MATH 731: HODGE THEORY

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These are notes from Math 731 taught by Professor Mircea Mustață in Fall 2019, \LaTeX’ed by Aleksander Horawa (who is the only person responsible for any mistakes that may be found in them).

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http://www-personal.umich.edu/~ahorawa/index.html

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The goal of the course is to give an introduction to the basic results in Hodge theory. The prerequisites are: familiarity with algebraic varieties and sheaf cohomology (no familiarity with scheme theory is required) and with smooth manifolds (the tangent bundle, differential forms, integration).

The course will not follow a single textbook but most of the material covered can be found in [Voi07]. Additional references will be given where appropriate throughout the notes.

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1. The classical topology on a complex algebraic variety

The set up is as follows. Let $X$ be an algebraic variety over $\mathbb{C}$, i.e. (the closed points of) a reduced scheme of finite type over $\mathbb{C}$.

Suppose $X$ is affine. Then there is a closed immersion $X \hookrightarrow A^N_{\mathbb{C}} = \mathbb{C}^N$. Note that $\mathbb{C}^N$ carries the Euclidean topology. By definition, the classical topology on $X$ is the induced subspace topology. This is well-defined: given two closed embeddings

there are polynomial functions $\mathbb{C}^N \to \mathbb{C}^{N'}$ and $\mathbb{C}^{N'} \to \mathbb{C}^N$ that make the triangle commute; since polynomial functions $\mathbb{C}^m \to \mathbb{C}^n$ are continuous with respect to the Euclidean topology, the two embeddings induce the same topology on $X$.

To give some more details, the two closed immersions correspond to surjective maps

and we may defined the map $\mathbb{C}[x_1, \ldots, x_N] \to \mathbb{C}[y_1, \ldots, y_N]$ (and vice versa) by taking any lifts from $\mathcal{O}(X)$.

Proposition 1.1.

1. The classical topology on $X$ is finer than the Zariski topology.
2. If $X$ is affine and $Z \hookrightarrow X$ is a closed subvariety, then in the classical topology on $Z$ is the subspace topology with respect to the classical topology on $X$.
3. The same is holds for an open subvariety $U \hookrightarrow X$ (if $U$ is affine).

Proof. It is enough to check (1) for $\mathbb{C}^N$; there, we use the definition of the Zariski topology and the fact that polynomial functions are continuous with respect to the Euclidean topology.

Part (2) follows directly from the definition.

To prove (3), we first note that by covering $U$ by principal affine open subsets, we may assume that $U$ is principal affine:

$$ U = \{ x \in X \mid f(x) \neq 0 \} $$
for some \( f \in \mathcal{O}(X) \). Indeed, if \( U = U_1 \cup \cdots \cup U_r \) and \( U_i \) is principal affine with respect to \( X \) (hence also \( U \)), then the principal case implies that the topology on \( U_i \) is the subspace topology on \( U_i \) with respect to both the classical topologies on \( U \) and on \( X \), and hence the assertion holds for \( U \).

Given a closed immersion \( X \hookrightarrow \mathbb{C}^N \) and \( f \in \mathcal{O}(X) \), choose \( g \in \mathbb{C}[x_1, \ldots, x_N] \) such that \( g|_X = f \) and embed \( U \) in \( \mathbb{C}^{N+1} \) as \( \{(u, t) \mid u \in X, g(u)t = 1\} \):

\[
\begin{array}{ccc}
X & \rightarrow & \mathbb{C}^N \\
\downarrow & & \\
U & \rightarrow & \mathbb{C}^{N+1}
\end{array}
\]

The two maps:

- \( \mathbb{C}^{N+1} \rightarrow \mathbb{C}^N \) given by projecting only the first component,
- \( \mathbb{C}^N \setminus V(g) \rightarrow \mathbb{C}^{N+1} \) given by \( u \mapsto (u, \frac{1}{g(u)}) \)

are both continuous with respect to the Euclidean topology. Thus the topology on \( U \) as a subspace of \( \mathbb{C}^N \) and \( \mathbb{C}^{N+1} \) coincide. \( \square \)

We now glue the above construction. Let \( X \) be an algebraic variety over \( \mathbb{C} \). Take an affine open cover \( X = \bigcup_{i=1}^r U_i \). Each \( U_i \) has the classical topology introduced above.

We know that:

1. \( U_i \cap U_j \) is open in both \( U_i \) and \( U_j \) with respect to the classical topology,
2. by covering \( U_i \cap U_j \) by affine open subsets, Proposition 1.1 (3) implies that the classical topology on \( U_i \) and \( U_j \) induce the same topology on \( U_i \cap U_j \).

It is now easy to check that in this case there is a unique topology on \( X \) such that the subspace topology on each \( U_i \) is the classical topology.

By definition, a subset \( U \subseteq X \) is classically open if \( U \cap U_i \) is open for all \( i \) in the classical topology. Note that each \( U_i \) is open in \( X \) in this topology.

It is easy to see that:

1. the definition is independent of the choice of cover,
2. Proposition 1.1 extends to an arbitrary complex variety.

**Definition 1.2.** We write \( X^{an} \) for the topological space \( X \) with the classical topology.

**Remarks 1.3.**

1. If \( f : X \rightarrow Y \) is a morphism of algebraic varieties, then \( f : X^{an} \rightarrow Y^{an} \) is continuous. Indeed, by covering \( X \) and \( Y \) by affines, we reduce to \( X \) and \( Y \) affine. Given closed immersions
there is a polynomial map $g$ that makes the square above commute. Since polynomial maps are continuous, this proves the assertion.

In particular, regular functions on $X$ are continuous with respect to the classical topology.

(2) Every point has a countable basis of open neighborhoods in the classical topology.

(3) If $X, Y$ are complex algebraic variety, then $(X \times Y)^{\text{an}} = X^{\text{an}} \times Y^{\text{an}}$ in the category of topological spaces (i.e. the classical topology of $X \times Y$ is the product of the classical topologies on $X$ and $Y$).

Indeed, we reduce again to $X, Y$ affine, and using the definition of classical topology, we reduce to $X = \mathbb{C}^m, Y = \mathbb{C}^n$ — then the claim follows since the Euclidean topology on $\mathbb{C}^{m+n}$ is the product topology.

**Theorem 1.4.** If $X$ is an irreducible complex algebraic variety and $U \subseteq X$ is Zariski open, nonempty, then $U$ is dense in the classical topology.

We will prove this theorem next time.

**Corollary 1.5.** Let $X$ be a complex algebraic variety and $U \subseteq X$ be Zariski open and dense in the Zariski topology, then $U$ is dense in the classical topology.

**Proof.** If $X = X_1 \cup \cdots \cup X_r$ is an irreducible decomposition and $U \subseteq X$ is Zariski dense, then $U \cap X_i \neq \emptyset$ for all $i$. Theorem 1.4 then implies that $U \cap X_i$ is dense in $X_i$ in the classical topology, and hence $U$ is dense in $X$ in the classical topology. \qed

The proof of Theorem 1.4 follows [Mum99].

**Proof of Theorem 1.4.** Step 1. We may assume that $X$ is affine. Indeed, given an open affine cover $X = U_1 \cup \cdots \cup U_r$ then $U \cap U_i \neq \emptyset$ for all $i$, and if we know $U \cap U_i$ is dense in $(U_i)^{\text{an}}$, then $U$ is dense in $X^{\text{an}}$.

**Step 2.** We apply Noether normalization to get a finite surjective map

$$
\pi : X \to \mathbb{C}^n.
$$

We need to show that given and $p \in Z = X \setminus U$, we can find a sequence $y_m \to p$ with $y_m \in U$.

Let $u = \pi(p)$. Since $\pi$ is finite, $\pi(Z)$ is a closed proper subset of $\mathbb{C}^n$, there exists $g \neq 0$ in $\mathbb{C}[x_1, \ldots, x_n]$ such that $\pi(Z) \subseteq V(g)$.

Consider

$$
\mathbb{R} \xrightarrow{\varphi} \mathbb{C}
$$

$$
t \mapsto g(tx + (1 - t)w)
$$

where $w \in \mathbb{C}^n$ is such that $g(w) \neq 0$. Since $\varphi(0) \neq 0$ and $\varphi$ is a polynomial, $\varphi$ only vanishes at finite many points. Thus there is a sequence $u_m \to u$, $g(u_m) \neq 0$. After passing to a
subsequence, we need to find $y_m \in \pi^{-1}(u_m)$ such that $y_m \to y$. Since $u_m \not\in V(g)$, $y_m \in U$, so $p \in \overline{U}$.

**Step 3.** Finding the $y_m$'s. Recall that we have a map
\[
\pi : X \to \mathbb{C}^n \ni u_m \to u
\]
and we want to find elements in fibers over $u_m$'s. Write
\[
\pi^{-1}(u) = \{p = p_1, p_2, \ldots, p_r\}.
\]
Choose $g \in \mathcal{O}(X)$ such that $g(p) = 0$ but $g(p_j) \neq 0$ for $j \geq 2$. Since $\pi$ is finite, there exists $F \in \mathcal{O}(\mathbb{C}^n)[t]$ monic such that $F(x, g) = 0$.

Write
\[
F = t^d + a_1(x)t^{d-1} + \cdots + a_r(x).
\]
Since $\mathcal{O}(X)$ is a domain, we may assume that $F$ is irreducible. The map $\pi$ factors as
\[
\begin{array}{ccc}
Y & \xrightarrow{\pi} & \mathbb{C}^n \\
| & | & | \\
\pi_2 & \downarrow & \pi_1 \\
 & \mathbb{C}^{n+1} & \supset V(F)
\end{array}
\]
where $\pi_2(x) = (\pi(x), g(x))$. Since $\pi$ is finite, $\pi_2$ is also finite. Since $\pi$ is surjective, this shows that $\pi_2$ is also surjective (otherwise, $\pi_2(X)$ has dimension less than $n$, and hence so does $\pi(X)$).

Recall that $g(p) = 0$ and hence $a_r(u) = 0$. Since $|a_r(x)|$ is the absolute value of the product of the roots of $F(x, -)$ and $u_m \to u$, we can choose $t_m$ such that $F(u_m, t_m) = 0$ for all $m$ and $t_m \to 0$.

Now choose $y_m \in \pi_2^{-1}(u_m, t_m)$ arbitrarily.

We claim that, after passing to a subsequence, we assume that $y_m$ converges to some $y$. Since $\pi(y_m) = u_m \to u$, we see that $y \in \pi^{-1}(U)$. Since $g(y) = \lim g(y_m) = \lim t_m = 0$, we have that $y \neq p_j$ for $j \geq 2$. Hence $y = p$.

We thus just need to prove this claim. Choose generators $h_1, \ldots, h_s$ of $\mathcal{O}(X)$ to get a closed immersion $X \hookrightarrow \mathbb{C}^s$ given by $(h_1, \ldots, h_s)$.

We use the fact that each $h_i$ satisfies a monic equation:
\[
t^{d_i} + a_{i,1}(x)t^{d_i-1} + \cdots = 0.
\]
We want to show that each $(h_i(y_m))_{m \geq 1}$ is bounded. Since $h_i$ satisfies the monic equation above, we just need to show that the coefficients $(a_{i,j}(\pi(y_m)))_{m \geq 1}$ is bounded. Since $\pi(y_m) = u_m$ is convergent, it is bounded, and hence $a_{i,j}(\pi(y_m))$ is bounded for all $i, j$. \hfill $\square$

**Corollary 1.6.** Let $X$ be an algebraic variety over $\mathbb{C}$ and $W \subseteq X$ is a constructible$^1$ subset. Then $\overline{W}^{\text{Zar}} = \overline{W}^{\text{an}}$.

$^1$Recall that a subset is constructible if it is a finite union of locally closed sets.
Proof. Since the classical topology is finer than the Zariski topology, \( \overline{W}^\text{an} \subseteq \overline{W}^\text{Zar} \).

Since \( W \) is constructible, there exists \( U \subseteq W \) such that \( U \) is open and dense in \( \overline{W}^\text{Zar} \).

By Corollary 1.5, \( U \) is dense in \( \overline{W}^\text{Zar} \) in the classical topology. This shows that \( \overline{W}^\text{Zar} \subseteq \overline{W}^\text{an} \). \( \square \)

Remark 1.7 (Chevalley’s theorem). Let \( f : X \to Y \) be a finite morphism. The image of a constructible set under \( f \) is constructible.

Theorem 1.8. Suppose \( X \) is a complex algebraic variety. Then

1. \( X \) is separated if and only if \( X^\text{an} \) is Hausdorff,
2. \( X \) is complete if and only if \( X^\text{an} \) is compact,
3. if \( f : X \to Y \) is a morphism of separated varieties, then \( f \) is proper (in the algebraic sense) if and only if \( f : X^\text{an} \to Y^\text{an} \) is proper (i.e. \( K \subseteq Y^\text{an} \) is compact implies that \( f^{-1}(K) \) is compact).

Proof. We first show (1). Recall that, in general, the diagonal map \( \Delta : X \to X \times X \) is a locally closed immersion. By definition, \( X \) is separated if \( \Delta \) is a closed immersion, i.e. \( \Delta(X) \) is closed in \( X \times X \) (in the Zariski topology).

By Corollary 1.6, \( \Delta(X) \) is Zariski closed in \( X \times X \) if and only if it is closed in the classical topology. But \( \Delta(X) \) is closed in \( X^\text{an} \times X^\text{an} \) if and only if \( X^\text{an} \) is Hausdorff.

We will now prove (2). Note first that \( (\mathbb{P}^n)^\text{an} \) is compact. Indeed, we have a continuous surjective map from the \( n \)-sphere which is compact:

\[
\left\{ z = (z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1} \left| \sum_{i=0}^{n} |z_i|^2 = 1 \right. \right\} \to (\mathbb{P}^n)^\text{an}.
\]

Suppose now that \( X \) is complete. In particular, it is separated, so we know that \( X^\text{an} \) is Hausdorff by (1). By Chow’s lemma, there is a surjective (birational) morphism \( \pi : \tilde{X} \to X \) with \( \tilde{X} \) projective, \( \tilde{X} \to \mathbb{P}^n \) Zariski-closed. Then \( (\tilde{X})^\text{an} \) is closed in \( (\mathbb{P}^n)^\text{an} \), hence compact. Therefore, \( \pi(\tilde{X}^\text{an}) = X^\text{an} \) is compact.

Conversely, suppose \( X^\text{an} \) is compact. In particular, it is Hausdorff and hence \( X \) is separated. We need to show that for every algebraic variety \( Y \), the projection map

\[
X \times Y \to Y
\]

is closed in the Zariski topology, i.e. if \( Z \subseteq X \times Y \) is Zariski-closed, \( f(Z) \) is Zariski-closed.

By Chevalley’s theorem 1.7, \( f(Z) \) is constructible. By Corollary 1.6, \( f(Z) \) is Zariski closed if and only if it is closed in the classical topology. Suppose \( y_n \in f(Z) \) is such that \( \lim_{n \to \infty} y_n = b \in Y \). Then there exists \( x_n \in X \) such that \( (x_n, y_n) \in Z \) for all \( n \). Since \( X^\text{an} \) is compact,

\(^2\)Recall that compact means quasicompact and Hausdorff.
after passing to a subsequence, we may assume that \( x_n \to a \in X \). Since \( Z \) is Zariski closed in \( X \times Y \), it is closed in the classical topology. Since \((x_n, y_n) \in Z, (a, b) \in Z\), and hence \( b \in f(Z) \).

\[ \square \]

**Remark 1.9.** A related result is that if \( X \) is an irreducible variety over \( \mathbb{C} \), then \( X^{an} \) is connected. We will come back to this when we discuss holomorphic function. A challenge exercise is to prove this statement directly.

**Exercise.** Show that if \( f : X \to Y \) is a morphism of separated algebraic varieties, then \( f \) is proper if and only if \( f^{an} \) is proper.

**Remark 1.10.** From now on, all varieties over \( \mathbb{C} \) will be assumed to be separated.

### 2. Holomorphic Functions

The reference for this section is [GH94].

**2.1. Holomorphic functions in one variable.** First, we consider the case of 1-variable functions.

**Setup.** Consider an open set \( U \subseteq \mathbb{C} = \mathbb{R}^2 \). All functions considered will be smooth \((C^\infty)\). Coordinate functions on \( U \) will be denoted by \( z = x + yi \), \( \bar{z} = x - yi \). We have

\[
    dz = dx + idy, \quad d\bar{z} = dz - idy,
    \]

\[
    dz \wedge d\bar{z} = (-2i)dx \wedge dy.
    
\]

Dually, we have

\[
    \frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)
\]

(this is dual to the basis of differentials given by \( dz, d\bar{z} \)).

These operators acts on \( z^m \) by

\[
    \frac{\partial}{\partial z}(z^m) = mz^{m-1}, \quad \frac{\partial}{\partial \bar{z}}z^m = 0
\]

(by product rule, it is enough to check these for \( m = 1 \)).

If \( f : U \to \mathbb{C} \) is smooth, we write

\[
    df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}.
\]

**Exercise.** Check that \( \frac{\partial f}{\partial \bar{z}} = \overline{\frac{\partial f}{\partial z}} \).

**Proposition 2.1** (Cauchy’s formula). Let \( \Delta \) be an open disc in \( \mathbb{C} \). If \( f \) is a smooth function on an open neighborhood of \( \overline{\Delta} \), then

\[
    f(z) = \frac{1}{2\pi i} \int_{\partial \Delta} \frac{f(w)}{w - z} dw + \frac{1}{2\pi i} \int_{\Delta} \frac{\partial f}{\partial w} \frac{dw \wedge dw}{w - z}
\]

for all \( z \in \Delta \).

**Remark 2.2.**
(1) The loop $\partial \Delta$ is oriented counterclockwise.
(2) Part of the statement is that the second integral is well-defined.
(3) We will define a holomorphic function to be annihilated by $\frac{\partial}{\partial \bar{w}}$. In particular, the second integral vanishes when $f$ is holomorphic.

Proof. Let $\Delta_\epsilon$ be a disc of radius $0 < \epsilon \ll 1$ around $z$.

We apply Stokes’ formula for
\[ \eta = \frac{f(w)}{w-z} \, dw \]
on $\overline{\Delta} \setminus \Delta_\epsilon$. Note that
\[ d\eta = -\frac{\partial}{\partial \bar{w}} \left( \frac{f(w)}{w-z} \right) \, dw \wedge \, d\bar{w}. \]

By the quotient rule and
\[ \frac{\partial}{\partial \bar{w}} \left( \frac{1}{w-z} \right) = 0, \]
we have that
\[ d\eta = -\frac{\partial f}{\partial \bar{w}} \, \frac{dw \wedge \, d\bar{w}}{w-z}. \]

Stokes’ theorem then says that
\[ -\int_{\overline{\Delta} \setminus \Delta_\epsilon} \frac{\partial f}{\partial \bar{w}} \, \frac{dw \wedge \, d\bar{w}}{w-z} = \int_{\partial \Delta} \frac{f(w)}{w-z} \, dw - \int_{\partial \Delta_\epsilon} f(w) \, dw. \]

We evaluate the last integral. We change variables to $w = z + \epsilon e^{i\theta}$ for $\theta \in [0, 2\pi]$ and $dw = \epsilon i e^{i\theta} d\theta$:
\[ \int_{\partial \Delta_\epsilon} \frac{f(w)}{w-z} \, dw = \int_0^{2\pi} f(z + \epsilon e^{i\theta}) \epsilon i e^{i\theta} \, d\theta = \epsilon i \int_0^{2\pi} f(z + \epsilon e^{i\theta}) \, d\theta. \]

For $\epsilon \to 0$, this converges (by the dominated convergence theorem, for example) to
\[ i \int_0^{2\pi} f(z) \, d\theta = 2\pi i f(z). \]
Finally, we deal with the integral on the left hand side of Stokes’ theorem. Again, we change
variables to $w = re^{i\theta} + z$ for $r \geq 0$ and $\theta \in [0, 2\pi]$ and
\[
dw = e^{i\theta} dr + ire^{i\theta}d\theta, \\
\overline{d}w = e^{-i\theta} dr - ire^{-i\theta}d\theta, \\
dw \wedge \overline{d}w = -2irdr \wedge d\theta.
\]
Then
\[
\frac{dw \wedge \overline{d}w}{w - z} = -2ie^{-i\theta}dr \wedge d\theta.
\]
This is integrable on any compact subset of $\mathbb{C}$ and
\[
\int_{\Delta} \frac{\partial f}{\partial \overline{w}} \frac{dw \wedge \overline{d}w}{w - z} = \lim_{\epsilon \to 0} \int_{\Delta \setminus \Delta_\epsilon} \frac{\partial f}{\partial \overline{w}} \frac{dw \wedge \overline{d}w}{w - z}.
\]
This completes the proof.

\textbf{Definition 2.3.} A smooth function $f : U \to \mathbb{C}$ is
\begin{itemize}
  \item \textit{holomorphic} if $\frac{\partial f}{\partial \overline{z}} = 0$ (if $f = u + iv$, this is equivalent to the Cauchy–Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$),
  \item \textit{analytic} if for any $a \in U$, there exists an open disc $\Delta_r(a)$ centered at $a$ inside $U$ such that
    \[ f(z) = \sum_{n \geq 0} c_n(z - a)^n \]
    for some $c_n \in \mathbb{C}$ where the convergence is absolute and uniform for $z \in \Delta_r(a)$.
\end{itemize}

\textbf{Theorem 2.4.} A function $f$ is holomorphic if and only if it is analytic.

\textit{Proof.} We start with the ‘only if’ implication. Given $a \in U$, let $\Delta$ be a disc centered at $a$ such that $\overline{\Delta} \subseteq U$. Cauchy’s formula 2.1 then shows that
\[
f(z) = \frac{1}{2\pi i} \int_{\partial \Delta} \frac{f(w)}{w - z} dw.
\]
We write
\[
\frac{f(w)}{w - z} = \frac{f(w)}{(w - a) - (z - a)} = \frac{f(w)}{(w - a) \left( 1 - \frac{z - a}{w - a} \right)}.
\]
If $R$ is the radius of $\Delta$, we fix a disc $\Delta'$ centered at $a$ of radius $R' < R$. Then
\[
\left| \frac{z - a}{w - a} \right| \leq \frac{R'}{R} < 1
\]
for $z \in \Delta'$. Then
\[
\frac{f(w)}{w - z} = \sum_{n \geq 0} \frac{f(w)}{(w - a)^{n+1}} (z - a)^n
\]
converges absolutely and uniformly for $z \in \Delta'$ and $w \in \partial \Delta$. Therefore,
\[
f(z) = \sum_{n \geq 0} c_n(z - a)^n
\]
is absolutely and uniformly convergence for \( z \in \Delta' \), where
\[
c_n = \frac{1}{2\pi i} \int_{\partial \Delta} \frac{f(w)}{(w - a)^{n+1}} \, dw.
\]

Hence \( f \) is analytic.

For the ‘if’ implication, suppose \( f \) is analytic and choose around \( a \in U \) a small disc \( \Delta \) such that \( \overline{\Delta} \subseteq U \) and
\[
f(z) = \sum_{n \geq 0} c_n(z - a)^n
\]
converges absolutely and uniformly for \( z \in \Delta \).

Since \( \frac{\partial P}{\partial z} = 0 \) for any polynomial \( P \), if \( P_n \) is the \( n \)th partial sum, by Cauchy’s theorem 2.1,
\[
P_n(z) = \frac{1}{2\pi i} \int_{\partial \Delta} \frac{P_n(w)}{w - z} \, dw,
\]
and hence
\[
f(z) = \frac{1}{2\pi i} \int_{\partial \Delta} \frac{f(w)}{w - z} \, dw.
\]

Therefore:
\[
\frac{\partial f}{\partial z} = \frac{1}{2\pi i} \int_{\partial \Delta} \frac{\partial}{\partial z} \left( \frac{f(w)}{w - z} \right) \, dw = 0
\]
because the integrand is 0. Hence \( f \) is holomorphic. \( \square \)

**Theorem 2.5** (\( \partial \)-lemma in 1 variable). Let \( U \subseteq \mathbb{C} \) and \( g: U \to \mathbb{C} \) be a smooth function. If \( \Delta \) is a disc such that \( \overline{\Delta} \subseteq U \) and we define
\[
f(z) = \frac{1}{2\pi i} \int_{\Delta} g(w) \frac{dw \land d\overline{w}}{w - z}, \quad \text{for } z \in \Delta,
\]
then \( f \) is a smooth function and
\[
\frac{\partial f}{\partial \overline{z}} = g \quad \text{on } \Delta.
\]

**Remark 2.6.** This theorem will later be used to compute Dolbeaux cohomology.

**Proof.** Given \( z_0 \in \Delta \), choose discs centered at \( z_0 \) such that \( \Delta' \subseteq \Delta'' \subseteq \Delta \) (and the closure of the previous is contained in the next).
We can write $g = g_1 + g_2$ with $g_1$, $g_2$ smooth on $U$ such that
\[
\begin{cases}
  g_1 = 0 & \text{inside } \Delta', \\
  g_2 = 0 & \text{outside } \Delta''. 
\end{cases}
\]

Consider separately
\[
f_i(z) = \frac{1}{2\pi i} \int_{\Delta} g_i(w) \frac{dw \wedge d\overline{w}}{w - z}, \quad \text{for } i = 1, 2.
\]

For $z \in \Delta'$, $f_1$ is clearly smooth and
\[
\frac{\partial f_1}{\partial z} = \frac{1}{2\pi i} \int_{\Delta} \frac{\partial g_1(w)}{\partial w} \frac{dw \wedge d\overline{w}}{w - z} = 0.
\]

Now note that
\[
f_2(z) = \frac{1}{2\pi i} \int_C g_2(w) \frac{dw \wedge d\overline{w}}{w - z}
\int_0^{2\pi} e^{-i\theta} \int_0^\infty g_2(z + re^{i\theta}) dr d\theta
\]

where $w = z + re^{i\theta}$ and $\frac{dw \wedge d\overline{w}}{w - z} = -2ie^{-i\theta} dr \wedge d\theta$

This implies that $f_2$ is smooth on $\Delta$. After going back via the change of variables, we see that
\[
\frac{\partial f_2}{\partial z} = \frac{1}{2\pi i} \frac{\partial g_2}{\partial w} \frac{dw \wedge d\overline{w}}{w - z}.
\]

Cauchy’s formula 2.1 for $g_2$ then shows that
\[
g_2(z) = \frac{1}{2\pi i} \int_{\partial \Delta} \frac{g_2(w)}{w - z} dw + \frac{1}{2\pi i} \int_{\Delta} \frac{\partial g_2}{\partial w} \frac{dw \wedge d\overline{w}}{w - z} = \frac{\partial f_2}{\partial z}
\]

on $\Delta$. Since $\frac{\partial f_1}{\partial z} = 0 = g_1$ on $\Delta'$, this shows that
\[
\frac{\partial f}{\partial z} = g \quad \text{on } \Delta'.
\]
This shows that $\frac{\partial f}{\partial z} = g$ for any $z \in \Delta$. \qed

**Remark 2.7.** The proof also shows that if $g$ is a smooth function of $U_1 \times \cdots \times U_n \subseteq \mathbb{C}^n$, so is $f$. Moreover, if $g$ is holomorphic (separately) in each of $z_2, \ldots, z_r$, so is $f$.

### 2.2. Holomorphic functions in several variables.

Let $U \subseteq \mathbb{C}^n$ be open with coordinate functions $z_1, \ldots, z_n$, $z_j = x_j + iy_j$.

**Definition 2.8.** A smooth function $f: U \to \mathbb{C}$ is

- **holomorphic** if it is holomorphic in each variable, i.e.
  \[
  \frac{\partial f}{\partial \bar{z}_i} = 0 \quad \text{on } U.
  \]
- **analytic** if for every $a \in U$, there is a polydisc $B = B_r(a) = \{z \mid |z_j - a_j| < r \text{ for all } j\}$ such that
  \[
  f(z) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha (z - a)^\alpha,
  \]
  where we use the multiindex notation:
  \[
  (z - a)^\alpha = \prod_{i=1}^n (z_i - a_i)^{\alpha_i}.
  \]

**Theorem 2.9.** If $f: U \to \mathbb{C}$ is a smooth function, the following are equivalent:

1. $f$ is holomorphic,
2. $f$ is analytic,
3. for every polydisc $\Delta = \prod_{i=1}^n \{z_i \mid |z_i - a_i| < \alpha_i\} \subseteq U$,
   \[
   f(z) = \left(\frac{1}{2\pi i}\right)^n \int_{|z_i - a_i| = \alpha_i} \frac{f(w)}{(w_1 - z_1) \cdots (w_n - z_n)} \, dw_1 \wedge \cdots \wedge dw_n,
   \]
   where the integral is over the product of circles with product orientation.

**Proof.** It is clear that if $f$ is analytic, it is analytic in each variable, hence holomorphic in each variable, i.e. $f$ is holomorphic. This proves that (2) implies (1). To prove (3) implies (2), we argue as in the proof of Theorem 2.4. We get

\[
 f(z) = \sum_{\beta \in \mathbb{N}^n} c_\beta (z - a)^\beta
\]

where

\[
 c_\beta = \left(\frac{1}{2\pi i}\right)^n \int_{|z_i - a_i| = \alpha_i} \frac{f(w)}{(w_1 - z_1)^{\beta_1+1} \cdots (w_n - z_n)^{\beta_n+1}} \, dw_1 \wedge \cdots \wedge dw_n.
\]

For (1) implies (3), use Cauchy’s formula 2.1 for holomorphic functions in each variable:

\[
 f(z) = \frac{1}{2\pi i} \int_{|z_n - a_n| = \alpha_n} \frac{f(z_1, \ldots, z_{n-1}, w_n)}{z_n - w_n} \, dw_n = \ldots
\]
and use that $f$ is continuous and Fubini’s theorem.

For an open subset $U \subseteq \mathbb{C}^n$, we write

$$\mathcal{O}(U) = \{ f : U \to \mathbb{C} \mid f \text{ holomorphic} \}.$$  

Then the following are true.

- The subset $\mathcal{O}(U) \subseteq C^\infty(U)$ is a $\mathbb{C}$-subalgebra. To prove this, use the fact that $\frac{\partial}{\partial z_j}$ are linear (so closed under $+$ and scalar multiplication) and derivations (so closed under product).
- If $f \in \mathcal{O}(U)$ and $f(z) \neq 0$ for all $z \in U$, then $\frac{1}{f} \in \mathcal{O}(U)$. Indeed, $\frac{\partial}{\partial z_j}$ satisfies the quotient rule.

**Definition 2.10.** A function $f = (f_1, \ldots, f_m) : U \to \mathbb{C}^m$ is holomorphic if all $f_j$ are holomorphic.

We start by checking that the composition of holomorphic functions is holomorphic.

Identifying $\mathbb{C}^n = \mathbb{R}^{2n}$ with coordinates $z_1, \ldots, z_n$ and $\mathbb{C}^m = \mathbb{R}^{2m}$ with coordinates $z_1', \ldots, z_m'$, and $f_j = u_j + iv_j$, we have a map

$$T_p \mathbb{R}^{2n} \xrightarrow{df_p} T_{f(p)} \mathbb{R}^{2m}$$

which can be written explicitly as

$$\frac{\partial}{\partial x_j}(p) \mapsto \sum_{k=1}^m \frac{\partial u_k}{\partial x_j}(p) \frac{\partial}{\partial x'_k}(f(p)) + \sum_{k=1}^m \frac{\partial v_k}{\partial x_j}(p) \frac{\partial}{\partial y'_k}(f(p))$$

$$\frac{\partial}{\partial y_j}(p) \mapsto \cdots$$

**Exercise.** Show that after we tensor with $\mathbb{C}$, we have the formulas

$$\frac{\partial}{\partial z_j}(p) \mapsto \sum_{k=1}^m \frac{\partial f_k}{\partial z_j}(p) \frac{\partial}{\partial z_k}(f(p)) + \sum_{k=1}^m \frac{\partial f_k}{\partial z_j}(p) \frac{\partial}{\partial z_k}(f(p)),$$

$$\frac{\partial}{\partial z_j}(p) \mapsto \sum_{k=1}^m \frac{\partial f_k}{\partial z_j}(p) \frac{\partial}{\partial z'_k}(f(p)) + \sum_{k=1}^m \frac{\partial f_k}{\partial z_j}(p) \frac{\partial}{\partial z'_k}(f(p)).$$

The upshot is that if $f$ is holomorphic, then $\frac{\partial f_k}{\partial z_j} = \frac{\partial f_k}{\partial z_j} = 0$. Therefore

$$\text{span} \left( \frac{\partial}{\partial z_j} \bigg| j \right) \to \text{span} \left( \frac{\partial}{\partial z'_k} \bigg| k \right),$$

$$\text{span} \left( \frac{\partial}{\partial z'_j} \bigg| j \right) \to \text{span} \left( \frac{\partial}{\partial z'_k} \bigg| k \right).$$
Consider maps $U \xrightarrow{f} V \xrightarrow{g} \mathbb{C}$. The upshot is that if $g$ is also holomorphic, then $g \circ f$ is also holomorphic. In fact, for any $g$, we have that
\[
\frac{\partial (g \circ f)}{\partial z_j}(p) = \sum_{k=1}^{m} \frac{\partial f_k}{\partial z_j'} \left( \frac{\partial g_k}{\partial z_k'} \circ g \right).
\]
This implies that
- if $f, g$ is holomorphic, then $g \circ f$ is also holomorphic,
- if $f$ is a holomorphic diffeomorphism $f : U \to V$ (for $U, V \subseteq \mathbb{C}^n$), $g \circ f$ is holomorphic, and the matrix $\left( \frac{\partial f_i}{\partial z_j} \right)_{i,j}$ is invertible at every point, then $g$ is holomorphic.

Moreover, if both $f$ and $g : V \to \mathbb{C}$ are holomorphic, then
\[
\frac{\partial (g \circ f)}{\partial z_j} = \sum_k \frac{\partial f_k}{\partial z_j} \left( \frac{\partial g_k}{\partial z_k'} \circ g \right).
\]

Remark 2.11. We only assume that $g : V \to \mathbb{C}$ to simplify the notation. The above assertions also hold for $g : V \to \mathbb{C}^p$ in general.

Let $U \subseteq \mathbb{C}^n$ and $f : U \to \mathbb{C}^n$. The next goal is to prove the inverse function theorem. We want to compare the real Jacobian of $f$ with $\det \left( \frac{\partial f_i}{\partial z_j} \right)$, and deduce it from the inverse function theorem for smooth functions.

Write $f = (f_1, \ldots, f_n)$ and $(z_1, \ldots , z_n)$ for the variables on $U$ and $\mathbb{C}^n$. One can compute that:
\[
f^*(dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n) = \left( \text{determinant of real Jacobian of } f \right) dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n,
\]
where $dz_j \wedge d\overline{z}_j = (dx_j + idy_j) \wedge (dx_j - idy_j) = (2i)dx_j \wedge dy_j$,
and hence (after tensoring with $\mathbb{C}$):
\[
f^*(dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_n) = \left( \text{determinant of real Jacobian of } f \right) dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_n.
\]
The left hand side is equal to
\[
df_1 \wedge df_1 \wedge \cdots \wedge df_n \wedge d\overline{f}_n.
\]

Recall that
\[
df = \sum_j \left( \frac{\partial f}{\partial x_j} dx_j + \frac{\partial f}{\partial y_j} dy_j \right) = \sum_j \left( \frac{\partial f}{\partial z_j} dz_j + \frac{\partial f}{\partial \overline{z}_j} d\overline{z}_j \right).
\]
In particular, if $f$ is holomorphic, then each $f_k$ is holomorphic, and hence
\[
df_k = \sum_{j=1}^{n} \frac{\partial f_k}{\partial z_j} dz_j.
\]
\[ d\overline{f}_k = \sum_{j=1}^{n} \frac{\partial f_k}{\partial z_j} dz_j. \]

Finally, this shows that
\[
df_1 \wedge df_1 \wedge \cdots \wedge df_n \wedge d\overline{f}_1 \wedge \cdots \wedge d\overline{f}_n = \left( \det \left( \frac{\partial f_j}{\partial z_k} \right) \right) \left( \det \left( \frac{\partial f_j}{\partial \overline{z}_k} \right) \right) dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_n
\]

The overall conclusion is that
\[
\left( \text{determinant of the real Jacobian matrix of } f \right) = \left| \text{determinant of the complex Jacobian matrix of } f \right|^2.
\]

In particular:

- the left hand size is \( \geq 0 \),
- the left hand size is \( = 0 \) if and only if the right hand side is.

**Theorem 2.12** (Holomorphic inverse function theorem). If \( U \subseteq \mathbb{C}^n \) is open and \( f: U \to \mathbb{C}^n \) is holomorphic. Then for \( p \in U \) such that \( \det \left( \frac{\partial f_j}{\partial z_k} (p) \right) \neq 0 \), there are open neighborhoods \( U' \subseteq U \) of \( p \) and \( V' \subseteq \mathbb{C}^n \) of \( f(p) \) such that \( f \) gives a bijective map \( U' \to V' \) and its inverse is holomorphic.

**Proof.** By the previous discussion, the hypothesis implies that the determinant of the real Jacobian of \( f \) is nonzero at \( p \). The inverse function theorem for smooth maps implies that there are open subsets \( U', V' \) as above such that \( U' \xrightarrow{f} V' \) is bijective and its inverse is smooth. We may assume that \( \det \left( \frac{\partial f_j}{\partial z_k} \right) \neq 0 \) on \( U' \), and hence \( g \) is holomorphic on \( V \) since \( f \) and \( g \circ f \) are. \( \square \)

**Remark 2.13.**

1. If \( f: U \to \mathbb{C} \) is holomorphic, then \( \frac{\partial^{|\alpha|} f}{\partial z^\alpha} \) is holomorphic for all \( \alpha \) (since \( \frac{\partial}{\partial z_j} \) and \( \frac{\partial}{\partial \overline{z}_k} \) commute).

2. If \( a \in U \) is such that \( \frac{\partial^{|\alpha|}}{\partial z^\alpha} (a) = 0 \) for all \( \alpha \), then \( f \equiv 0 \) in a neighborhood of \( a \). Indeed, if \( B = \{ z \mid |z_i - a_i| < \epsilon \text{ for all } i \} \) is such that \( \overline{B} \subseteq U \), then
\[
f(z) = \left( \frac{1}{2\pi i} \right)^n \int_{|z_i - a_i| = \epsilon} f(w) \frac{dw_1 \wedge \cdots \wedge dw_n}{(w_1 - z_1) \cdots (w_n - z_n)},
\]
and using this we got
\[
\sum_{\alpha \in \mathbb{N}^n} c_\alpha (z - a)^\alpha
\]
for \( z \in B \), where

\[
c_\alpha = \left( \frac{1}{2\pi i} \right)^n \int_{|z_i - a_i| = \alpha_i} \frac{f(w)}{(w - 1 - z_1)^{\alpha_1+1} \cdots (w_n - z_n)^{\alpha_n+1}} \, dw_1 \wedge \cdots \wedge dw_n
\]

\[
= \frac{1}{\alpha!} \frac{\partial^{|\alpha|}}{\partial z^\alpha}(a)
\]

\[
= 0.
\]

**Proposition 2.14.** Suppose \( f : U \to \mathbb{C} \) is holomorphic and \( U \) is connected. If \( f = 0 \) on some \( V \subseteq U \) open, then \( f = 0 \).

**Proof.** Let \( U' = \{ z \in U \mid f = 0 \text{ on some open neighborhood of } z \} \). This set is non-empty by hypothesis and clearly open. It is enough to show that it is closed. If \( z_n \in U' \) converges to \( a \), then for every \( \alpha \)

\[
\frac{\partial^{|\alpha|}}{\partial z^\alpha}(z_n) = 0
\]

and hence

\[
\frac{\partial^{|\alpha|}}{\partial z^\alpha}(a) = 0.
\]

This holds for all \( \alpha \), so \( f = 0 \) in a neighborhood of \( a \). Thus \( a \in U' \). \( \square \)

The next goal is to state and prove the maximum modulus principle.

**Theorem 2.15 (Maximum modulus principle).** If \( U \subseteq \mathbb{C}^n \) is open and connected, and \( f : U \to \mathbb{C} \) is a holomorphic function such that \( |f| \) has a local max at \( a \in U \), then \( f \) is constant.

**Proof.** By Proposition 2.14, it is enough to show that there is an open neighborhood \( U_0 \) of \( a \) such that \( f \) is constant on \( U_0 \).

We first reduce to the case \( n = 1 \). Take \( U_0 \) to be an polydisc containing \( a \),

\[
U_0 = \{ z \mid |z_i - a_i| < \epsilon \text{ for all } i \}.
\]

For any \( z \in U_0 \), consider the 1-variable function

\[
\mathbb{C} \ni w \mapsto f(wa + (1 - w)z) \in \mathbb{C} \quad \text{for } |wa_i - (1 - w)z_i - a_i| < \epsilon.
\]

This function is defined on an open subset of \( \mathbb{C} \) containing 0 and 1. It is a holomorphic function and its has absolute value has a local maximum at \( w = 1 \). The 1-variable case then implies that this is constant, and hence \( f(z) = f(a) \).

We now prove the theorem for \( n = 1 \). Let \( \Delta = B_R(a) \) be a disc centered at \( a \) such that \( \Delta \subseteq U \). Cauchy’s formula 2.1 implies that

\[
f(a) = \frac{1}{2\pi i} \int_{\partial \Delta} \frac{f(w)}{w - a} \, dw = \frac{1}{2\pi i} \int_0^1 f(a + Re^{2\pi i \theta}) Re^{2\pi i \theta} \cdot 2\pi i \, d\theta = \int_0^1 f(a + Re^{2\pi i \theta}) \, d\theta.
\]
where \( w = a + \Re^{2\pi i \theta} \). Therefore,

\[
|f(a)| \leq \int_0^1 |f(a + \Re^{2\pi i \theta})| d\theta \overset{(*)}{\leq} |f(a)| \int_0^1 d\theta = |f(a)|,
\]

assuming that \( |f(z)| \leq |f(a)| \) in a neighborhood of \( \overline{\Delta} \) (this is true for \( R \) small enough). Therefore, the above inequalities are all equalities. Since \((*)\) is an equality and \( f \) is continuous, we conclude that \( |f(z)| = |f(a)| \) for all \( z \in \partial \Delta \).

The same holds for any \( 0 < R' \leq R \), so \( |f(z)| \) is constant in an open neighborhood of \( a \). \( \square \)

**Exercise.** Show that if \( f = u + iv \) is holomorphic on some open connected subset and \( u^2 + v^2 \) is constant, then \( f \) is constant. (Apply \( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \), and Cauchy–Riemann equations.)

### 3. Complex manifolds

If \( U \subseteq \mathbb{C}^n \) is an open subset, consider the sheaf \( \mathcal{O}_U \) of \( \mathbb{C} \)-algebras on \( U \) defined by

\[
\mathcal{O}_U(V) = \{ f: V \to \mathbb{C} \mid f \text{ holomorphic} \}.
\]

We check that this is indeed a sheaf:

1. have restriction maps: if \( V_1 \subseteq V_2 \) and \( f \) is holomorphic on \( V_2 \), then \( f \) is holomorphic on \( V_1 \),
2. if \( V = \bigcup V_i \) and \( \varphi_i: V_i \to \mathbb{C} \) are holomorphic functions such that \( \varphi_i|_{V_i \cap V_j} = \varphi_j|_{V_i \cap V_j} \), then there exists a unique \( \varphi: V \to \mathbb{C} \) such that \( \varphi|_{V_i} = \varphi_i \) for all \( i \); indeed, if \( \varphi \) is such that \( \varphi|_{V_i} \) is holomorphic for all \( i \), \( \varphi \) is holomorphic.

**Definition 3.1.** A complex manifold of dimension \( n \) is a pair \( (X, \mathcal{O}_X) \) where

1. \( X \) is a topological space, assumed Hausdorff and having a countable basis of open subsets,
2. \( \mathcal{O}_X \subseteq \mathcal{C}_{X,\mathbb{C}} \) is a subsheaf of the sheaf of continuous \( \mathbb{C} \)-valued functions on \( X \), such that \( X \) can be written as

\[
X = \bigcup_i U_i, \quad U_i \subseteq X \text{ open}
\]

such that each \( (U_i, \mathcal{O}_{U_i}) \cong (V_i, \mathcal{O}_{U_i}) \) for some \( V_i \subseteq \mathbb{C}^n \) is open with \( \mathcal{O}_{V_i} \) is the sheaf of holomorphic functions on \( V_i \).

**Remark 3.2.** Suppose \( V_1, V_2 \subseteq \mathbb{C}^n \) are open and \( f: (V_1, \mathcal{O}_{V_1}) \to (V_2, \mathcal{O}_{V_2}) \) is an isomorphism, i.e. a homeomorphism \( f: V_1 \to V_2 \) which induces an isomorphism of sheaves: for all \( U \subseteq V_2 \),

\[
\mathcal{O}(V_2) \xrightarrow{\cong} \mathcal{O}(f^{-1}(V_2)),
\]

\( \varphi \mapsto \varphi \circ f \).

This forces \( f \) and \( f^{-1} \) to be holomorphic functions. The converse is also true.

**Definition 3.3.**

1. If \( (X, \mathcal{O}_X) \) is a complex manifold, the section of \( \mathcal{O}_X \) are the holomorphic functions on \( X \).
(2) If \((X, \mathcal{O}_X)\) and \((Y, \mathcal{O}_Y)\) are complex manifolds, then a **holomorphic map**

\[
(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y)
\]

is a continuous map \(f: X \to Y\) that induces a map of sheaves, i.e. for any \(V \subseteq Y\) open and \(\varphi \in \mathcal{O}_Y(V)\), we have that \(\varphi \circ f \in \mathcal{O}(f^{-1}(V))\).

**Remark 3.4.** If \(X \subseteq \mathbb{C}^n\) and \(Y \subseteq \mathbb{C}^m\) are open subsets, this coincides with the previous definition.

**Remark 3.5.** If \(U \subseteq \mathbb{C}^n\), \(p \in U\),

\[
\mathcal{O}_{\mathbb{C}^n, p} = \lim_{\substack{\longrightarrow \nu \ni p}} \mathcal{O}_U(V).
\]

To check that this is a local ring, we note that we have a map

\[
\mathcal{O}_{U,p} = \lim_{\substack{\longrightarrow \nu \ni p}} \mathcal{O}_U(V) \twoheadrightarrow \mathbb{C} \quad (V, \varphi) \mapsto \varphi(p)
\]

whose kernel \(\{(V, \varphi) \mid \varphi(p) = 0\} = \mathfrak{m}\) is the unique maximal ideal. Indeed, if \((V, \varphi) \notin \mathfrak{m}\), we may assume that \(\varphi(z) \neq 0\) for all \(z \in V\), and hence \(\frac{1}{\varphi} \in \mathcal{O}(V)\). Hence \((\mathcal{O}_{U,p}, \mathfrak{m})\) is a local ring.

**Remark 3.6.** All such local rings for manifolds of fixed dimension are isomorphic. This is very different from the algebraic case.

**Remark 3.7.** One can define complex manifolds using atlases: \(X\) is a topological space with suitable properties and \(X = \bigcup_i U_i\) is an open cover together with homeomorphisms \(\varphi_i: U_i \cong V_i \subseteq \mathbb{C}^n\), where \(V_i \subseteq \mathbb{C}^n\) are open, such that for all \(i, j\) the map

\[
\varphi_i(U_i \cap U_j) \xrightarrow{\varphi_i^{-1}} \varphi_j(U_i \cap U_j)
\]

is biholomorphic.

We identify two such objects \((X, \mathcal{A}), (X, \mathcal{A}')\) if \(\mathcal{A}\) and \(\mathcal{A}'\) are compatible.

**Remark 3.8.** It is clear from the definition via atlases (Remark 3.7), using that holomorphic maps \(\mathbb{C}^m \supseteq U \to \mathbb{C}^n\) are smooth, that every complex manifold of dimension \(n\) has an underlying real smooth manifold structure of dimension \(2n\). To avoid confusion, we will write \(X_\mathbb{R}\) for this real smooth manifold (if necessary). We have an inclusion of sheaves

\[
\mathcal{O}_X \subseteq C_{\mathbb{R}, \mathbb{C}}^\infty.
\]

Next, we will discuss:

- vector bundles in the smooth/holomorphic category,
- submanifolds,
- complex manifold associated to a smooth complex algebraic variety.
3.1. **Vector bundles.** If $M$ is a smooth real manifold, a *real* (or *complex*) vector bundle of rank $r$ on $M$ is a smooth manifold $E$ with a smooth map $E \to M$ such that for any $x \in M$, $\pi^{-1}(x)$ has the structure of a vector space over $\mathbb{R}$ (respectively $\mathbb{C}$) of dimension $r$ such that there is an open cover $M = \bigcup U_i$ such that we have isomorphisms

$$
\pi^{-1}(U_i) \cong U_i \times \mathbb{R}^r \quad \text{(resp. } U_i \times \mathbb{C}^r)\quad (\text{respectively, } U_i \times \mathbb{C}^r),
$$

inducing linear maps on the fibers.

Given such $E$, we get a sheaf $\mathcal{E}$ on $M$ such that

$$
\mathcal{E}(U) = \{s: U \to E \text{ smooth} \mid \pi \circ s = 1_U\}.
$$

This gives an equivalence of categories

$$
\begin{align*}
\left\{ \text{real (complex) vector bundles on } M \text{ (of rank } r) \right\} & \leftrightarrow \left\{ \text{locally free sheaves (of rank } r)\text{ of } \mathcal{C}_M^{\infty} \text{-modules} \right\}.
\end{align*}
$$

We will consider the corresponding notion in the category of complex manifolds. For *complex vector bundles*, we assume that $E$ is a complex manifold, $\pi$ is holomorphic.

These correspond to locally free sheaves of $\mathcal{O}_M$-modules. Note that associated to such $E$, we will have: sheaves of smooth sections and sheaves of holomorphic sections.

**Definition 3.9.** Let $X$ be a complex manifold of dimension $n$. A *closed submanifold* of $X$ of codimension $r$ is a closed subset $Y \subseteq X$ such that for all $p \in Y$, there is a chart $p \in U \xrightarrow{\varphi} V \subseteq \mathbb{C}^n$ such that

$$
\varphi(U \cap Y) = \{z \in V \mid z_1 = \cdots = z_r = 0\}.
$$

It is easy to see that by restricting such charts to $Y$, we get a holomorphic atlas on $Y$, making it a complex manifold of dimension $n - r$.

The universal property of submanifolds is: given a holomorphic $g: Z \to X$ such that $g(Z) \subseteq Y$, there is a unique holomorphic map $g': Z \to Y$ such that $\text{incl} \circ g' = g$.

**Proposition 3.10.** If $U \subseteq \mathbb{C}^n$ is open and $f_1, \ldots, f_r \in \mathcal{O}(U)$ are such that

$$
\text{rank} \left( \frac{\partial f_i}{\partial z_j}(p) \right) = r \leq n
$$

for all $p \in U$, then

$$
Y = \{z \in U \mid f_1(z) = f_2(z) = \cdots = f_r(z) = 0\}
$$

is a closed submanifold of $U$ of codimension $r$. 

Proof. Given \( p \in Y \), we may assume that \( \text{rank} \left( \frac{\partial f_i}{\partial z_j}(p) \right)_{1 \leq i,j \leq n} \neq 0 \). Define

\[ \varphi: U \to \mathbb{C}^n, \]

\[ z \mapsto (f_1(z), \ldots, f_r(z), z_{r+1}, \ldots, z_n). \]

Then

\[ \det \left( \frac{\partial \varphi_i}{\partial z_j}(p) \right) \neq 0 \]

and we apply the Inverse Function Theorem 2.12 to see that \( \varphi \) is biholomorphic in some neighborhood of \( p \). In the neighborhood, \( \varphi \) is the desired chart. \( \square \)

Basic properties of holomorphic functions we discussed extend to this setting. We recall a few of them for completeness. Let \( X \) be a complex manifold.

1. If \( f \in \mathcal{O}(X) \) is such that \( f|_U = 0 \) for some \( U \subseteq X \) open, and \( X \) is connected, then \( f = 0 \).
2. (Maximum modulus principle) If \( f \in \mathcal{O}(X) \) is such that \( |f| \) has a local max, \( X \) is connected, then \( f \) is constant.

**Corollary 3.11.** If \( X \) is a compact connected complex manifold, then \( \Gamma(X, \mathcal{O}_X) = \mathbb{C} \).

*Proof.* Since \( X \) is compact, for any \( f \in \mathcal{O}(X) \), \( |f| \) has a maximum. Then the maximum modulus principle implies that \( f \) is constant. \( \square \)

3.2. **The complex manifold associated to a smooth complex algebraic variety.** Let \( X \) be a smooth complex algebraic variety of pure dimension \( n \). Choose an affine open subset \( U \subseteq X \) and let \( U \hookrightarrow \mathbb{C}^N \) be a closed immersion, \( r = N - n \). Since \( U \) and \( \mathbb{C}^N \) are smooth, can cover \( \mathbb{C}^N \) by open subsets \( V_i \) (in the Zariski topology) such that if \( V_i \cap U \neq \emptyset \) then \( V_i \cap U \hookrightarrow V_i \) is cut out by \( r \) equations \( f_1, \ldots, f_r \in \mathcal{O}(V_i) \) with

\[ \text{rank} \left( \frac{\partial f_i}{\partial z_j}(p) \right) = r \]

for all \( p \in V_i \cap U \).

Applying Proposition 3.10, each \( V_i \cap U \hookrightarrow V_i \) is a closed complex submanifold of codimension \( r \).

**Exercise.** Check that the resulting transition maps are holomorphic, using the fact that rational maps are holomorphic.

**Exercise.** Show that if \( f: X \to Y \) is a morphism between smooth complex algebraic varieties, then the induced map \( X^{\text{an}} \to Y^{\text{an}} \) is holomorphic.

We now discuss an application.

**Theorem 3.12.** If \( X \) is a connected complex algebraic variety, then \( X^{\text{an}} \) is connected.

We first prove this theorem when \( X \) is a smooth connected projective curve over \( \mathbb{C} \).
Proof when $X$ is a smooth, projective curve. We first prove this when $X$ is a smooth connected projective curve over $\mathbb{C}$. We know that $X^{an} = U \cup V$ is a disjoint union with $U, V$ open in $X^{an}$ and nonempty. Take $P \in U$. If $n \gg 0 \ (n \geq 2 \cdot \text{genus}(X))$, $\mathcal{O}_X(nP)$ is globally generated. Then there exists $s \in \Gamma(X, \mathcal{O}_X(nP))$ which does not vanish at $P$. Then $nP \sim P_1 + \cdots + P_n$ for $P_i \neq P$. So there exists $\phi \in \mathcal{O}^*(X)$ such that $\text{div}(\phi) = (P_1 + \cdots + P_n) - nP$, so $\phi$ gives a regular function $X \{P\} \to \mathbb{C}$. Note that it is holomorphic. By restricting to $V$, we get a holomorphic map $V \xrightarrow{g\phi \mid V} \mathbb{C}$. Since $V$ is a compact complex manifold, $g$ is constant by Corollary 3.11. In particular, $\phi$ takes the same value infinitely many times, so $\phi$ is constant, and hence $\phi = 0$. This is a contradiction. □

To reduce the general case to dim $X = 1$, we use the following result.

**Proposition 3.13.** Let $X$ be an algebraic variety over $k = \overline{k}$. For any $x_1, x_2 \in X$, there is an irreducible curve $C \subseteq X$ such that $x_1, x_2 \in C$.

**Proof.** We may assume that $n = \text{dim} X \geq 2$.

(1) By Chow’s lemma, there is a surjective morphism $\pi: \tilde{X} \to X$ where $\tilde{X}$ is irreducible and quasi-projective. If $\tilde{x}_1, \tilde{x}_2$ lie above $x_1, x_2$, it is enough to find a curve $\tilde{C}$ on $\tilde{X}$ through $\tilde{x}_1, \tilde{x}_2$ and take $C = \pi(\tilde{C})$. We may hence assume $X$ is quasi-projective.

(2) Choose a locally closed immersion $X \hookrightarrow \mathbb{P}^N$. It is enough to prove the statement for $X$. We may hence assume that $X$ is projective.

Consider the blow up of $X$ at $\{x_1, x_2\}$:

$$ Y = \text{Bl}_{\{x_1, x_2\}} X \hookrightarrow E_i = \pi^{-1}(x_i) $$

where $\text{dim} E_i = n - 1$.

The variety $Y$ is projective since $X$ is, so we may choose an embedding $Y \hookrightarrow \mathbb{P}^N$. Cut $Y$ with $n - 1$ general hyperplane $H_1, \ldots, H_{n-1}$. Since $\text{dim} E_i = n - 1$,

$$ E_i \cap H_1 \cap \cdots \cap H_{n-1} \neq \emptyset \quad \text{for } i = 1, 2. $$

If $Z = Y \cap H_1 \cap \cdots \cap H_{n-1}$, the curve $C = \pi(Z)$ satisfies the requirements. Using Bertini’s Theorem: a general hyperplane section of an irreducible projective variety of dimension $\geq 2$ is irreducible, and hence $Z$ is irreducible.

We need to assume that $\text{dim}(Z \cap E_i) = 0$ for $i = 1, 2$. This is okay since the $H_i$s are general. □

We can now finish the proof of Theorem 3.12.
Proof of Theorem 3.12. We just have to reduce to the smooth, projective curve case from the general case.

First, we may assume that $X$ is irreducible (since by hypothesis we can go from any irreducible component to any other one via points of intersection).

For an irreducible algebraic variety $X$ over $k = \overline{k}$, for any $x, y \in X$, there is an irreducible curve $C$ such that $x, y \in C$ by Proposition 3.13. We may hence assume that $X$ is an irreducible curve.

If $\tilde{X} \to X$ is the normalization, it is enough to show that $\tilde{X}^{an}$ is connected. We may hence assume that $X$ is smooth.

Finally, let $X \subseteq \overline{X}$ where $\overline{X}$ is a smooth, projective, connected curve. We showed last time that $\overline{X}^{an}$ is connected.

We now use that if $M$ is smooth real manifold of dimension $\geq 2$, $p \in M$, and $M$ is connected, then $M \setminus \{p\}$ is also connected.

Indeed, if $M \setminus \{p\} = U \cup V$ is a disjoint union of open non-empty sets, then $p \in U \cap V$ because $M$ is connected. Choose a neighborhood $W$ of $p$ such that $W$ is isomorphic to a ball. Then $W \setminus \{p\}$ is disconnected. This is a contradiction, since it is clearly path-connected. \[\Box\]

3.3. More examples of complex manifolds. Suppose $X$ is a complex manifold and $G$ is a group acting on $X$ via holomorphic maps. Suppose

(1) for all $x \in X$, there exists an open neighborhood $U \ni x$ such that $U \cap gU \neq \emptyset$ implies $g = e$ (this is sometimes called a properly discontinuous action),

(2) for all $x, y \in X$ such that $x, y$ are not in the same orbit, there exist open neighborhoods $U \ni x, V \ni y$ such that $gU \cap V = \emptyset$ for all $g$.

Note that (1) implies that the quotient map $\pi: X \to X/G$ is a covering space. Moreover, since the transition maps are holomorphic, there is a unique complex manifold structure on $X/G$ such that $\pi$ is holomorphic. Condition (2) implies that $X/G$ is Hausdorff.

Example 3.14 (Complex tori). Let $V$ be an $n$-dimensional complex vector space and $\Lambda \subseteq V$ be a lattice (i.e. a free abelian group of rank $\leq 2n$ such that $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} \cong V$). The natural action of $\Lambda$ by translations satisfies (1) and (2) above, and hence

$$V \xrightarrow{\pi} Z = V/\Lambda$$

gives a complex manifold $Z$. Note that

$$V/\Lambda \cong \mathbb{R}^{2n}/\mathbb{Z}^{2n} \cong (S^1)^{2n},$$

and hence topologically, $Z$ is a $2n$-dimensional torus.

When $n = 1$, the resulting $Z$ is an analytic construction of elliptic curves, which are algebraic. We will see that for $n \geq 2$, most of these do not come from algebraic varieties. However, they are still Kähler manifold.

Example 3.15 (Hopf surface). Consider the action of $\mathbb{Z}$ on $\mathbb{C}^2 \setminus \{(0, 0)\}$, where the generator $\gamma$ of $\mathbb{Z}$ acts by $(z_1, z_2) \mapsto (2z_1, 2z_2)$.
This clearly satisfies conditions (1) and (2), so we get a complex manifold structure on the quotient.

We have a diffeomorphism:

\[
\mathbb{C}^2 \setminus \{(0, 0)\} \xrightarrow{\sim} S^3 \times \mathbb{R}
\]

\[
(z_1, z_2) \mapsto \left(\frac{1}{\sqrt{|z_1|^2 + |z_2|^2}}(z_1, z_2), \log \sqrt{|z_1|^2 + |z_2|^2}\right)
\]

under which the action of \( \gamma \) translates to

\[
(u, t) \mapsto (u, t + \log 2).
\]

Therefore, the Hopf surface is topologically

\[
\mathbb{C}^2 \setminus \{(0, 0)\}/\mathbb{Z} \cong S^3 \times S^1.
\]

We will later see these manifolds are not even Kähler, and hence do not come from algebraic surfaces.

3.4. Orientation. If \( V \) is a 1-dimensional real vector space, an orientation on \( V \) is a choice of element in \( V/\mathbb{R}^*_>0 \). Note that an orientation of \( V \) is the same as an orientation of \( V^* \).

If \( V \) is an \( n \)-dimensional vector space, an orientation on \( V \) is an orientation on \( \Lambda^n V \).

If \( X \) is a smooth real manifold and \( E \) is a real vector bundle on \( X \), an orientation on \( E \) is a compatible system of orientations on \( E_x \) for all \( x \in X \), i.e. locally have trivializations \( \pi^{-1}(U) \cong U \times \mathbb{R}^r \) where \( \pi : E \to X \), preserving the orientations on the fibers.

Note that an orientation on \( E \) corresponds to an orientation on \( E^* \).

**Definition 3.16.** An orientation on a smooth real manifold \( X \) is an orientation on the tangent bundle \( TX \) (or equivalently on the cotangent bundle \( T^*X \)).

Giving an orientation is equivalent to giving a system of charts such that for all transition maps

\[
f = (f_1, \ldots, f_n) : U \to \mathbb{R}^n, \quad \det \frac{\partial f_i}{\partial x_j} > 0.
\]

Note that if \( X \) is a complex manifold and we consider the smooth manifold structure, we saw that if we take a system of holomorphic charts, then for the transition maps \( f = (f_1, \ldots, f_n) : U \to \mathbb{C}^n \) where \( U \subseteq \mathbb{C}^n \),

\[
\det(\text{real Jacobian}) = \left| \det \frac{\partial f_i}{\partial z_j} \right|^2 > 0.
\]

Therefore, we have a canonical orientation on \( X \).

By convention: given chart \( f : U \to \mathbb{C}^n \), the orientation on \( U \) corresponds to the orientation on \( \mathbb{C}^n \) given by the top form \( dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n \).
3.5. **The analytic space associated to an algebraic variety.** We first discuss the local model. For \( U \subseteq \mathbb{C}^n \) open in the classical topology, \( f_1, \ldots, f_r \in \mathcal{O}(U) \), consider

\[
Z = \{ u \in U \mid f_1(u) = \cdots = f_r(u) = 0 \}.
\]

Consider on \( Z \) the sheaf given by

\[
\mathcal{O}_Z(V) = \{ f : V \to \mathbb{C} \mid \text{locally } f \text{ extends to a holomorphic function on an open subset in } \mathbb{C}^n \}.
\]

If \( Z \hookrightarrow U \) is the inclusion, we get a map

\[
\mathcal{O}_U \twoheadrightarrow j_* \mathcal{O}_Z
\]

and the kernel is \( \mathcal{I}_{Z/U} \) given by

\[
\Gamma(V, \mathcal{I}_{Z/U}) = \{ f : V \to \mathbb{C} \mid f|_{V \cap Z} = 0 \}.
\]

Note that \((Z, \mathcal{O}_Z)\) is a locally ringed space.

**Definition 3.17.** A (reduced) **analytic space** is a locally ringed space \((X, \mathcal{O}_X)\) such that

1. \( X \) is a Hausdorff topological space with a countable basis for the topology,
2. there is an open cover \( X = \bigcup_i W_i \) such that each \((W_i, \mathcal{O}_{W_i})\) is isomorphic as a locally ringed space to a local model as above.

The sections of \( \mathcal{O}_X \) are called **holomorphic functions** on \( X \).

**Definition 3.18.** A **holomorphic map** between analytic spaces \((X, \mathcal{O}_X)\) and \((Y, \mathcal{O}_Y)\) is a continuous map \( f : X \to Y \) such that for any \( V \subseteq Y \) open and \( \varphi \in \mathcal{O}_Y(V) \), we have \( \varphi \circ f \in \mathcal{O}_X(\varphi^{-1}(V)) \).

**Examples 3.19.**

1. Every complex manifold is canonically an analytic space.
2. If \( X \) is a separated algebraic variety, we have a sheaf \( \mathcal{O}_{X^{an}} \) on \( X^{an} \) that makes it an analytic space. We do it locally. Choose affine open subsets covering \( X \); each such open subspace covering \( U \) has a closed immersion \( U \hookrightarrow \mathbb{C}^N \) (cut out by finitely many polynomials), so we have a sheaf \( \mathcal{O}_{U^{an}} \) on \( U^{an} \) making it an analytic space. It is easy to check that these sheaves are compatible on intersections, so we get a sheaf \( \mathcal{O}_{X^{an}} \) on \( X^{an} \).

We get in this way a functor

\[
\{ \text{complex algebraic varieties} \} \to \{ \text{analytic spaces} \}.
\]

However, in this class, we deal with smooth varieties, and hence we only have to work with complex manifolds.

3.6. **Comparison results.** Let \( X \) be a complex algebraic variety. As we saw above, it has an associated analytic space \( X^{an} \).

We have a morphism of locally ringed spaces:

\[
(\varphi, \varphi^\#) : (X^{an}, \mathcal{O}_{X^{an}}) \to (X, \mathcal{O}_X)
\]
defined by \( \varphi(x) = x \) and
\[
\varphi^\#: \mathcal{O}_X \to \varphi_* \mathcal{O}_{X^{an}} \\
\mathcal{O}_X(U) \to \mathcal{O}_{X^{an}}(U) \\
f \mapsto f
\]
(since every regular function on \( U \) is holomorphic). The corresponding ring homomorphism \( \mathcal{O}_{X,x} \to \mathcal{O}_{X^{an},x} \) is a local homomorphism.

Given an \( \mathcal{O}_X \)-module \( \mathcal{F} \), let
\[
\mathcal{F}^{an} = \varphi^*(\mathcal{F}) = \varphi^{-1}(\mathcal{F}) \otimes_{\varphi^{-1}(\mathcal{O}_X)} \mathcal{O}_{X^{an}},
\]
which is an \( \mathcal{O}_{X^{an}} \)-module.

In particular, for every \( x \in X \), we have a canonical isomorphism
\[
(\mathcal{F}^{an})_x \cong \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X^{an},x}.
\]
We will see later that \( \mathcal{O}_{X^{an},x} \) is a Noetherian ring and the morphism \( \mathcal{O}_{X,x} \to \mathcal{O}_{X^{an},x} \) is flat. In particular, this will imply that the functor \( \mathcal{F} \mapsto \mathcal{F}^{an} \) is exact.

Note that we have canonical maps:
\[
\begin{align*}
\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) & \to \text{Hom}_{\mathcal{O}_{X^{an}}}(\mathcal{F}^{an}, \mathcal{G}^{an}), \\
H^i(X, \mathcal{F}) & \to H^i(X^{an}, \mathcal{F}^{an}), \\
\text{more generally, } \text{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) & \to \text{Ext}^i_{\mathcal{O}_X}(\mathcal{F}^{an}, \mathcal{G}^{an}).
\end{align*}
\]

**Theorem 3.20** (GAGA, part 1). If \( X \) is a complete variety, then the functor \( \mathcal{F} \mapsto \mathcal{F}^{an} \) is fully faithful on coherent sheaves. Moreover, for all \( \mathcal{F}, \mathcal{G} \) coherent, the map
\[
\text{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \text{Ext}^i_{\mathcal{O}_X}(\mathcal{F}^{an}, \mathcal{G}^{an})
\]
is an isomorphism.

The theorem is due to Serre when \( X \) is projective and due to Grothendieck when \( X \) is complete. There is also a relative version for proper morphisms. We will prove this theorem only when \( X \) is projective.

The category which is the target of this functor (i.e. which \( \mathcal{F}^{an} \) belongs to) still have to be defined.

**Definition 3.21.** In general, if \( (X, \mathcal{O}_X) \) is a locally ringed space, an \( \mathcal{O}_X \)-module, \( \mathcal{F} \) is **locally finitely generated** if for any \( x \in X \), there is an open neighborhood \( U \ni x \) and \( s_1, \ldots, s_n \in \mathcal{F}(U) \) such that
\[
s_1.y, \ldots, s_n.y \in \mathcal{F}_y
\]
generate \( \mathcal{F}_y \) over \( \mathcal{O}_{X,y} \) for all \( y \in U \).

**Definition 3.22.** An \( \mathcal{O}_X \)-module \( \mathcal{F} \) is **coherent** if
\[
\begin{align*}
&\text{it is locally finitely generated}, \\
&\text{for every open subset } U \subseteq X, s_1, \ldots, s_r \in \mathcal{F}(U), \text{ the kernel of the induced map } \\
&\ker(\mathcal{O}_U^{pr} \to \mathcal{F})
\end{align*}
\]
is locally finitely generated.
Exercise. Check that on algebraic varieties, this coincides with the definition in Hartshorne [Har77].

Theorem 3.23 (Oka). If $X$ is an analytic space, then $\mathcal{O}_X$ is coherent. (In particular, also locally free $\mathcal{O}_X$-modules of finite rank are coherent)

If $X$ is an algebraic variety over $\mathbb{C}$, then any coherent sheaf on $\mathcal{F}$ on $X$ has a finite presentation, so $\mathcal{F}^{\text{an}}$ is coherent.

Theorem 3.24 (GAGA, part 2). If $X$ is complete, the functor

$$\{\text{coherent } \mathcal{O}_X\text{-modules}\} \to \{\text{coherent } \mathcal{O}_X^{\text{an}}\text{-modules}\}$$

$$\mathcal{F} \mapsto \mathcal{F}^{\text{an}}$$

is an equivalence of categories.

Remark 3.25. In particular, in this case we have an equivalence of categories

$$\{\text{locally free } \mathcal{O}_X\text{-modules}\} \to \{\text{locally free } \mathcal{O}_X^{\text{an}}\text{-modules}\}.$$  

To show that $\mathcal{F}$ is locally free if $\mathcal{F}^{\text{an}}$ is, use the fact that $\mathcal{O}_{X,x} \to \mathcal{O}_{X^{\text{an}},x}$ is faithfully flat.


1. In general, $(\mathcal{O}_X)^{\text{an}} = \mathcal{O}_{X^{\text{an}}}.$

2. If $X$ is a complex algebraic variety and $E \xrightarrow{\pi} X$ is an algebraic vector bundle with sheaf of sections $\mathcal{E}$, the holomorphic vector bundle $E^{\text{an}} \xrightarrow{\pi^{\text{an}}} X^{\text{an}}$ has the sheaf of sections $\mathcal{E}^{\text{an}}$.

3. Applying the theorem for coherent ideal sheaves, in the setting of the theorem, every closed analytic subspace of $X^{\text{an}}$ is equal to $Y^{\text{an}}$ for some closed subvariety $Y \subseteq X$. (When $X = \mathbb{P}^N$, this was known as Chow’s Theorem.)

4. Using (3) and the graph, any morphism $f: X^{\text{an}} \to Y^{\text{an}}$ comes from a morphism $X \to Y$. Therefore, the functor

$$\{\text{complete algebraic varieties}\} \to \{\text{compact analytic spaces}\}$$

$$X \mapsto X^{\text{an}}.$$ 

3.7. The ring $\mathcal{O}_{\mathbb{C}^n,0}$.

Definition 3.27. An element $f \in \mathbb{C}[z_1, \ldots, z_n]$ is convergent if there is an $R$ such that

$$f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha$$

converges uniformly and absolutely for $|z_i| < R$ for all $i$. We write

$$\mathbb{C}\{z_1, \ldots, z_n\} = \{f \in \mathbb{C}[z_1, \ldots, z_n] \mid f \text{ is convergent}\} \subseteq \mathbb{C}[z_1, \ldots, z_n].$$

It is easy to check that $f = \sum a_\alpha z^\alpha$ is convergent if and only if there exists $R > 0$ such that $\{|a_\alpha|R^\alpha\}_\alpha$ is bounded. This is also equivalent to

$$\limsup_{|\alpha| \to \infty} |a_\alpha|^{1/|\alpha|} < \infty$$

(by the Cauchy-Hadamard Theorem).
We have a map

\[ \mathcal{O}_{\mathbb{C}^n, 0} \to \mathbb{C}[z_1, \ldots, z_n] \]

\[ f \mapsto \sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha \quad \text{where} \quad a_\alpha = \frac{1}{\alpha!} \frac{\partial|f|}{\partial z^\alpha}(0). \]

Here \( \mathcal{O}_{\mathbb{C}^n, 0} \) is the ring of germs of holomorphic functions at 0. Recall that if \( f \) is holomorphic at 0, then

\[ f(z) = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \frac{\partial|f|}{\partial z^\alpha}(0)z^\alpha \]

converges absolutely and uniformly in a neighborhood of 0. By definition, the image of the above map is hence

\[ \mathbb{C}\{z_1, \ldots, z_n\} \]

and it is clear it is injective. Moreover, it is clearly a ring homomorphism.

**Conclusion.** If \( p \in M \) where \( M \) is a complex manifold, then

\[ \mathcal{O}_{M, p} \cong \mathbb{C}\{z_1, \ldots, z_n\} \]

where \( n = \dim X \).

The next goal is to show that \( \mathbb{C}\{z_1, \ldots, z_n\} \) is Noetherian. The idea is to proceed by induction and the key ingredient is the Weierstrass Preparation Theorem.

**Definition 3.28.** A **Weierstrass polynomial with respect to** \( z_n \) is an element of \( \mathbb{C}\{z_1, \ldots, z_n\} \) of the form

\[ z_n^d + a_1(z_1, \ldots, z_{n-1})z_n^{d-1} + \cdots + a_d(z_1, \ldots, z_{n-1}) \]

such that \( a_0(0) = 0 \) for \( 1 \leq i \leq d \).

**Theorem 3.29** (Weierstrass Preparation Theorem). *Given \( f \in \mathbb{C}\{z_1, \ldots, z_n\} \) such that \( f(0, \ldots, 0, z_n) \neq 0 \), there exist unique \( g, h \in \mathbb{C}\{z_1, \ldots, z_n\} \) such that \( h(0) \neq 0 \), \( g \) is a Weierstrass polynomial, and

\[ f = g \cdot h. \]

**Remark 3.30.**

(1) If \( n = 1 \) and \( f \in \mathbb{C}\{z\} \), \( f \neq 0 \), then

\[ f = z^d h \]

where \( h(0) \neq 0 \). Weierstrass Preparation Theorem 3.29 is a generalization of this statement to more variables.

(2) Note that (1) implies that (still for \( n = 1 \)) if \( f \in \mathcal{O}(U) \), the zeroes of \( f \) do not accumulate in \( U \).

(3) The condition that \( f(0, \ldots, 0, z_n) \neq 0 \) can always be achieved (if \( f \neq 0 \)) by a linear change of variables.

Recall (a special case of) the Residue Theorem. Suppose \( f \in \mathcal{O}(U \setminus \{a_1, \ldots, a_r\}) \) and there is a disc \( \Delta \subseteq U \) such that \( a_i \in \Delta \).
Then
\[
\frac{1}{2\pi i} \int_{\partial \Delta} \varphi(z) \, dz = \sum_{i=1}^{r} \text{Res}_{a_i}(\varphi_i)
\]

In fact, we will only need this when \(\varphi\) is meromorphic at \(a_i\) with pole of order \(\leq 1\). Using \(d(\varphi(z)dz) = 0\) and Stokes’ Theorem, we can reduce the computation of the integral to the case \(r = 1\) by cutting out small discs around \(a_1, \ldots, a_r\).

In this case, we may write \(\varphi = \frac{\psi}{z - a}\) and then \(\text{Res}_{a}(\varphi) = \psi(a)\). Then
\[
\frac{1}{2\pi i} \int_{|w-a|=r} \frac{\psi(w)}{w-a} \, dw = \psi(a)
\]
by Cauchy’s formula 2.1.

In our case, we will take \(f \in \mathcal{O}(U), \Delta \subseteq U\), and consider
\[
\frac{1}{2\pi i} \int_{\partial \Delta} z^j \frac{f'(z)}{f(z)} \, dz
\]
where \(f\) has no zeroes on \(\partial \Delta\). Suppose \(a\) is a zero of \(f\) and write \(f = (z - a)^m h\), \(h(a) \neq 0\). Then
\[
z^j \frac{f'(z)}{f(z)} = z^j \left( \frac{m}{z-a} + \frac{h'}{h} \right)
\]
which implies that
\[ \text{Res}_a z^j f'(z) f(z) = ma^j. \]

Overall, the conclusion is that:
\[ \frac{1}{2\pi i} \int_{\partial \Delta} z^j f'(z) f(z) \, dz = \lambda_1^j + \cdots + \lambda_m^j \]
if \( \lambda_1, \ldots, \lambda_m \) are the roots of \( f \) in \( \Delta \), listed with multiplicity.

**Proof of Weierstrass Preparation Theorem 3.29.** Let \( z' = (z_1, \ldots, z_{n-1}) \) and write
\[ f_{z'}(z_n) = f(z', z_n) \]
where \( f \) is a holomorphic function on \( \mathbb{C}^n \supseteq U \ni 0 \). Let \( \epsilon_n > 0 \) be such that \( f(0, \ldots, 0, z_n) \neq 0 \) for \( 0 < |z_n| \leq \epsilon_n \).

Choose \( \epsilon' > 0 \) such that if \( z' \) satisfies that if \( |z_i| < \epsilon' \) for \( 1 \leq i \leq n-1 \) and \( |z_n| = \epsilon_n \), then \( f(z', z_n) \neq 0 \), and
\[ \{ z \mid |z_i| < \epsilon' \text{ for } i \leq n-1, \text{ } |z_n| < \epsilon_n \} \subseteq U. \]
Otherwise, one can choose \( z_i \to 0 \) such that \( f(z', z_n) = 0 \) and continuity of \( f \) will contradict the way we chose \( \epsilon_n \).

Given \( z' \) such that \( |z_i| < \epsilon' \) for \( i \leq n-1 \), let
\[ \lambda_1(z'), \ldots, \lambda_m(z') \]
be the zeroes of \( f_{z'} \) in
\[ \{ z_n \mid |z_n| < \epsilon_n \}, \]
listed with multiplicities. By equation (1),
\[ \sum_{i=1}^m \lambda_i(z')^j = \frac{1}{2\pi i} \int_{|w| = \epsilon_n} w^j \cdot \frac{\partial f(z', w)}{\partial z_n} f(z', w) \, dw. \]

Note that the right hand side is a holomorphic function as a function of \( z' \). For \( j = 0 \), the left hand side is an integer, and hence constant. This shows that
\[ m = \text{ord}_{z_n} f(0, \ldots, 0, z_n) \]
by taking \( z' = 0 \).

If \( \sigma_1(z'), \ldots, \sigma_m(z') \) are the symmetric functions of \( \lambda_1(z'), \ldots, \lambda_m(z') \), then each \( \sigma_i \) is holomorphic for \( |z_j| < \epsilon' \), \( j \leq n-1 \) and \( \sigma_i(0) = 0 \) for \( 1 \leq i \leq m \). Let
\[ g = z_n^m - \sigma_1(z') z_n^{m-1} + \cdots + (-1)^m \sigma_m(z') \]
which is a Weierstrass polynomial.

It is clear that the function \( \frac{f}{g} \) is well-defined and holomorphic in
\[ \{ z \mid |z_j| < \epsilon' \text{ for } j \leq n-1, \text{ } |z_n| < \epsilon_n \} \setminus \{ g = 0 \}. \]
For every \( z' \), \( \frac{f(z', -)}{g(z', -)} \) extends to a holomorphic function of \( z_n \) for \( |z_n| < \epsilon_n \).
Exercise. Check that therefore $h = \frac{f}{g}$ is in fact holomorphic in a neighborhood of 0 and $h(0) \neq 0$.

This proves existence.

Uniqueness is straightforward. If $f = g' \cdot h'$ as in the theorem and $g' = z_n^{d'} + \cdots$, we see that $f(0, \ldots, 0, z_n) = z_n^{d'} \cdot h(0, \ldots, 0, z_n)$ which implies that $d' = m$. For every $z'$, $f(z', -)$ has $d$ roots in $|z_n| < \epsilon_n$, so $g'(z', -)$ vanishes on these with the right multiplicities. For degree reasons, this implies that $g' = g$. □

References


