xrightarrow{\text{conj}} H^{n-p,n-q}_{\Delta'}(M).$$

The same computation we have done for $\Delta$ implies that

$$\sigma_2(\Delta'\omega)(v) = -\frac{1}{2} \|v\|^2 \cdot \text{Id} = \sigma_2(\Delta''\omega)(v).$$

Hence, like $\Delta$, the operators $\Delta'$ and $\Delta''$ are elliptic operators. We may hence apply the Fundamental Theorem of Elliptic Operators 5.20 to $\Delta'$ and $\Delta''$.

Part (1) implies that

$$H^{p,q}_{\Delta',x}(M), H^{p,q}_{\Delta''}(M)$$

are both finite-dimensional over $\mathbb{C}$. Part (2) for $\Delta''$ gives an orthogonal decomposition

$$\mathcal{A}^{p,q}(M) = H^{p,q}_{\Delta''}(M) \perp \text{Im}(\Delta'': \mathcal{A}^{p,q}(M) \to \mathcal{A}^{p,q}(M))$$

$$= H^{p,q}_{\Delta'}(M) \perp \bar{\partial}(\mathcal{A}^{p,q-1}(M)) \perp \partial^{\ast}(\mathcal{A}^{p,q+1}(M)).$$

Moreover,

$$\ker(\bar{\partial}: \mathcal{A}^{p,q}(M) \to \mathcal{A}^{p,q+1}(M)) = H^{p,q}_{\Delta'}(M) \perp \bar{\partial}(\mathcal{A}^{p,q-1}(M)).$$
The conclusion is that the Dolbeaux cohomology group
\[ H^{p,q}(M) = \mathcal{H}^{q}(\mathcal{A}^{p,\cdot}(M), \bar{\partial}) \cong H^{q}(M, \Omega^p) \]
is isomorphic to
\[ \mathcal{H}^{p,q}_{\Delta'}(M). \]
Here is an application of the above theory. We have a pairing
\[ H^{q}(X, \Omega^p) \times H^{n-p}(X, \Omega^{n-p}) \to \mathbb{C} \]
\[ ([\alpha], [\beta]) \mapsto \int_M \alpha \wedge \beta \]
where \( \alpha \) is a \((p, q)\)-form such that \( \bar{\partial} \alpha = 0 \), \( \beta \) is an \((n - p, n - q)\)-form such that \( \bar{\partial} \beta = 0 \). We claim that this is a non-degenerate pairing. To check this, put a metric \( h \) on \( M \). Given a non-zero element in \( H^{q}(X, \Omega^p) \), choose a \( \bar{\partial} \)-harmonic representative \( \alpha \). If \( \beta = \ast \alpha \) is harmonic (so \( \bar{\partial} \beta = 0 \)), and
\[ \int_M \alpha \wedge \beta = \langle \langle \alpha, \alpha \rangle \rangle > 0. \]

7. Kähler manifolds

7.1. Linear algebra background: Kähler metrics. Let \( V \) be a finite-dimensional vector space over \( \mathbb{C} \). We write \( J \) for multiplication by \( i \). We have a decomposition \( V_{\mathbb{C}} = V' \oplus V'' \).

Giving a Hermitian form \( h \) on \( V \) is equivalent to giving a bilinear alternating form \( A: V_\mathbb{R} \times V_\mathbb{R} \to \mathbb{R} \) such that
\[ A(u, v) = A(Ju, Jv) \quad \text{for all } u, v. \]
Indeed, given \( h \), we may define \( A = \text{Im}(h) \), and given \( A \), we may define \( S(u, v) = A(Ju, v) \) and \( h = S + iA \).

Given \( A \in \bigwedge^2 V_\mathbb{R}^* \), let \( \tilde{A} \in \bigwedge^2 V_{\mathbb{C}}^* \) be the corresponding alternating bilinear form on \( V_{\mathbb{C}} \). By definition, \( \tilde{A} \) is real.

We claim that
\[ A(Ju, Jv) = (u, v) \text{ if and only if } \tilde{A} \text{ is a } (1, 1) \text{-form}. \]
Indeed, \( \tilde{A} \) is a \((1, 1)\) form if and only if \( \tilde{A}(V' \times V') = 0, \tilde{A}(V'' \times V'') = 0 \). Since \( \tilde{A}(\pi, \nu) = \tilde{A}(u, v) \), it is enough to check that \( \tilde{A}(V' \times V') = 0 \).

Recall that \( V' = \{ u - iJu \mid u \in V \} \). We have that
\[ \tilde{A}(u - iJu, v - iJv) = (A(u, v) - A(Ju, Jv)) - i(A(Ju, v) + A(u, Jv)). \]
From this, the ‘only if’ implication is clear and the ‘if’ implication follows since \( A \) takes real values.

**Definition 7.1.** The *fundamental form* of the Hermitian metric \( h \) is the real \((1, 1)\) form \( \omega_h = -\tilde{A} \).
We now describe the fundamental form in a basis. Let $x_1, \ldots, x_n$ be a basis of $V$ over $\mathbb{C}$. Let $y_j = Jx_j$ so that
\[
x_1, \ldots, x_n, y_1, \ldots, y_n
\]
is a basis of $V_{\mathbb{R}}$.

Then a basis of $V'$ is $e_1, \ldots, e_n$ and a basis of $V''$ is $\overline{e}_1, \ldots, \overline{e}_n$ for
\[
e_j = \frac{1}{2}(x_j - iy_j).
\]

Given a Hermitian form $h$, we let $h_{i,j} = h(x_i, x_j)$ so that for $v = \sum v_ie_i$, $w = \sum w_je_j$,
\[
h(v, w) = \sum h_{i,j}v_iw_j.
\]

Write $\omega_h$ as
\[
\sum_{j<k} \lambda_{j,k} e_j^* \wedge \overline{e}_k^*
\]
for some $\lambda_{j,k}$. We compute it
\[
\lambda_{j,k} = \omega_h(e_j, \overline{e}_k) = -\frac{1}{4} \tilde{A}(x_j - iy_j, x_k + iy_k) = -\frac{1}{4} \left( \frac{A(x_j, x_k) + A(y_j, y_k)}{A(x_j, x_k)} + i \frac{A(x_j, x_k) - A(y_j, x_k)}{S(x_k, x_j)} \right) = -\frac{1}{2} \cdot (A(x_j, x_k) - iS(x_j, x_k)) = \frac{i}{2} (S(x_j, x_k) + iA(x_j, x_k)) = \frac{i}{2} h_{j,k}.
\]

Therefore,
\[
\omega_h = \frac{i}{2} \sum_{j<k} h_{j,k} e_j^* \wedge \overline{e}_k^*.
\]

Conclusions.

1. This implies that $h$ is a metric if and only if $-i\omega_h(v, v) > 0$ for all $v \neq 0$.

   Hence: giving a Hermitian metric on $V$ is equivalent to giving a real $(1, 1)$ form $\omega$ with $i\omega(v, v) > 0$ for all $v$.

2. Suppose $x_1, \ldots, x_n$ is an orthonormal basis of $V$. Then $h_{j,k} = \delta_{j,k}$, so
\[
\omega_h = \frac{i}{2} \sum_{j=1}^{n} e_j^* \overline{e}_j^*.
\]

Recall that $S$ gives a top form $dV$. If $x_1, y_1, \ldots, x_n, y_n$ is a positive orthonormal basis for $S$,
\[
dV = x_1^* \wedge y_1^* \wedge \cdots \wedge x_n^* \wedge y_n^*.
\]
On the other hand, the formula above gives
\[ \omega_n^h = \left( \frac{i}{2} \right)^n n! \, e_1^* \wedge \overline{e_1^*} \wedge \cdots \wedge e_n^* \wedge \overline{e_n^*} \]

Moreover,
\[ e_j^* \wedge \overline{e_j^*} = (x_j^* + iy_j^*) \wedge (x_j^* - iy_j^*) = -2ix_j^* \wedge y_j^*. \]
The conclusion is that
\[ \omega_n^h = n! \cdot dV. \]

**Definition 7.2.** Let \( M \) be a complex manifold. A **Hermitian metric** \( h \) on \( M \) is **Kähler** if the (real (1, 1) form) \( \omega = \omega_h \) is closed, i.e. \( d\omega = 0 \).

**Example 7.3** (Trivial example). The manifold \( \mathbb{C}^n \) with the standard metric is Kähler. With respect to the standard basis which is orthonormal for \( h \),
\[ \omega_h = \frac{i}{2} \sum_{j=1}^n dz_j \wedge d\overline{z}_j \]

This is clearly closed.

**Remark 7.4.** The existence of a Kähler metric is a **global** property. The issue is that we cannot glue such metrics using partitions of unity any more. Why? If \( h \) is a Kähler metric with form \( \omega \) and \( f \) is a smooth everywhere positive function, then \( f \cdot h \) is a metric and \( \omega_{fh} = f \cdot \omega_h \). However,
\[ d(f\omega_h) = \left( \frac{df}{\omega_h} \right) + f d\omega_h. \]

**Example 7.5** (Important example: the Fubini–Study metric on \( \mathbb{P}^n \)). Let \( z_0, \ldots, z_n \) be homogeneous coordinates on \( \mathbb{P}^n \) and \( U_j = (z_j \neq 0) \). Let
\[ \omega_j = \frac{i}{2} \frac{\partial \overline{\partial}}{\partial \log \left( \sum_{k=0}^n \left| z_k \overline{z_j} \right|^2 \right)} \in A^{1,1}(U_j). \]

These glue to a global (1, 1) form. Indeed, on \( U_j \cap U_i \):
\[ \sum \left| \frac{z_k}{z_j} \right|^2 = \left( \sum \left| \frac{z_k}{z_\ell} \right|^2 \right) \cdot \left| \frac{z_\ell}{z_j} \right|^2 \]
so
\[ \log \left( \sum \left| \frac{z_k}{z_j} \right|^2 \right) = \log \left( \sum \left| \frac{z_k}{z_\ell} \right|^2 \right) + \log \left| \frac{z_\ell}{z_j} \right|^2. \]
It is enough to note that \( \frac{\partial \overline{\partial}}{\log |w_j|^2} = 0 \) if \( w_1, \ldots, w_n \) are coordinates on \( \mathbb{C}^n \). This is true because \( |w_j|^2 = w_j \overline{w_j} \), so
\[ \log |w_j|^2 = \log w_j + \log \overline{w_j}, \]
so
\[ \partial \overline{\partial} \log |w_j|^2 = 0 \quad \text{on} \quad \mathbb{C}^n \setminus \{w_j = 0\}. \]
We have hence shown that
\[ \omega_j = \frac{i}{2} \partial \overline{\partial} \log \left( \sum_{k=0}^n \left| \frac{z_k}{z_j} \right|^2 \right) \]
glue to
\[ \omega \in A^{1,1}(\mathbb{P}^n). \]

Since \( \partial^2 = 0, \bar{\partial}^2 = 0, \partial \bar{\partial} + \bar{\partial} \partial = 0 \), we see that \( \partial \omega = \bar{\partial} \omega = 0 \), and since \( d = \partial + \bar{\partial}, d\omega = 0 \). Therefore, \( \omega \) is closed.

We finally need to check that \( \omega \) defines a Hermitian metric. Note that
\[ \omega_j = -i 2 \bar{\partial} \log \left( \sum_{k=0}^{n} \frac{z_k}{z_j} |z_j|^{-2} \right) = i 2 \partial \bar{\partial} \log \left( \sum_{k=0}^{n} \frac{z_k}{z_j} |z_j|^{-2} \right). \]

Let us check that \( i \omega(v, v) > 0 \) if \( v \neq 0 \). Work on \( U_j \cong \mathbb{C}^n, \frac{z_j}{z_j} = w_k \) (and reorder coordinates to assume that \( j = 0 \)). Then
\[ \omega_j = i 2 \partial \bar{\partial} \log \left( 1 + \sum_{k=1}^{n} |w_k|^2 \right). \]

Note that
\[ \bar{\partial} \log(1 + \sum |w_k|^2) = \sum_{k=1}^{n} \frac{w_k d\bar{w}_k}{1 + \sum_{\ell=1}^{n} |\omega_\ell|^2} \]
and
\[ \frac{\partial}{\partial w_j} \left( \frac{w_k}{1 + \sum_{\ell} |w_\ell|^2} \right) = \delta_{jk} \frac{1 + \sum |w_\ell|^2}{1 + \sum_{\ell} |w_\ell|^2} - \frac{w_k \bar{w}_j}{(1 + \sum |w_\ell|^2)^2} \]
which shows that
\[ \omega_j = i 2 \left( \sum_k \frac{dw_k \wedge d\bar{w}_k}{1 + \sum_{\ell} |w_\ell|^2} - \sum_{k,j} \frac{w_k \bar{w}_j dw_j \wedge d\bar{w}_k}{(1 + \sum |w_\ell|^2)^2} \right). \]

Write
\[ a_{jk} = (1 + \sum_{\ell} |w_\ell|^2) \delta_{jk} - w_k \bar{w}_j. \]

We need to show that if \( v \neq 0 \), then \( v \cdot (a_{jk}) v' > 0 \). Writing \( \langle \cdot, \cdot \rangle \) for the standard Hermitian metric on \( \mathbb{C}^n \), we see that
\[
\begin{align*}
v \cdot (a_{jk}) v' &= (1 + (w, w)) (v, v) - \sum_{j,k} v_j w_k \bar{w}_j v_k \\
&= (v, v) + \langle (w, w) \cdot (v, v) - |(v, w)|^2 \rangle_{(v, w), (w, v)} > 0 \text{ by Cauchy–Schwartz inequality}
\end{align*}
\]
Therefore, \( \omega \) is a Kähler metric on \( \mathbb{P}^n \).

**Remark 7.6.** If \( h \) is a Hermitian metric on \( M \) and \( M' \hookrightarrow M \) is a submanifold, then the restriction \( h' \) of \( h \) to \( TM' \) is a Hermitian metric on \( M' \), and \( \omega_{h'} = \omega_h|_{M'} \). In particular, if \( h \) is Kähler, \( h' \) is also Kähler.
Upshot. If $X$ is a smooth quasi-projective complex algebraic variety, we have a locally closed immersion $X \hookrightarrow \mathbb{P}^N$ such that $X^{an}$ is a submanifold of $(\mathbb{P}^N)^{an}$. By restriction of the Fubini–Study metric to $X^{an}$, $X^{an}$ has Kähler metrics.

Example 7.7 (Complex tori). Let $M = V/\Lambda$ where $V$ is an $n$-dimensional complex vector space and $\Lambda \subseteq V$ is a lattice (i.e. $V \cong \mathbb{C}^n$ and $\Lambda \cong \mathbb{Z}^{2n} \subseteq \mathbb{C}^n$).

If $h$ is the standard Hermitian metric on $\mathbb{C}^n$, with the form $\omega = \frac{i}{2} \sum_{k=1}^{n} z_k \wedge \overline{z_k}$, and $\gamma_\lambda : V \to V$ is the translation by some $\lambda \in \Lambda$, then

$$\gamma_\lambda^*(\omega) = \omega.$$ 

Then $h$ induces a metric on the quotient $M$, which is again Kähler.

Note that we will later see that for $\Lambda$ general, $M$ is not algebraic. This hence gives an example of Kähler manifolds which are not algebraic.

The next goal is to show that Kähler metrics are not far from the standard one. Suppose $p \in M$. Choose coordinates $z_1, \ldots, z_n$ in a chart around $P$ such that $z_i(P) = 0$ for all $i$.

Suppose $h$ is a Hermitian metric on $M$, with fundamental form

$$\omega = \frac{i}{2} \sum_{j,k} h_{j,k} dz_j \wedge d\overline{z_k}.$$ 

We will say that $\omega$ osculates to order 2 to the standard metric at $P$ (in these coordinates) if

$$h_{j,k}(0) = \delta_{jk},$$

$$\frac{\partial h_{j,k}}{\partial z_\ell}(0) = 0,$$

$$\frac{\partial h_{j,k}}{\partial \overline{z_\ell}}(0) = 0.$$

Proposition 7.8. Given a Hermitian metric $h$ with fundamental form $\omega$, $h$ is Kähler if and only if for all $p \in M$, there is a chart as above such that $\omega$ osculates to order 2 with the standard metric.

Proof. Write $\omega = \frac{i}{2} \sum_{j,k} h_{j,k} dz_j \wedge d\overline{z_k}$. Then

$$d\omega = \frac{i}{2} \sum_{j,k,\ell} \frac{\partial h_{j,k}}{\partial z_\ell} dz_\ell \wedge dz_j \wedge d\overline{z_k} - \frac{i}{2} \sum_{j,k,\ell} \frac{\partial h_{j,k}}{\partial \overline{z_\ell}} d\overline{z_\ell} \wedge dz_j \wedge d\overline{z_k}.$$ 

It is clear that if $\frac{\partial h_{j,k}}{\partial z_\ell}(p) = 0 = \frac{\partial h_{j,k}}{\partial \overline{z_\ell}}(0)$, this implies that $d\omega(p) = 0$. If this holds at every $p$, $\omega$ is closed.

Conversely, suppose $d\omega = 0$. It is easy to see that there is a linear change of variables such that $h_{j,k}(0) = \delta_{jk}$. We will assume that this holds. Let

$$a_{j,k,\ell} = \frac{\partial h_{j,k}}{\partial z_\ell}(p), \quad a'_{j,k,\ell} = \frac{\partial h_{j,k}}{\partial \overline{z_\ell}}(p).$$
If $d\omega(p) = 0$, we must have
\begin{equation}
    a_{jk\ell} = a_{\ell kj}, \quad a'_{jk\ell} = a'_{j\ell k}.
\end{equation}
Moreover, since $\omega$ is real, we have that $h_{jk} = \overline{h_{kj}}$, so
\[ \frac{\partial h_{jk}}{\partial z_\ell} = \frac{\partial h_{kj}}{\partial \overline{z_\ell}}, \]
which shows that
\begin{equation}
    \overline{a_{jk\ell}} = a'_{kj\ell}.
\end{equation}
Now, do the change of variables
\[ w_j = z_j + \frac{1}{2} \sum_{k,\ell=1}^{n} a_{kj\ell} z_k z_\ell. \]
We want to compare $\omega$ with $i \frac{i}{2} \sum_{j=1}^{n} dw_j \wedge d\overline{w_j}$.

We have that
\[ dw_j = dz_j + \frac{1}{2} \sum_{k,\ell=1}^{n} a_{kj\ell} (z_k dz_\ell + z_\ell dz_k) \]
\[ = dz_j + \sum_{k,\ell=1}^{n} a_{kj\ell} z_k dz_\ell \quad \text{by equation (7)} \]
\[ d\overline{w_j} = d\overline{z_j} + \sum_{k,\ell=1}^{n} a'_{kj\ell} \overline{z_k} d\overline{z_\ell} \quad \text{by equation (8)}. \]
Therefore,
\[ \frac{i}{2} \sum_{j=1}^{n} dw_j \wedge d\overline{w_j} = \frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\overline{z_j} \]
\[ + \frac{i}{2} \sum_{k,\ell,j} a'_{kj\ell} \overline{z_k} dz_j \wedge d\overline{z_\ell} + \frac{i}{2} \sum_{k,\ell,j} a_{kj\ell}(p) z_k dz_\ell \wedge d\overline{z_j} \]
\[ = \int_{\partial h_{kj\ell}(p)} \frac{\partial h_{kj\ell}}{\partial z_k} (p) \]
\[ + \text{terms vanish at } p \text{ with order } \geq 2. \]
Therefore, $\omega = \frac{i}{2} \sum_{j=1}^{n} dw_j \wedge d\overline{w_j} + \text{terms vanish at } p \text{ with order } \geq 2. \quad \square$

7.2. Operators on Kähler manifolds. We have $\star, d, \partial, \overline{\partial}$ and the adjoints $d^*, \partial^*, \overline{\partial^*}$. Given the Kähler metric $h$, with fundamental form $\omega$, the Lefschetz operator is
\[ L = \omega \wedge - : \mathcal{A}^{p,q}_M \to \mathcal{A}^{p+1,q+1}_M. \]
We also define $\Lambda = \star^{-1}L^* : \mathcal{A}^{p,q}_M \to \mathcal{A}^{p-1,q-1}_M$. 
Note that since $\omega$ is a real form, $L$ is a real operator (i.e. it commutes with conjugation). Since $*$ is a real operator, $\Lambda$ is a real operator.

**Lemma 7.9.** The operator $\Lambda$ is the adjoint of $L$, i.e. 

$$\langle L\alpha, \beta \rangle = \langle \alpha, \Lambda \beta \rangle$$

for every $(p,q)$-form $\alpha$ and $(p+1,q+1)$-form $\beta$.

**Proof.** Recall that $\alpha \wedge *\beta = \langle \alpha, \beta \rangle dV$. Then

$$\langle \alpha, \Lambda \beta \rangle dV = \alpha \wedge *\left(\frac{-1}{i}L*\right)\beta \wedge *\beta = \langle \alpha \wedge \omega \wedge \beta \rangle$$

$$= L\alpha \wedge \beta = \langle L\alpha, \beta \rangle dV.$$ 

□

**Theorem 7.10 (Kähler Identities).** We have:

1. $[\overline{\partial}^*, L] = i\partial$,
2. $[\partial^*, L] = i\overline{\partial}$,
3. $[\Lambda, \partial] = i\overline{\partial}^*$,
4. $[\Lambda, \overline{\partial}] = -i\overline{\partial}^*$.

**Proof.** We only need to prove (1). The other ones follow from the formulas:

$$[P, Q]^* = [Q^*, P^*], \quad (\lambda P)^* = \overline{\lambda} P^*.$$ 

We prove (1). Suppose first that we deal with a (rescaling by 2) of the standard metric on $\mathbb{C}^n$, i.e.

$$\omega = i \sum_{j=1}^{n} dz_j \land d\overline{z}_j.$$ 

A similar computation to that for $d^*$ gives the expression

$$\overline{\partial}^* \left( \sum_{I,J} f_{I,J} dz_I \land d\overline{z}_J \right) = - \sum_{I,J} \frac{\partial f_{I,J}}{\partial z_k} \cdot i \frac{\partial}{\partial \overline{z}_k} (dz_I \land d\overline{z}_J).$$

Writing $\eta = \sum_{I,J} f_{I,J} dz_I \land d\overline{z}_J$, we have that

$$[\overline{\partial}^*, L] \eta = \overline{\partial}^* (\omega \land \eta) - \omega \land \overline{\partial}^* \eta.$$
Hence
\[\bar{\partial}^* \eta = -i \sum_{I,J,k,j} \frac{\partial f_{I,J}}{\partial z_k} \left( i \frac{\partial}{\partial \overline{z}_k} (dz_j \wedge d\overline{z}_j) \right) + i \sum_{I,J,k,j} \frac{\partial f_{I,J}}{\partial z_k} dz_j \wedge d\overline{z}_j \wedge i \frac{\partial}{\partial \overline{z}_k} (dz_I \wedge d\overline{z}_J).\]

Since \(i \frac{\partial}{\partial \overline{z}_k}\) is a derivation, this gives
\[\bar{\partial}^* \eta = -i \sum_{I,J,k,j} \frac{\partial f_{I,J}}{\partial z_k} \left( i \frac{\partial}{\partial \overline{z}_k} (dz_j \wedge d\overline{z}_j) \right) - \delta_{jk} dz_j \wedge dz_I \wedge d\overline{z}_J = i \partial \eta.\]

This gives the result for the standard metric. In the general case, to check that \([\bar{\partial}^*, L] = i \partial\) at \(p \in M\), choose a chart where \(\omega\) osculates to order 2 with the standard metric (Proposition 7.8), i.e.
\[\omega = i \sum_{j,k} h_{j,k} dz_j \wedge d\overline{z}_k\]
and
\[h_{j,k}(P) = \delta_{jk}, \quad \frac{\partial h_{j,k}}{\partial z_\ell}(P) = 0, \quad \frac{\partial h_{j,k}}{\partial \overline{z}_\ell}(P) = 0.\]

Since both \(\bar{\partial}^*\) and \(\partial\) are differential operators of order 1, the difference with respect to the computation for the standard metric will only involve the derivatives \(\frac{\partial h_{j,k}}{\partial z_\ell}(P), \frac{\partial h_{j,k}}{\partial \overline{z}_\ell}(P)\). These vanish, which gives the result.

**Corollary 7.11.** If \((M, h)\) is a Kähler manifold, then \(\Delta' = \Delta'' = \frac{1}{2} \Delta\).

**Proof.** We compute:
\[\Delta'' = \bar{\partial}^* \bar{\partial} + \bar{\partial}^* \bar{\partial} = i(\overline{\partial}(\partial \Lambda - \Lambda \partial)) + i(\partial \Lambda - \Lambda \partial)\overline{\partial} = i \overline{\partial} \partial \Lambda - i \Lambda \partial \overline{\partial} + i(\partial \Lambda \overline{\partial} - \overline{\partial} \Lambda \partial).\]

Note that \(i \overline{\partial} \partial \) is a real operator, since \(i \overline{\partial} \partial = -i \overline{\partial} \partial = i \overline{\partial} \). Moreover, \(L, \Lambda\) are both real operators and so is \(i(\partial \Lambda \overline{\partial} - \overline{\partial} \Lambda \partial)\). Overall, this shows that \(\Delta''\) is a real operator. Since we know that \(\Delta'' = \Delta'\), this shows that \(\Delta' = \Delta''\).
Let us now compute $\Delta$:

$$\Delta = dd^* + d^*d$$

$$= (\partial + \overline{\partial})(\partial^* + \overline{\partial}^*) + (\partial^* + \overline{\partial}^*)(\partial + \overline{\partial})$$

$$= (\partial \partial^* + \partial^* \partial) + (\overline{\partial} \overline{\partial}^* + \overline{\partial}^* \overline{\partial}) + (\partial \overline{\partial}^* + \overline{\partial} \partial^* + \partial^* \overline{\partial} + \overline{\partial}^* \partial).$$

We now show that $\partial \overline{\partial}^* + \overline{\partial}^* \partial = 0$. Indeed,

$$\partial \overline{\partial}^* + \overline{\partial}^* \partial = i\partial(\partial \Lambda - \Lambda \partial) + i(\partial \Lambda - \Lambda \partial)\partial = -i\partial \Lambda \partial + i\partial \Lambda \partial = 0.$$

Then the above computation shows that

$$\Delta = \Delta' + \Delta'' + (\partial \overline{\partial}^* + \overline{\partial}^* \partial) + (\overline{\partial} \overline{\partial}^* + \overline{\partial}^* \overline{\partial}) = \Delta' + \Delta''.$$

Using $\Delta' = \Delta''$, this completes the proof.

7.3. **Consequence: the Hodge decomposition.** We finally talk about the consequences. Recall that we have

$$H^m(M, \mathbb{C}) = \text{space of complex } m\text{-harmonic forms on } M$$

$$= H^m(M) \otimes_{\mathbb{R}} \mathbb{C}$$

$$= \text{null}(\Delta_M).$$

Since $\Delta = 2\Delta' = (2\Delta'')$, the decomposition of $A^m(M) \otimes_{\mathbb{R}} \mathbb{C}$ into $(p, q)$-parts, induces a decomposition

$$H^m(M, \mathbb{C}) = \bigoplus_{p+q=m} H^{p,q}(M)$$

which are harmonic $(p, q)$-forms.

We saw (Hodge Theorem 5.21) that the inclusion of $H^m(M)$ into closed real $m$-forms induces an isomorphism

$$H^m(M) \cong H^m_{\text{dR}}(M, \mathbb{R}).$$

Tensoring with $\mathbb{C}$, this gives

$$H^m(M, \mathbb{C}) \cong H^m_{\text{dR}}(M, \mathbb{C}).$$

If $H^{p,q}(M)$ is the image of $H^{p,q}(M)$, then we get the Hodge decomposition

$$H^m_{\text{dR}}(M, \mathbb{C}) = \bigoplus_{p+q=m} H^{p,q}.$$

Recall that $\overline{H^{p,q}_{\Delta'}} = H^{q,p}_{\Delta'}$. Since $\Delta' = \Delta''$, this shows that

$$\overline{H^{p,q}} = H^{q,p}.$$

We also know that $H^{p,q}_{\Delta''}(M) \cong H^q(M, \Omega^p)$, so

$$H^{p,q} \cong H^q(M, \Omega^p).$$

**Definition 7.12.** The **Betti numbers** of $M$ are $b_m = \dim_{\mathbb{C}} H^m(M, \mathbb{C})$.

The **Hodge numbers** of $M$ are $h^{p,q} = \dim_{\mathbb{C}} H^q(M, \Omega^p)$. 
Numerically, the above assertions imply that
\[ b_m = \sum_{p+q=m} h^{p,q} \]
and the Hodge symmetry:
\[ h^{p,q} = h^{q,p}. \]
In particular, \( b_m(M) \) is even for all \( m \) odd.
Therefore, computing these numbers for a manifold might show that it is not Kähler.

**Example 7.13.** Recall the Hopf surface
\[ M = \mathbb{C}^2 \setminus \{(0,0)\} \longrightarrow (z_1, z_2) \sim (2z_1, 2z_2) \]
from Example 3.15. We checked that \( M \) is diffeomorphic to \( S^3 \times S^1 \) and Künneth formula then shows that \( b_3(M) = 1 \), so \( M \) is not Kähler.

### 7.4. Independence of metric.

The next goal is to show that the decomposition \( H^m(M, \mathbb{C}) = \bigoplus_{p+q=m} H^{p,q} \) is independent of the choice of metric.

**Definition 7.14.** The Bott–Chern cohomology of \( M \) is
\[ H^{p,q}_{BC}(M) = \{ (p,q)\text{-forms } u \mid \partial u = 0, \bar{\partial} u = 0 \} / \partial \bar{\partial} A^{p-1,q-1}(M) \].

Note that \( \partial \bar{\partial} v = d(\bar{\partial} v) \), so we have a canonical map
\[ H^{p,q}_{BC}(M) \to H^{p+q}_{dR}(M, \mathbb{C}). \]

**Theorem 7.15.** This map gives an isomorphism of \( H^{p,q}_{BC}(M) \) and \( H^{p,q}(M) \).

**Lemma 7.16 (\( \partial \bar{\partial} \)-Lemma).** Let \( M \) be a Kähler manifold and \( u \) be a global form on \( M \) such that \( \partial u = 0 \) and \( \bar{\partial} u = 0 \). Then the following are equivalent:

1. \( u \in \text{im}(d) \),
2. \( u \in \text{im}(\partial) \),
3. \( u \in \text{im}(\bar{\partial}) \),
4. \( u \in \text{im}(\partial \bar{\partial}) \).

**Proof.** Note that \( \partial \bar{\partial} v = d(\bar{\partial} v) \) and \( \partial \bar{\partial} = -\bar{\partial} \partial \), so clearly (4) implies (1), (2), (3).

It suffices to prove that (1), (2), (3) imply (4). First, write \( u = \partial v \) for some \( v \). By Hodge Theorem 5.21 for \( \bar{\partial} \), we may write
\[ v = v_1 + \bar{\partial} v_2 + \bar{\partial}^* v_3 \]
for some \( v_1 \) harmonic and some \( v_2, v_3 \). Then
\[ u = \partial \bar{\partial} v_2 + \partial \bar{\partial}^* v_3. \]
By Corollary 7.11, $\Delta = \Delta' + \Delta''$ so $\partial \bar{\partial}^* = -\bar{\partial}^* \partial$. Hence $\bar{\partial} u = 0$, equation (9) implies that
\[
0 = \bar{\partial} \bar{\partial} v_2 + \bar{\partial} \bar{\partial}^* v_3 = \bar{\partial} \bar{\partial}^* v_3.
\]
Hence $\bar{\partial} \partial^* v_3 = 0$. If $\bar{\partial} \eta = 0$, $0 = \langle \bar{\partial} \eta, \eta \rangle = \langle \partial^* \eta, \partial^* \eta \rangle$ so $\partial^* \eta = 0$. This shows that $\partial^* \partial v_3 = 0$, so $u = \partial \bar{\partial} v_2$.

If $u \in \text{im}(\bar{\partial})$, apply the previous argument for $\bar{u}$ to shows that $u \in \text{im}(\partial \bar{\partial})$.

Finally, if $u = d(w) = \partial w + \bar{\partial} w$, we see that $\partial u = 0$ implies that $\partial \bar{\partial} w = 0$ and $\bar{\partial} \partial w = 0$. By the (3) implies (4) implication for $\partial w$, we see that $\partial w \in \text{im}(\partial \bar{\partial})$. Similarly, $\bar{\partial} u = 0$ implies that $\bar{\partial} w \in \text{im}(\partial \bar{\partial})$, so $u \in \text{im}(\partial \bar{\partial})$. \hfill $\Box$

Recall that $H^{p,q}_{BC}(M)$, which was defined as
\[
\{\text{global } (p,q)\text{-forms } u \mid \partial u = 0, \bar{\partial} u = 0\}.\]

We hence have a map
\[
\varphi: H^{p,q}_{BC} \to H^{p+q}_{DR}(M, \mathbb{C})
\]
\[
u \mapsto [\nu].\]

Recall that
\[
H^{p,q}(M) = \{\alpha \in H^{p+q}_{DR}(M, \mathbb{C}) \mid \alpha = [\eta], \eta \text{ is a harmonic } (p,q)\text{-form}\}.
\]

We check that $H^{p,q}_{BC}$ lands in this piece of de Rham cohomology. Since $\partial u = 0$, the Hodge Theorem 5.21 for $\partial$ implies that
\[
u = v + \partial w
\]
for some harmonic $(p,q)$-form $v$. Then $0 = \bar{\partial} u = \bar{\partial} \partial w$ and $\partial \partial w = 0$, so the $\partial \bar{\partial}$-Lemma 7.16 for $\partial w$ shows that $\partial w \in \text{im}(d)$, so $[\nu] = [\nu]$ in $H^{p+q}_{DR}(M, \mathbb{C})$.

This shows that $\text{im}(\varphi) \subseteq H^{p,q}(M)$. Moreover, since a harmonic $(p,q)$-form satisfies $\partial \eta = 0$, $\bar{\partial} \eta = 0$, $\varphi$ is surjective by Hodge Theorem 5.21.

Finally, $\varphi$ is injective by the implication (1) implies (4) in $\partial \bar{\partial}$-Lemma 7.16.

**Corollary 7.17.** The map $\varphi$ induces an isomorphism $H^{p,q}_{BC}(M) \cong H^{p,q}(M)$.

**Corollary 7.18.**

1. If $\alpha \in H^{p,q}(M)$, $\beta \in H^{p',q'}(M)$, $\alpha \cup \beta \in H^{p+p',q+q'}(M)$.
2. If $f: M' \to M$ is a holomorphic map of complex manifolds of Kähler type, the pullback maps on cohomology
\[
f^* : H^m_{\text{dR}}(M, \mathbb{C}) \to H^m_{\text{dR}}(M', \mathbb{C})
\]
map each $H^{p,q}(M)$ to $H^{p,q}(M')$.  

Proof. For (1), Corollary 7.17 allows us to write
\[ \alpha = [u] \text{ for a } (p,q)\text{-form } u \text{ such that } \partial u = 0, \bar{\partial} u = 0, \]
\[ \beta = [v] \text{ for a } (p',q')\text{-form } v \text{ such that } \partial v = 0, \bar{\partial} v = 0. \]

We then have that
\[ \partial (u \wedge v) = \partial u \wedge v \pm u \wedge \partial v = 0, \]
\[ \bar{\partial} (u \wedge v) = 0, \]
and \[ \alpha \cup \beta = [u \wedge v] \in H^{p+p',q+q'}(M) \] by Corollary 7.17.

In (2), for \( \alpha \in H^{p,q}(M) \), use Corollary 7.17 to write \( \alpha = [u] \) for a \( (p,q)\)-form \( u \) such that \( \partial u = 0, \bar{\partial} u = 0 \). Then
\[ f^* \alpha = [f^* u] \]
and
\[ \partial (f^* u) = f^*(\partial u) = 0, \]
\[ \bar{\partial} (f^* u) = 0. \]

Using Corollary 7.17, this shows that \( f^* \alpha \in H^{p,q}(M') \).

We prove one more consequence which will be useful later, when we discuss applications to the Kodaira Vanishing Theorem.

Corollary 7.19. Suppose \( M \) is a compact manifold of Kähler type. Consider the map \( \mathbb{C} \rightarrow \mathcal{O}_M \). The induced morphisms in cohomology
\[ H^q(M, \mathbb{C}) \rightarrow H^q(M, \mathcal{O}_M) \]
are surjective.

Proof. Compute the maps in cohomology using soft resolutions (cf. Theorem 4.26). We have
\[
\begin{array}{ccc}
\mathbb{C} & \longrightarrow & (\mathcal{A}_M^\bullet \otimes \mathbb{C}, d), \\
\downarrow & & \downarrow \text{proj} \\
\mathcal{O}_M & \longrightarrow & (\mathcal{A}_M^{0,\bullet}, \bar{\partial}).
\end{array}
\]

For every \( \alpha \in H^q(M, \mathcal{O}_M) \), there is a harmonic \((0,q)\)-form \( u \) such that \( \alpha = [u] \). In particular, \( du = 0 \) implies that \( \alpha \in \text{im}(H^q(M, \mathbb{C}) \rightarrow H^q(M, \mathcal{O}_M)) \).

8. HARD LEFSCHETZ THEOREM

8.1. Lefschetz decomposition. Let \( M \) be a compact Kähler manifold and \( \omega \) be the fundamental form of the Kähler metric. We have two operators (that play a role in the Kähler
Identities 7.10):
\[ L: \mathcal{A}^*_M,\mathcal{C} \to \mathcal{A}^{*+2}_M,\mathcal{C} \]
real and takes \((p, q)\)-forms to \((p + 1, q + 1)\)-forms,
\[ \eta \mapsto \omega \wedge \eta, \]
\[ \Lambda: \mathcal{A}^*_M,\mathcal{C} \to \mathcal{A}^{*-2}_M,\mathcal{C} \]
\[ \Lambda = \ast^* L \ast \text{ is the adjoint of } L. \]

Proposition 8.1. We have \([L, \Lambda] = H\) where \(A^k_{M,\mathcal{C}} \to A^k_{M,\mathcal{C}}\) is \((k - n)\text{Id}\) where \(n = \text{dim } M\).

Remarks 8.2. Both \(L\) and \(\Lambda\) are linear operators and we only need to check this point-wise, so it is enough to check it when we have a complex vector space \(V\) with a Hermitian metric \(h\) and \(\omega = -\text{Im}(h)\). Moreover, these are real operators, so it will be enough to consider their effect on \(\bigwedge^m V^*\). The ideas is to

1. check this for \(\text{dim}_\mathbb{C} V = 1\),
2. show that if \(V = V' \oplus V''\) and if we know the assertion for \(V', V''\), then we get it for \(V\).

We remark that \(L, \Lambda, H\) give the generators of the Lie algebra of \(\text{SL}_2\). We might discuss this later in the class.

References


