These are notes from Math 632: Algebraic geometry II taught by Professor Mircea Mustaţă in Winter 2018, \LaTeX ed by Aleksander Horawa (who is the only person responsible for any mistakes that may be found in them).

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http://www-personal.umich.edu/~ahorawa/index.html

If you find any typos or mistakes, please let me know at ahorawa@umich.edu.

The problem sets, homeworks, and official notes can be found on the course website:


This course is a continuation of Math 631: Algebraic Geometry I. We will assume the material of that course and use the results without specific references. For notes from the classes (similar to these), see:

http://www-personal.umich.edu/~ahorawa/math_631.pdf

and for the official lecture notes, see:


The focus of the previous part of the course was on algebraic varieties and it will continue this course. Algebraic varieties are closer to geometric intuition than schemes and understanding them well should make learning schemes later easy. The focus will be placed on sheaves, technical tools such as cohomology, and their applications.

Contents

1. Sheaves 2
   1.1. Quasicoherent and coherent sheaves on algebraic varieties 2
   1.2. Locally free sheaves 7
   1.3. Vector bundles 11
   1.4. Geometric constructions via sheaves 12
   1.5. Geometric vector bundles 16

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1. Sheaves

1.1. Quasicoherent and coherent sheaves on algebraic varieties. The object we will consider is a ringed space $(X, \mathcal{O}_X)$ where $X$ is an algebraic variety and $\mathcal{O}_X$ is the sheaf of regular functions on $X$.

Recall that $\mathcal{O}_X$-module is a sheaf $\mathcal{F}$ such that for any open subset $U \subseteq X$, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module and if $U \subseteq V$ is an open subset, then

$$\mathcal{F}(V) \to \mathcal{F}(U)$$

is a morphism of $\mathcal{O}_X(V)$-modules, where the $\mathcal{O}_X(V)$-module structure on $\mathcal{F}(U)$ is given by the restriction map $\mathcal{O}_X(V) \to \mathcal{O}_X(U)$. 
Fact 1.1.1. If $X$ has a basis of open subsets $U$, closed under finite intersection, giving an $\mathcal{O}_X$-module on $X$ is equivalent to giving $\mathcal{O}_X(U)$-modules $\mathcal{F}(U)$ for any $U \in U$ with restriction maps between these which satisfy the sheaf axiom.

Example 1.1.2. If $X$ is an affine variety and $A = \mathcal{O}(X)$, $M$ is an $A$-module, then we obtain an $\mathcal{O}_X$-module $\tilde{M}$ such that for any $f \in A$, $\Gamma(D_X(f), \tilde{M}) = M_f$ with the restriction maps induced by localization. (The sheaf axiom was checked in Math 631.)

We get a functor
\[
\{A\text{-modules}\} \to \{\mathcal{O}_X\text{-modules}\}
\]
\[M \mapsto \tilde{M}.
\]

Definition 1.1.3. Suppose $X$ is affine. An $\mathcal{O}_X$-module $\mathcal{F}$ is quasicoherent (coherent) if $\mathcal{F} \cong \tilde{M}$ for some (finitely-generated) $A$-module $M$.

Example 1.1.4. The sheaf of regular functions $\mathcal{O}_X$ on $X$ is a coherent sheaf with $\mathcal{O}_X \cong \tilde{A}$.

If $V \subseteq X$ is irreducible and closed with $p = I_X(V)$,
\[
\tilde{M}_V = \lim_{\substack{\text{U open} \\ U \cap V \neq \emptyset}} \Gamma(U, \tilde{M}) = \lim_{\substack{f \not\in p}} M_f = M_p.
\]

Remarks 1.1.5.

(1) Given any $\mathcal{O}_X$-module $\mathcal{M}$, we have a canonical morphism of $\mathcal{O}_X$-modules:
\[
\Phi_M : \Gamma(X, \mathcal{M}) \to \mathcal{M}
\]
given on $D_X(f)$ by the unique morphism of $A_f$-modules
\[
\Gamma(X, \mathcal{M})_f \to \Gamma(D_X(f), \mathcal{M})
\]
induced by the restriction map.

Then the following are equivalent:

- $\mathcal{M}$ is quasicoherent,
- $\Phi_{\mathcal{M}}$ is an isomorphism,
- for any $f \in A$, the canonical map
  \[
  \Gamma(X, \mathcal{M})_f \to \Gamma(D_X(f), \mathcal{M})
  \]
  is an isomorphism.

(2) If $\mathcal{M}$ is quasicoherent (coherent) on $X$, then $\mathcal{M}|_{D_X(f)}$ is quasicoherent (coherent) on $D_X(f)$ for any $f \in A$.

The following proposition shows that for affine varieties all the operations on modules have natural analogues in $\mathcal{O}_X$-modules.

Proposition 1.1.6.

(1) If $M_1, \ldots, M_n$ are $A$-modules, then
\[
\bigoplus_{i=1}^n \tilde{M}_i \cong \bigoplus_{i=1}^n M_i.
\]
The functor $M \mapsto \tilde{M}$ is exact.

(3) Given a morphism $\varphi: \tilde{M} \to \tilde{N}$ of $\mathcal{O}_X$-modules and $u = \varphi_X: M \to N$ induced by $\varphi$ on global sections, we have that $\varphi = \tilde{u}$.

(4) If $\varphi: \tilde{M} \to \tilde{N}$ is a morphism of quasicoherent (coherent) sheaves, then $\ker(\varphi)$, $\operatorname{coker}(\varphi)$, $\operatorname{im}(\varphi)$ are quasicoherent (coherent).

(5) The functor $M \mapsto \tilde{M}$ gives an equivalence of categories between $A$-modules and quasicoherent sheaves on $X$ with the inverse given by $\Gamma(X, -)$.

(6) If $M, N$ are $A$-modules, then $\tilde{M} \otimes_A N \cong \tilde{M} \otimes_{\mathcal{O}_X} \tilde{N}$.

(7) If $M$ is an $A$-module, then

\[ \tilde{S}^p(M) \cong S^p(\tilde{M}), \]
\[ \bigwedge^p M \cong \bigwedge^p (\tilde{M}). \]

(8) If $M, N$ are $A$-modules, then

\[ \tilde{\operatorname{Hom}}_A(M, N) \cong \mathcal{H}om_{\mathcal{O}_X}(\tilde{M}, \tilde{N}) \]

if $M$ is a finitely-generated $A$-module.

(9) Let $f: X \to Y$ be a morphism of affine varieties and $\varphi = f^\# : A = \mathcal{O}(Y) \to B = \mathcal{O}(X)$.

Then

\[ f_*(B\tilde{M}) \cong_A \tilde{M} \quad \text{if } M \text{ is a } B\text{-module} \]

and

\[ f^*(\tilde{N}) \cong B \otimes_A N \quad \text{if } N \text{ is an } A\text{-module}. \]

**Proof.** For (1), we have that

\[ \Gamma(X, \bigoplus_{i=1}^n \tilde{M}_i) \cong \bigoplus_{i=1}^n \Gamma(X, \tilde{M}_i) = \bigoplus_{i=1}^n M_i. \]

It is hence enough to show that $\bigoplus_{i=1}^n M_i$ is quasicoherent. We have that

\[ \Gamma \left( X, \bigoplus_{i=1}^n \tilde{M}_i \right)_f \xrightarrow{\cong} \Gamma \left( D_X(f), \bigoplus_{i=1}^n \tilde{M}_i \right) \]
\[ \bigoplus_{i=1}^n M_i \xrightarrow{\cong} \bigoplus_{i=1}^n M_i = \bigoplus_{i=1}^n \Gamma(D_X(f), \tilde{M}_i) \]

where the bottom arrow is an isomorphism since localization commutes with direct sums.
To prove (2), it is enough to show that if \( M' \to M \to M'' \) is an exact sequence, then
\[
\begin{array}{c}
\widetilde{M'}_x \to \widetilde{M}_x \to \widetilde{M''}_x \\
= M'_{m_x} \to M_{m_x} \to M''_{m_x}
\end{array}
\]
is exact for all \( x \in X \), where \( m_x \) is the maximal ideal corresponding to \( x \). This is clear since localization is an exact functor.

In (3), it is enough to check that the two maps agree on \( D_X(f) \). By definition, we have a commutative diagram
\[
\begin{array}{ccc}
M & \xrightarrow{u=\varphi_X} & N \\
\downarrow & & \downarrow \\
M_f & \xrightarrow{\varphi_{D_X(f)}} & N_f
\end{array}
\]
and \( \varphi_{D_X(f)} \) is a morphism of \( A_f \)-modules. Then \( \varphi_{D_X(f)} \) is induced by \( u \) by passing to the localization, which completes the proof.

For (4), we know by (3) that \( \varphi = \tilde{u} \) for some \( u: M \to N \), so (2) shows that
\[
\ker(\varphi) = \widetilde{\ker(u)}.
\]
Similarly for \( \text{coker}(\varphi) \) and \( \text{im}(\varphi) \). The assertion about coherence follows from a more general fact: if \( \mathcal{M} \) is coherent, then any quasicoherent subsheaf or quotient sheaf of \( \mathcal{M} \) is coherent (to show this, use the fact that \( A \) is Noetherian).

In (5), we already know that we have a functorial isomorphism \( \Gamma(X, \widetilde{M}) \cong M \). Then
\[
\text{Hom}_A(M, N) \to \text{Hom}_{\mathcal{O}_X-\text{mod}}(\widetilde{M}, \widetilde{N})
\]
is injective, and we saw in (3) that it is surjective, so it is bijective. The result then follows.

We show (6). By definition of \( \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \), we have a canonical map
\[
M \otimes N \to \Gamma(X, \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}).
\]
Therefore, we get maps
\[
\widetilde{M} \otimes \widetilde{N} \to \Gamma(X, \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}) \xrightarrow{\Phi_{\widetilde{M} \otimes \widetilde{N}}} \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}
\]
where \( \Phi_{\widetilde{M} \otimes \widetilde{N}} \) is given by Remark 1.1.5 (1). It is now enough to show that it is an isomorphism by checking it on stalks: if \( x \in X \) corresponds to \( m \subseteq A \), then
\[
(M \otimes_A N)_x = (M \otimes_A N)_m = (M \otimes_A N) \otimes_A A_m,
\]
\[
(\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N})_x = \widetilde{M}_x \otimes_{\mathcal{O}_X,x} \widetilde{N}_x = M_m \otimes_{A_m} N_m,
\]
and we have that
\[
M \otimes_A N \otimes_A A_m \cong M \otimes_A N_m \cong M \otimes_A A_m \otimes_A N \cong M_m \otimes_{A_m} N_m.
\]
Part (7) follows by a similar argument. Part (8) is given as a homework problem.

In part (9), note that
\[
f^*(\widetilde{N}) = f^{-1}(\widetilde{N}) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X
\]
and proving the assertion about $f^*$ is hence similar to $\otimes$ and left as an exercise. For $f_*$, if $a \in A$, we have

$$\Gamma(D_Y(a), f_*(\widetilde{M})) = \Gamma(f^{-1}(D_Y(a)), \widetilde{M}) = M_{\varphi(a)} = M_a.$$  

This shows the isomorphism for $f_*$.

So far, we have assumed that $X$ is finite. The next goal is to globalize the definitions and results to the general case. Thus, let now $X$ be any algebraic variety.

**Proposition 1.1.7.** If $\mathcal{F}$ is an $\mathcal{O}_X$-module, the following are equivalent:

1. For any affine open $U \subseteq X$, $\mathcal{F}|_U$ is quasicoherent,
2. For any affine open $U \subseteq X$ and every $f \in \mathcal{O}_X(U)$, the canonical map $\mathcal{F}(U)_f \to \mathcal{F}(D_U(f))$ is an isomorphism,
3. There is an affine open cover $X = U_1 \cup \cdots \cup U_n$ such that $\mathcal{F}|_{U_i}$ is quasicoherent for all $i$.

Moreover (1) is equivalent to (3) if we replace “quasicoherent” by “coherent”.

**Proof.** Note that (1) and (2) are clearly equivalent, and (1) trivially implies (3). We only need to show that (3) implies (1). Choose an affine open subset $U \subseteq X$. We know that the restriction of $\mathcal{F}$ to any principal affine open subset of some $U_i$ is quasicoherent. We showed in Math 631 that we can cover $U$ by open subsets that are principal with respect to both $U$ and some $U_i$.

It is hence enough to show that if $X$ is an affine variety, $\mathcal{F}$ is an $\mathcal{O}_X$-module, and $X = \bigcup_{i=1}^r D_X(f_i)$, where each $\mathcal{F}|_{D_X(f_i)}$ is quasicoherent, then $\mathcal{F}$ is quasicoherent. Consider $a \in A = \mathcal{O}(X)$. We have the commutative diagram with exact rows

$$
\begin{array}{ccc}
0 & \to & \mathcal{F}(X)_a \\
\downarrow & & \downarrow \cong \\
\prod_i \mathcal{F}(D_X(f_i))_a & \to & \prod_{i,j} \mathcal{F}(D_X(f_if_j))_a \\
\downarrow \cong & & \downarrow \\
0 & \to & \mathcal{F}(D_X(a)) \\
\end{array}
$$

Since the rows are exact and the second and third vertical maps are isomorphism, the first map is an isomorphism (we can add another two zeros on the left and apply the Five Lemma).

Finally, in the coherent case, it is enough to show that if $X = D_X(f_1) \cup \cdots \cup D_X(f_m)$ is affine, $M$ is an $A$-module for $A = \mathcal{O}(X)$, and $M_{f_i}$ is a finitely-generated $A_{f_i}$-module for all $i$, then $A_M$ is finitely-generated. This was already proved in Math 631, so we leave it as an exercise here. \qed
Definition 1.1.8. An $\mathcal{O}_X$-module $\mathcal{F}$ is quasi-coherent (coherent) if it satisfies the equivalent properties in Proposition 1.1.7 (for coherent: replace “quasi-coherent” by “coherent” in (1) and (3)).

The categories $\text{Qcoh}(X)$, $\text{Coh}(X)$ are closed under:

- finite direct sums,
- $\ker$, $\text{coker}$, $\text{im}$ (so they are abelian categories),
- tensor products, symmetric powers, exterior powers,
- if $\mathcal{F}, \mathcal{G}$ are quasi-coherent, $\mathcal{F}$ is coherent, then $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is quasi-coherent, and if $\mathcal{G}$ is coherent then $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is also coherent (this is a homework problem).

Proposition 1.1.9. For a morphism of algebraic varieties $f : X \to Y$:

1. if $\mathcal{F}$ is quasi-coherent (coherent) on $Y$, then $f^*(\mathcal{F})$ is quasi-coherent (coherent) on $X$,
2. if $\mathcal{G}$ is quasi-coherent on $X$, then $f_!\mathcal{G}$ is quasi-coherent on $Y$ (this is not true for general coherent sheaves, but it is true when $f$ is a finite morphism$^1$).

Proof. For (1), choose for any $x \in X$, affine open neighborhood $V$ of $f(x)$, affine open neighborhood $U \subseteq f^{-1}(V)$ of $x$. We then have

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\uparrow & & \uparrow \\
U & \xrightarrow{g} & V
\end{array}
$$

and hence

$$f^*(\mathcal{F})|_U \cong g^*(\mathcal{F}|_V).$$

Thus, we are done by the affine case.

Part (2) was discussed during the problem session and we include it here for completeness. \qed

1.2. Locally free sheaves. Let $X$ be an algebraic variety. Recall that an $\mathcal{O}_X$-module $\mathcal{F}$ is locally free if there exists an open cover $X = \bigcup_i U_i$ such that $\mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus r_i}$. If all $r_i$ are equal to $r$, then $\mathcal{F}$ is locally free of rank $r$.

Note that:

- every such $\mathcal{F}$ is coherent,
- if $X$ is connected, every locally free sheaf has well-defined rank.

Proposition 1.2.1. Suppose $X$ is affine and $\mathcal{O}(X) = A$. If $A\mathcal{M}$ if finitely-generated, then the following conditions are equivalent:

1. $\widetilde{\mathcal{M}}$ is locally free,
2. for any prime ideal $p$, $M_p$ is free over $A_p$,
3. for any maximal ideal $m$, $M_m$ is free over $A_m$.

$^1$In fact, this even holds when $f$ is proper. This will be proved later in the class.
Recall that \( M \) is projective if \( \text{Hom}(M, -) \) is exact.

**Remark 1.2.2.** If \( _AM \) is finitely-generated. Then there exists a free finitely generated module \( F \) with a surjective map \( F \to M \). Then \( M \) is projective if and only if there is a splitting \( M \to F \), which is equivalent to saying that \( M \) is a direct summand of a free module.

**Proof.** We first show that \( _AM \) is projective if and only if \( M_p \) is projective over \( A_p \) for all prime (maximal) ideals \( p \). Indeed, choose a free finitely generated module \( F \) with \( F \to M \). Then \( M \) is projective if and only if \( \text{Hom}(M, F) \to \text{Hom}(M, M) \) is surjective. Since \( \text{Hom} \) commutes with localization and a morphism is surjective if and only if it is surjective after localizing at every prime (maximal) ideal.

We now show that if \( _RM \) is finitely generated and projective and \( (R, \mathfrak{m}) \) is local, then \( _RM \) is free. Choose a minimal set \( u_1, \ldots, u_n \) of generators of \( M \) and consider the exact sequence

\[
0 \longrightarrow N \longrightarrow R^n \longrightarrow M \longrightarrow 0
\]

This exact sequence is split since \( M \) is projective. Since the sequence is split, tensoring with \( R/\mathfrak{m} \) gives an exact sequence (since tensor products commute with direct sums)

\[
0 \longrightarrow N/\mathfrak{m}N \longrightarrow (R/\mathfrak{m})^n \longrightarrow M/\mathfrak{m}M \longrightarrow 0
\]

Hence \( N/\mathfrak{m}N = 0 \). By Nakayama’s Lemma, this shows that \( N = 0 \), so \( M \cong R^n \).

Altogether, we have shown that (2), (3), and (4) are all equivalent. To see that (1) implies (3), note that if \( \tilde{M} \) is locally free, then for any \( x \in X \) corresponding to the maximal ideal \( \mathfrak{m} \):

\[
M_\mathfrak{m} = \tilde{M}_x \cong O_{X,x}^{\oplus r} = A_\mathfrak{m}^{\oplus r}
\]

for some \( r \).

Let us now show that (3) implies (1). Fix \( x \in X \) corresponding to \( \mathfrak{m} \subset A \). We know that \( M_\mathfrak{m} \) is free of rank \( r \), so we may choose a basis \( u_1, \ldots, u_r \). Then the map

\[
\varphi: A^{\oplus r} \to M
\]

\[
e_i \mapsto u_i
\]

becomes an isomorphism after localizing at \( \mathfrak{m} \), i.e. \( (\ker \varphi)_\mathfrak{m} = 0 = (\text{coker} \varphi)_\mathfrak{m} \). Since \( \ker \varphi \) and \( \text{coker} \varphi \) are finitely generated over \( A \), there exists \( f \) such that \( (\ker \varphi)_f = 0 = (\text{coker} \varphi)_f \). Therefore, \( \varphi \otimes_A A_f \) is an isomorphism, and hence

\[
\tilde{M}_{D_X(f)} \cong O_{D_X(f)}^{\oplus r}.
\]

Since \( D_X(f) \) is a neighborhood of \( x \), this completes the proof. \( \square \)
Definition 1.2.3. Given a coherent sheaf $\mathcal{F}$, the fiber of $\mathcal{F}$ at $x \in X$ is
\[ \mathcal{F}(x) := \mathcal{F}_x / m_x \mathcal{F}_x \]
where $m_x \subseteq \mathcal{O}_{X,x}$ is the maximal ideal.

Note that

1. for $i : \{x\} \hookrightarrow X$ for $x \in X$, we have that
\[ \mathcal{F}(x) \cong i^* \mathcal{F}, \]
   since for a maximal ideal $\mathfrak{m} \subseteq A$, we have that
\[ M_{\mathfrak{m}} / \mathfrak{m} M_{\mathfrak{m}} \cong M / \mathfrak{m} M, \]

2. by Nakayama’s Lemma, we have that $\dim_k \mathcal{F}(x)$ is the minimal number of generators of $\mathcal{F}_x$.

Proposition 1.2.4. A coherent sheaf $\mathcal{F}$ is locally free of rank $r$ if and only if $\dim_k \mathcal{F}(x) = r$ for all $x \in X$.

Proof. The ‘only if’ implication is clear: if $\mathcal{F}$ is locally free of rank $r$, then $\mathcal{F}_x \cong \mathcal{O}_{X,x}^{\oplus r}$, and hence
\[ \mathcal{F}(x) \cong k^{\oplus r}. \]

By choosing an affine open neighborhood around each point, we may assume that $X$ is affine, $A = \mathcal{O}(X)$, and $\mathcal{F} = \tilde{M}$ for a finitely-generated module $M$. Fix $x \in X$ corresponding to the maximal ideal $\mathfrak{m} \subseteq A$. Then $M_{\mathfrak{m}}$ has the minimal number of generators equal to $r$. Choose a morphism
\[ \varphi : A^{\oplus r} \to M \]
which becomes surjective after localizing at $\mathfrak{m}$. Replacing $A$ by $A_f$ for some $f \notin \mathfrak{m}$ (i.e. replacing the affine neighborhood of $x$ by a smaller one), we may assume that $\varphi$ is surjective. Consider the short exact sequence
\[ 0 \longrightarrow N \longrightarrow A^{\oplus r} \longrightarrow M \longrightarrow 0. \]

For every maximal ideal $\mathfrak{n}$, the minimal number of generators $M_{\mathfrak{n}}$ is still $\mathfrak{n}$. Then, localizing the exact sequence at $\mathfrak{n}$, we get an exact sequence
\[ 0 \longrightarrow N_{\mathfrak{n}} \longrightarrow A_{\mathfrak{n}}^{\oplus r} \longrightarrow M_{\mathfrak{n}} \longrightarrow 0 \]
and tensoring with $A_{\mathfrak{n}} / \mathfrak{n} A_{\mathfrak{n}}$, we obtain
\[ N_{\mathfrak{n}} / \mathfrak{n} N_{\mathfrak{n}} \longrightarrow (A_{\mathfrak{n}} / \mathfrak{n} A_{\mathfrak{n}})^{\oplus r} \longrightarrow M_{\mathfrak{n}} / \mathfrak{n} M_{\mathfrak{n}} \longrightarrow 0. \]

Therefore, $N_{\mathfrak{n}} \subseteq \mathfrak{n} \cdot A_{\mathfrak{n}}^{\oplus r}$, so $N \subseteq \mathfrak{n} \cdot A^{\oplus r}$. Hence
\[ N \subseteq \left( \bigcap_{\mathfrak{n} \in \text{MaxSpec}(A)} \mathfrak{n} \right)^{\oplus r} = 0, \]
showing that $M \cong A^\oplus r$ is free.

**Proposition 1.2.5.** Given a short exact sequence

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

with $\mathcal{F}''$ locally free, for every $\mathcal{O}_X$-module $\mathcal{G}$ the sequence

$$0 \longrightarrow \mathcal{F}' \otimes_{\mathcal{O}_X} \mathcal{G} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \longrightarrow \mathcal{F}'' \otimes_{\mathcal{O}_X} \mathcal{G} \longrightarrow 0$$

is exact. In particular, for any $x \in X$, we have an exact sequence

$$0 \longrightarrow \mathcal{F}'(x) \longrightarrow \mathcal{F}(x) \longrightarrow \mathcal{F}''(x) \longrightarrow 0$$

**Proof.** For the first assertion, take stalks and use the fact that $\mathcal{F}' \otimes_{\mathcal{O}_x} \mathcal{G}$ is split exact (since $\mathcal{F}'' \otimes_{\mathcal{O}_x} \mathcal{G}$ is free), and hence tensoring preserves exactness.

The second assertion follows by taking $\mathcal{G} = k(x)$ where

$$k(x)(U) = \begin{cases} U & \text{if } x \in U, \\ 0 & \text{otherwise.} \end{cases}$$

Note that for $i: \{x\} \hookrightarrow X$, $k(x) = i_* \mathcal{O}_{\{x\}}$. □

**Corollary 1.2.6.** If $0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$ is exact with $\mathcal{F}''$ locally free, then $\mathcal{F}'$ is locally free if and only if $\mathcal{F}$ is locally free. If two of the above have well-defined rank, then so does the third, and

$$\text{rank}(\mathcal{F}) = \text{rank}(\mathcal{F}') + \text{rank}(\mathcal{F}'').$$

**Proof.** Work on the connected components of $X$ and apply Propositions 1.2.4 and 1.2.5. □

The following operations preserve locally free sheaves:

1. if $\mathcal{M}_1, \ldots, \mathcal{M}_n$ are locally free (with $\text{rank}(\mathcal{M}_i) = r_i$), then $\mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_n$ is locally free (with equal to $r_1 + \cdots + r_n$),
2. if $\mathcal{E}$, $\mathcal{F}$ are locally free, then $\mathcal{E} \otimes \mathcal{F}$ is locally free,
3. if $\mathcal{E}$ is locally free, then $S^p(\mathcal{E})$ and $\wedge^p(\mathcal{E})$ are locally free.

**Definition 1.2.7.** If $\mathcal{E}$ is a coherent sheaf, then the coherent sheaf

$$\mathcal{E}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$$

is the **dual of** $\mathcal{E}$.

The assignment $\mathcal{E} \rightarrow \mathcal{E}^\vee$ is a contravariant functor.

**Proposition 1.2.8.**
(1) For every coherent sheaves $\mathcal{E}, \mathcal{F}$, there exists a morphism
$$\mathcal{E}^\vee \otimes \mathcal{F} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$$
and it is an isomorphism if one of $\mathcal{E}, \mathcal{F}$ is locally free.
(2) If $\mathcal{E}$ and $\mathcal{F}$ are locally free coherent sheaves, then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$ is locally free.
(3) For every coherent sheaf $\mathcal{E}$, there is a canonical morphism $\mathcal{E} \to (\mathcal{E}^\vee)^\vee$ which is an isomorphism if $\mathcal{E}$ is locally free.
(4) For every coherent sheaf $\mathcal{E}$, there is a canonical morphism
$$\mathcal{O}_X \to \mathcal{H}om(\mathcal{E}, \mathcal{E})$$
which is an isomorphism if $\mathcal{E}$ is locally free of rank 1.

Proof. The proof of this proposition is left as an exercise. \(\square\)

1.3. Vector bundles. We will use the terminology:
- vector bundle for a locally free sheaf,
- line bundle for a locally free sheaf of rank 1,
- trivial vector bundle of rank $r$ for a sheaf isomorphic to $\mathcal{O}_X^\oplus r$.

Definition 1.3.1. The Picard group of $X$, denoted $\text{Pic}(X)$ is the set of isomorphism classes of line bundles on $X$ with the operation
$$(\mathcal{L}, \mathcal{M}) \mapsto \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M}$$
(which is clearly associative and commutative) with identity given by $\mathcal{O}_X$ and the inverse given by $\mathcal{L}^\vee$.

For now, we do not have the necessary tools to compute any nontrivial Picard groups, but we will get to this later, when we discuss divisors.

Remark 1.3.2. If $f: X \to Y$ is a morphism and $\mathcal{E}$ is locally free on $Y$, then $f^* \mathcal{E}$ is locally free on $X$.

In particular, we get a morphism of abelian groups
$$f^*: \text{Pic}(Y) \longrightarrow \text{Pic}(X)$$
$$\mathcal{L} \longmapsto f^* \mathcal{L}$$

Our next goal will be to provide some geometric intuition for vector bundles. We first discuss the description of locally free sheaves in terms of transition maps.

Let $X$ be an algebraic variety and $\mathcal{E}$ a vector bundle of rank $r$. Then there is an open cover
$$X = \bigcup_{i \in I} U_i$$
such that we have trivialization maps
$$\varphi_i: \mathcal{E}|_{U_i} \cong \mathcal{O}_{U_i}^\oplus r.$$
Given $i, j$, consider the transition maps
\[ \varphi_{i,j} = \varphi_i \circ \varphi_j^{-1} : \mathcal{O}_{U_i \cap U_j} \to \mathcal{O}_{U_i \cap U_j} \]
satisfying the cocycle condition:
\[ \varphi_{i,i} = \text{id} \text{ for all } i, \]
\[ \varphi_{i,j} \circ \varphi_{j,k} = \varphi_{i,k} \text{ on } U_i \cap U_j \cap U_k \text{ for all } i, j, k. \]

Note that an isomorphism
\[ \mathcal{O}_{U_i \cap U_j} \to \mathcal{O}_{U_i \cap U_j} \]
is given by an invertible matrix $M \in M_r(\mathcal{O}_X(U_i \cap U_j))$.

**Exercise.** Given an open cover
\[ X = \bigcup_{i \in I} U_i \]
and invertible matrices $(a_{i,j})_{i,j \in I}$, $a_{i,j} \in M_r(\mathcal{O}_X(U_i \cap U_j))$ satisfying the cocycle condition, there is a locally free sheaf $\mathcal{E}$, unique up to canonical isomorphism, that has this as the family of transition maps.

One can say precisely when the vector bundles corresponding to two covers and transition maps are isomorphic.

**Example 1.3.3.** Suppose $r = 1$. Consider the sheaf $\mathcal{O}_X^*$ of abelian groups given by
\[ \mathcal{O}_X^*(U) = \mathcal{O}_X(U)^* = \{ \text{invertible elements of } \mathcal{O}_X(U) \}. \]

A line bundle $\mathcal{L}$ on $X$ is described by an open cover $\mathcal{U} = (U_i)_{i \in I}$ and elements $(a_{i,j} \in \mathcal{O}_X^*(U_i \cap U_j))_{i,j \in I}$ that satisfy the cocycle condition.

Suppose $\mathcal{L}$ corresponds to the family $(a_{i,j})$. Then $\mathcal{L} \otimes m$ corresponds to $(a_{i,j}^m)$ and $\mathcal{L}^\vee$ corresponds to $(a_{i,j}^{-1})$.

If $\mathcal{L}$ is given with respect to $\mathcal{U} = (U_i)_{i \in I}$ by transition maps $(a_{i,j} \in \mathcal{O}_X^*(U_i \cap U_j))$, then
\[ \Gamma(X, \mathcal{L}) \cong \left\{ (f_i)_{i \in I} \in \prod_{i \in I} \mathcal{O}_X(U_i) \mid f_i = a_{i,j} f_j \text{ on } U_i \cap U_j \text{ for all } i, j \in I \right\}. \]

Proving this is left as an exercise.

### 1.4. Geometric constructions via sheaves.

The language of sheaves is used to deal with global objects which are “locally nice” (for example, manifolds, affine varieties etc). In this section, we will discuss how one can use sheaves for certain geometric constructions.

**Definition 1.4.1.** Let $X$ be an algebraic variety. An $\mathcal{O}_X$-algebra is a sheaf of commutative rings $\mathcal{A}$ on $X$ with a morphism of sheaves of rings $\mathcal{O}_X \to \mathcal{A}$. A morphism of $\mathcal{O}_X$-algebras is defined in the obvious way.

Note that every $\mathcal{A}$-module becomes an $\mathcal{O}_X$-module. In particular, $\mathcal{A}$ is an $\mathcal{O}_X$-module.

We will be interested in cases when:

- $\mathcal{A}$ is quasicoherent,
• \( \mathcal{A} \) is reduced: for any open affine \( U \subseteq X, \mathcal{A}(U) \) is reduced,
• \( \mathcal{A} \) is of finite type over \( \mathcal{O}_X \): for any open affine \( U \subseteq X, \mathcal{A}(U) \) is a finitely generated \( \mathcal{O}_X(U) \)-algebra (or equivalently, \( \mathcal{A}(U) \) is a finitely-generated \( k \)-algebra).

**Example 1.4.2.** Let \( \mathcal{F} \) be a quasicoherent sheaf. Then set

\[
S^\bullet(\mathcal{F}) := \bigoplus_{m \geq 0} S^m(\mathcal{F}).
\]

Given an open subset \( X \supseteq U \):

\[
S^i(\mathcal{F}(U)) \otimes_{\mathcal{O}_X(U)} S^j(\mathcal{F}(U)) \to S^{i+j}(\mathcal{F}(U))
\]

and passing to the associated sheaves, we get a map

\[
S^i(\mathcal{F}) \otimes S^j(\mathcal{F}) \to S^{i+j}(\mathcal{F})
\]

making \( S^\bullet(\mathcal{F}) \) and \( \mathcal{O}_X \)-algebra.

Note that:

1. \( S^\bullet(\mathcal{F}) \) is quasicoherent,
2. if \( \mathcal{F} \) is coherent, then \( S^\bullet(\mathcal{F}) \) is a finite type \( \mathcal{O}_X \)-algebra.

The \( \mathcal{O}_X \)-algebra \( S^\bullet(\mathcal{F}) \) has the following universal property: if \( A \) is any \( \mathcal{O}_X \)-algebra:

\[
\text{Hom}_{\mathcal{O}_X\text{-alg}}(S^\bullet(\mathcal{F}), A) \cong \text{Hom}_{\mathcal{O}_X\text{-alg}}(\mathcal{F}, A).
\]

Suppose \( f: Y \to X \) is any morphism. Then \( \mathcal{O}_X \to f_*\mathcal{O}_Y \) makes \( f_*\mathcal{O}_Y \) an \( \mathcal{O}_X \)-algebra. It is:

- quasicoherent,
- reduced,
- a finitely-generated \( \mathcal{O}_X \)-algebra if \( f \) is affine: for an affine open subset \( U \subseteq X, f^{-1}(U) \) is affine and

\[
\Gamma(U, f_*\mathcal{O}_Y) \cong \Gamma(f^{-1}(U), \mathcal{O}_Y)
\]

is a finitely-generated \( k \)-algebra.

Suppose \( g: Z \to X \) is another variety over \( X \) and we have a morphism \( h: Z \to Y \) over \( X \), i.e. the diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{h} & Y \\
\downarrow{g} & & \downarrow{f} \\
X & \xleftarrow{f} &
\end{array}
\]

commutes. We then have the map

\[
\mathcal{O}_Y \to h_*\mathcal{O}_Z,
\]

and hence a morphism of \( \mathcal{O}_X \)-algebras:

\[
f_*\mathcal{O}_Y \to f_*h_*\mathcal{O}_Z = g_*\mathcal{O}_Z.
\]

Altogether, this shows that there is a contravariant functor:
The goal is to construct an inverse functor. Let $\mathcal{A}$ be a quasicoherent, finitely generated, reduced $\mathcal{O}_X$-algebra. Given an affine open subset $U \subseteq X$, consider $\mathcal{A}(U)$. We then have affine varieties over $U$:

$$Y_U = \text{MaxSpec}(\mathcal{A}(U)) \xrightarrow{\pi_U} U$$

with the map induced by $\mathcal{O}_X(U) \to \mathcal{A}(U)$. We claim that these can be glued together into a variety. In particular, we will show that if $V \subseteq U$ is an affine open subset, the commutative diagram

$$\begin{array}{ccc}
Y_V & \longrightarrow & Y_U \\
\downarrow{\pi_V} & & \downarrow{\pi_U} \\
V & \leftarrow & U
\end{array}$$

is Cartesian (where the top map is induced by $\mathcal{A}(U) \to \mathcal{A}(V)$), i.e.

$$\alpha: Y_V \to \pi_V^{-1}(V)$$

is an isomorphism.

This is clear if $V$ is a principal affine open subset in $U$, since for $V = D_U(\varphi)$, $\mathcal{A}(V) = \mathcal{A}(U)_{\varphi}$. In the general case, write $V = V_1 \cup \cdots \cup V_r$ for principal affine open subsets $V_i \subseteq U$. Then

$$Y_{V_i} = \alpha^{-1}(\pi_U^{-1}(V_i)) \to \pi_U^{-1}(V_i)$$

is an isomorphism, so $\alpha$ is an isomorphism.

Given any two affine open subsets $U, V \subseteq X$, we get an isomorphism

$$\pi_U^{-1}(U \cap V) \cong Y_{U \cap V} \cong \pi_V^{-1}(U \cap V).$$

Therefore, we can glue $Y_U = \text{MaxSpec}(\mathcal{A}(U))$ along these isomorphisms to get

$$\text{MaxSpec}(\mathcal{A}).$$

Gluing the $\pi_U$, we get a map

$$\pi_X: \text{MaxSpec}(\mathcal{A}) \to X$$

such that for any open affine subset $U \subseteq X$, we have

$$\pi_X^{-1}(U) \cong \text{MaxSpec}(\mathcal{A}(U)).$$

Then the map $\pi_X$ is affine and there is a canonical isomorphism $(\pi_X)_* \mathcal{O}_Y \cong \mathcal{A}$. Moreover, this is functorial (which can be proved in similar fashion).

**Exercise.** The functor $A \mapsto \text{MaxSpec}(\mathcal{A})$ gives an inverse to the functor

$$\Phi: (Y \xrightarrow{\pi} X) \mapsto \pi_* \mathcal{O}_Y.$$ 

We still need to check separatedness of the variety $\text{MaxSpec}(\mathcal{A})$. We first make a definition.

**Definition 1.4.3.** If $f: X \to Y$ is a morphism of prevarieties, $f$ is *separated* if for
we have that $\Delta(X)$ is closed in $X \times_Y X$.

Note that if $Y$ is a point, this just means that $X$ is separated.

**Remark 1.4.4.** If $X$ is separated then any $f : X \to Y$ is separated.

**Remark 1.4.5.** If there is an open cover $Y = \bigcap_i V_i$ such that each $f^{-1}(V_i) \to V_i$ is separated, then $f$ is separated. Indeed,

$$\Delta(X) \subseteq \bigcup_{i \in I} f^{-1}(V_i) \times_{V_i} f^{-1}(V_i)$$

and the intersection of $\Delta(X)$ with $f^{-1}(V_i) \times_{V_i} f^{-1}(V_i)$ is $\Delta(f^{-1}(V_i))$, which is closed in $f^{-1}(V_i) \times_{V_i} f^{-1}(V_i)$.

In particular, if each $f^{-1}(V_i)$ is separated, then $f$ is separated.

In our case, this implies that

$$\pi : \mathcal{M}ax\text{Spec}(A) \xrightarrow{\pi_X} X$$

is separated, since for each affine subset $U \subseteq X$, we have that $\pi_X^{-1}(U)$ is affine, hence separated.

**Exercise.** A composition of separated morphisms is separated.

As a consequence, $\mathcal{M}ax\text{Spec}(A)$ is a variety, since the map

$$\mathcal{M}ax\text{Spec}(A) \to X \to \{\ast\}$$

is separated.

**Exercise.** Let $\pi : \mathcal{M}ax\text{Spec}(A) = Y \to X$. Given a map of varieties over $X$:

$$Z \xrightarrow{h} Y \xleftarrow{\pi} \mathcal{M}ax\text{Spec}(A)$$

we get a map $A = \pi_\ast \mathcal{O}_Y \to g_\ast \mathcal{O}_Z$. Show that the resulting morphism

$$\text{Hom}_{\text{Var}}(Z, \mathcal{M}ax\text{Spec}(A)) \to \text{Hom}_{\mathcal{O}_X}(A, g_\ast \mathcal{O}_Z)$$

is a bijection. Note that by the adjointness property:

$$\text{Hom}_{\mathcal{O}_X}(A, g_\ast \mathcal{O}_Z) = \text{Hom}_{\mathcal{O}_Z}(g^\ast A, \mathcal{O}_Z).$$

**Remark 1.4.6.** The map $\mathcal{M}ax\text{Spec}(A) \to X$ is finite if and only if $A$ is a coherent $\mathcal{O}_X$-module.
1.5. **Geometric vector bundles.** Let $X$ be a variety.

**Definition 1.5.1.** A *geometric vector bundle on $X$* is a morphism

\[ \pi : E \rightarrow X \]

such that for any $x \in X$, the fiber

\[ E(x) = \pi^{-1}(x) \]

has a $k$-vector space structure, which is *locally trivial*, i.e. there is an open cover $X = \bigcup U_i$ such that there is an isomorphism

\[
\begin{array}{ccc}
\pi^{-1}(U_i) & \xrightarrow{\alpha} & U_i \times k^{r_i} \\
\downarrow & & \downarrow \\
U_i & & 
\end{array}
\]

which is linear on each fiber.

If all $r_i$ are equal to $r$, we say that $E$ has *rank* $r$.

**Definition 1.5.2.** The category $\text{Vect}(X)$ of geometric vector bundles over $X$ is a category whose

- objects are geometric vector bundles,
- morphisms are morphisms of varieties over $X$ which are linear on the fibers.

Fix a geometric vector bundle $\pi : E \rightarrow X$. Define a sheaf $\mathcal{E}$ of sets on $X$ given by

\[ \mathcal{E}(U) = \{ \text{morphisms } s : U \rightarrow E \text{ such that } \pi \circ s = \text{id}_U \} \]

Defining:

\[(s_1 + s_2)(x) = s_1(x) + s_2(x),\]

\[fs(x) = f(x)s(x) \text{ for } f \in \mathcal{O}_X,\]

(both of these defined in $\pi^{-1}(x)$ which is a $k$-vector space) makes $\mathcal{E}$ into an $\mathcal{O}_X$-module. Indeed, suppose $X = \bigcup U_i$, $\pi^{-1}(U_i) \cong U_i \times k^{r_i}$. Then

\[ \mathcal{E}(U \cap U_i) \cong \mathcal{O}_X(U \cap U_i)^{\oplus r_i} \]

is an isomorphism $\mathcal{O}_X(U_i \cap U_i)$-modules. This shows that

- the operations are well-defined,
- $\mathcal{E}$ is an $\mathcal{O}_X$-module,
- $\mathcal{E}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus r_i}$.

Therefore, $\mathcal{E}$ is a locally free sheaf.

This assignment is actually functorial. Given a morphism

\[
\begin{array}{ccc}
E & \xrightarrow{f} & F \\
\downarrow & & \downarrow \\
X & & 
\end{array}
\]
In $\text{Vect}(X)$, if $\mathcal{E}$, $\mathcal{F}$ correspond to $E$, $F$, we get a map $\mathcal{E} \to \mathcal{F}$ given by $s: U \to E$ goes to $f \circ s$. We hence have a functor

$$\Phi: \left\{ \text{geometric vector bundles over } X \right\} \to \left\{ \text{full subcategory of locally free sheaves on } X \right\}$$

In the opposite direction, suppose $\mathcal{E}$ is a locally free sheaf on $X$, and define

$$V(\mathcal{E}) = \text{MaxSpec}(S^* (\mathcal{E})) \xrightarrow{\pi} X.$$ 

If $\mathcal{E}|_U \cong \mathcal{O}^{\oplus r}_U$, then

$$S^*(\mathcal{E})(U) \cong \mathcal{O}_U[t_1, \ldots, t_r],$$

hence $S^*(\mathcal{E})$ is a reduced algebra. We then have an isomorphism

$$\pi^{-1}(U) \xrightarrow{\cong} U \times k^r \xrightarrow{s} U$$

which gives a vector space structure on the fibers of $\pi$ (independent of the choice of $\mathcal{E}|_U \cong \mathcal{O}^{\oplus r}_U$). Therefore, $\pi$ is a geometric vector bundle on $X$.

Note that $\mathcal{E} \mapsto V(\mathcal{E})$ is a contravariant functor.

We want to work out what the sheaf of sections $\mathcal{F}$ of $V(\mathcal{E})$ is. Consider

$$\xymatrix{ V(\mathcal{E}) \ar[d]^\pi \ar@{-->}[dr]^s \ar[rd]^\cong & \ar@{<->}[rd] & U \ar[ld] & X \ar[l] }$$

Then

$$\mathcal{F}(U) = \text{Hom}_{\text{Var}_X}(U, \text{MaxSpec}(S^*(\mathcal{E}))) = \text{Hom}_{\mathcal{O}_U, \text{-alg}}(S^*(\mathcal{E})|_U, \mathcal{O}_U).$$

By the universal property of $S^*(\mathcal{E}|_U)$, this is isomorphic to

$$\text{Hom}_{\mathcal{O}_U, \text{-mod}}(\mathcal{E}|_U, \mathcal{O}_U) = \mathcal{E}^\vee(U).$$

Altogether, we see that the sheaf of section of $V(\mathcal{E})$ is isomorphic to $\mathcal{E}^\vee$.

It is now easy to see that the functor mapping $E$ to the sheaf of sections is an equivalence of categories with inverse $\mathcal{E} \mapsto V(\mathcal{E}^\vee)$.

We previously constructed $\text{MaxSpec}(A)$ given a locally free sheaf $A$. Next, we construct $\text{MaxProj}(S)$ for a graded $\mathcal{O}_X$-algebra $S$. Let $X$ be an algebraic variety and $S$ be an $\mathbb{N}$-graded $\mathcal{O}_X$-algebra: an $\mathcal{O}_X$-algebra together with a decomposition

$$S = \bigoplus_{m \geq 0} S_m$$

such that $S_i \cdot S_j \subseteq S_{i+j}$. We also assume that $S$ is quasi-coherent, reduced, and locally generated by $S_1$, which is a coherent sheaf, i.e. for any open subset $U \subseteq X$, $S(U)$ is generated as an $\mathcal{O}_X(U)$-algebra by $S_1(U)$ which is a finitely-generated $\mathcal{O}_X$-module.
Note that this condition implies that the map
\[ \mathcal{O}_X \to \mathcal{S} \to \mathcal{S}_0 \]
is surjective.

Given such an \( \mathcal{S} \), for every affine open subset \( U \subseteq X \), consider
\[
\begin{array}{c}
\text{MaxProj}(\mathcal{S}(U)) \\
\downarrow \quad \pi_U \\
\text{MaxSpec}(\mathcal{S}(U)) \\
\downarrow \\
U
\end{array}
\]
where \( \text{MaxSpec}(\mathcal{S}_0(U)) \to U \) is the closed immersion induced by \( \mathcal{O}_X(U) \to \mathcal{S}_0(U) \).

One checks that if \( V \subseteq U \) is an affine open subset, then the commutative diagram
\[
\begin{array}{ccc}
\text{MaxProj}(\mathcal{S}(V)) & \longrightarrow & \text{MaxProj}(\mathcal{S}(U)) \\
\downarrow & & \downarrow \\
V & \longleftarrow & U
\end{array}
\]
is Cartesian, so we can glue these together to get
\[ \pi: \text{MaxProj}(\mathcal{S}) \to X. \]

Note that if \( U \subseteq X \) is an affine open subset, then \( \text{MaxProj}(\mathcal{S}(U)) \) is separated, and hence \( \pi \) is a separated morphism and \( X \) is separated, so \( \text{MaxProj}(\mathcal{S}) \) is separated.

**Examples 1.5.3.**

1. **The blow-up of \( X \) along a coherent ideal.** If \( \mathcal{I} \subseteq \mathcal{O}_X \) is a coherent ideal sheaf, we define
\[ \mathcal{R}(\mathcal{I}) = \bigoplus_{m \geq 0} \mathcal{I}^m t^m \subseteq \mathcal{O}_X[t]. \]
The blow up of \( X \) along \( \mathcal{I} \) is \( \text{MaxProj}(\mathcal{R}(\mathcal{I})) \to X \).

2. **Projective bundles.** If \( \mathcal{E} \) is a locally free sheaf on \( X \), then consider
\[ \mathcal{S} = \bigoplus_{m \geq 0} \mathcal{S}^m(\mathcal{E}) \]
and define the projective bundle associated to \( \mathcal{E} \) to be
\[ \mathbb{P}(\mathcal{E}) := \text{MaxProj}(\mathcal{S}) \xrightarrow{\pi} X. \]

Note that if \( \mathcal{E}|_U \cong \mathcal{O}_U^{\oplus n} \) where \( U \) is affine, then
\[ \mathcal{S}|_U \cong \mathcal{O}_U[x_1, \ldots, x_n], \]
so \( \pi^{-1}(U) \cong U \times \mathbb{P}^{n-1} \).

**Definition 1.5.4.** A projective morphism \( f: Y \to X \) is a morphism such that there exists a commutative diagram.
It is clear that a projective morphism is proper.

1.6. The cotangent sheaf. We begin with the local case. Let $R$ be a commutative ring and $A$ be a commutative $R$-algebra.

**Definition 1.6.1.** If $M$ is an $A$-module, an $R$-derivation $D: A \rightarrow M$ is a map such that

1. $D$ is $R$-linear,
2. (Leibniz rule): $D(ab) = aD(b) + bD(a)$.

**Remark 1.6.2.** If $D$ satisfies (2), then (1) is satisfies if and only if $D$ is additive and $D = 0$ on $\text{im}(R \rightarrow A)$.

We denote set of $R$-derivations $A \rightarrow M$ by

$$\text{Der}_R(A, M) = \{R\text{-derivations } A \rightarrow M\} \subseteq \text{Hom}_R(A, M).$$

We note that this is an $A$-submodule.

Given $u \in \text{Hom}_A(M, N)$, we get a map

$$\text{Der}_R(A, M) \rightarrow \text{Der}_R(A, N),$$

$$D \mapsto u \circ D,$$

so $\text{Der}_R(A, -)$ is a covariant functor from $A$-mod to itself.

**Proposition 1.6.3.** The functor $\text{Der}_R(A, -)$ is representable, i.e. there exists an $A$-module $\Omega_{A/R}$ and an $R$-derivation $d = d_{A/R}: A \rightarrow \Omega_{A/R}$ that induces a bijection

$$\text{Hom}_A(\Omega_{A/R}, M) \rightarrow \text{Der}_R(A, M)$$

$$\varphi \mapsto \varphi \circ d.$$

**Proof.** Take $\Omega_{A/R}$ to be the quotient of the free $A$-module with basis $\{d(a) \mid a \in A\}$ by the following elements:

1. $d(a_1 + a_2) - d(a_1) - d(a_2)$ for $a_1, a_2 \in A$,
2. $d(ar) - rd(a)$ for $a \in A$, $r \in R$,
3. $d(ab) - ad(b) - bd(a)$ for $a, b \in A$,

and define $d_{A/R}(a)$ to be the image of $d(a)$ in the quotient. It is easy to check that this satisfies the required universal property. □

**Definition 1.6.4.** The module $\Omega_{A/R}$ defined above is called the module of Kähler differentials.
Remark 1.6.5. The construction implies that \( \{ d_{A/R}(a) \mid a \in A \} \) generate \( \Omega_{A/R} \). In fact, if \( (a_i)_{i \in I} \) generate \( A \) as an \( R \)-algebra, then \( \{ d_{A/R}(a_i) \mid i \in I \} \) generate \( \Omega_{A/R} \). This is because \( d(a_{i_1}, \ldots, a_{i_r}) \) lies in the linear span of \( d(a_{i_1}), \ldots, d(a_{i_r}) \) by the Leibniz rule.

In particular, if \( A \) is a finitely-generated \( R \)-algebra, then \( \Omega_{A/R} \) is finitely-generated.

Examples 1.6.6.

1. We have that \( \Omega_{R/R} = 0 \).

2. We know that \( \Omega_{R[x_1, \ldots, x_n]/R} \) is generated by \( dx_1, \ldots, dx_n \). We claim that these are linearly independent. Indeed, consider

\[
\partial_i : R[x_1, \ldots, x_n] \to R[x_1, \ldots, x_n]
\]

\[
f \mapsto \frac{\partial f}{\partial x_i}.
\]

This maps \( x_j \mapsto 0 \) for \( j \neq i \) and \( x_i \mapsto 1 \), so the corresponding morphism

\[
\Omega_{R[x_1, \ldots, x_n]/R} \to R[x_1, \ldots, x_n]
\]

is given by \( dx_j \mapsto 0 \) for \( j \neq i \) and \( dx_i \mapsto 1 \). This shows that \( dx_1, \ldots, dx_n \) are linearly independent.

Note that \( df = \bigoplus_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \).

Proposition 1.6.7. If \( S \subseteq A \) is a multiplicative system, then we have a canonical isomorphism

\[
\Omega_{S^{-1}A/R} \cong S^{-1} \Omega_{A/R}.
\]

Proof. If \( M \) is an \( S^{-1}A \)-module, we have a map

\[
\text{Der}_A(S^{-1}A, M) \to \text{Der}_R(A, M)
\]

induced by \( A \to S^{-1}A \). We claim that this is an isomorphism, i.e. given an \( R \)-derivation \( D : A \to M \), there is a unique extension \( \bar{D} : S^{-1}A \to M \). This is given by the quotient rule:

\[
\bar{D} \left( \frac{a}{s} \right) = \frac{1}{s} D(a) - \frac{a}{s^2} D(s).
\]

It is easy to check this is well-defined and gives a derivation.

This means that for any \( S^{-1}A \)-module \( M \),

\[
\text{Hom}_{S^{-1}A}(\Omega_{S^{-1}A/R}, M) \cong \text{Hom}_R(\Omega_{A/R}, M) \cong \text{Hom}_{S^{-1}A}(S^{-1} \Omega_A, M).
\]

This is functorial in \( M \), so it comes from an isomorphism \( \Omega_{S^{-1}A/R} \cong S^{-1} \Omega_{A/R} \). \qed

Proposition 1.6.8. Let \( A \) and \( B \) be finitely-generated \( B \) algebras. Then

\[
\Omega_{A \otimes_R B/R} \cong \Omega_{A/R} \otimes_R B.
\]
Proof. We have a series of functorial isomorphisms

\[ \text{Hom}_{A \otimes_R B}(\Omega_{A \otimes_R B/R}, M) \cong \text{Der}_B(A \otimes_R B, M) \]
\[ \cong \text{Der}_R(A, M) \quad \text{(induced by restriction)} \]
\[ \cong \text{Hom}_A(\Omega_{A/R}, M) \]
\[ \cong \text{Hom}_{A \otimes_R B}(\Omega_{A/R} \otimes_A (A \otimes_R B), M) \]
\[ \cong \text{Hom}_{A \otimes_R B}(\Omega_{A/R} \otimes_R B, M) \]

where last isomorphism is given by \( \Omega_{A/R} \otimes_R B \cong \Omega_{A/R} \otimes_A (A \otimes_R B) \).

Consider ring homomorphisms \( R \xrightarrow{\varphi} A \xrightarrow{\psi} B \).

**Proposition 1.6.9.** There exists an exact sequence

\[ \Omega_{A/R} \otimes_A B \xrightarrow{\alpha} \Omega_{B/R} \xrightarrow{\beta} \Omega_{B/A} \rightarrow 0 \]
\[ d_{A/R}(a) \otimes b \rightarrow b \cdot d_{B/R}(\psi(a)) \]
\[ d_{B/R}(b) \rightarrow d_{B/A}(b) \]

**Proof.** The proof is given as a homework exercise. \( \square \)

**Proposition 1.6.10.** Suppose \( \psi \) is a surjective map with \( \ker \psi = I \) (note that in this case, \( \Omega_{B/A} = 0 \)). Then the sequence

\[ I/I^2 \xrightarrow{\delta} \Omega_{A/R} \otimes_A B \xrightarrow{\alpha} \Omega_{B/R} \rightarrow 0 \]
\[ \bar{a} \rightarrow d_{A/R}(a) \otimes 1 \]

is exact.

**Proof.** The proof is given as a homework exercise. \( \square \)

**Remark 1.6.11.** Suppose \( B \) is a finitely-generated \( R \)-algebra. Choose generators \( b_1, \ldots, b_n \in B \), and suppose

\[ A = R[x_1, \ldots, x_n] \xrightarrow{\varphi} B \]
\[ x_i \mapsto b_i \]

has \( \ker \varphi = I \). Then by Proposition 1.6.10, we have an exact sequence

\[ I/I^2 \xrightarrow{\delta} \Omega_{A/R} \otimes_A B \xrightarrow{\alpha} \Omega_{B/R} \rightarrow 0 \]
\[ \bigoplus_{i=1}^n Bdx_i \]
and hence $\Omega_{B/R}$ is the quotient of the free module $\bigoplus_{i=1}^{n} Be_i$ by the relations:

$$f \in I \leadsto \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(b_1, \ldots, b_n)e_i.$$ 

**Remark 1.6.12.** If $A \to S^{-1}A \to B$, then by Propositions 1.6.9 and 1.6.7

$$\Omega_{S^{-1}A/A} \otimes B \longrightarrow \Omega_{B/A} \longrightarrow \Omega_{B/S^{-1}A} \longrightarrow 0$$

which shows that $\Omega_{B/A} \cong \Omega_{B/S^{-1}A}$.

Next, we want to define similar invariants associated to morphisms of algebraic varieties $f: X \to Y$. Explicitly, we want to glue the modules of differentials to get a quasi-coherent sheaf on $X$.

**Lemma 1.6.13.** Let $X$ be an algebraic variety. Suppose we have a map $\alpha$ that assigns to each affine open subset $U \subseteq X$, an $O_X$-module $\alpha(U)$, together with restriction maps: for all affine open subsets $V \subseteq U$, we have a restriction map $\alpha(U) \to \alpha(V)$ which is a morphism of $O_X(U)$-modules, which satisfy the usual compatibility condition, and if $V = D_U(f)$, then

$$\alpha(U)f \to \alpha(V)$$

is an isomorphism. Then there is a quasi-coherent sheaf $\mathcal{F}$ on $X$ with isomorphisms for $U \subseteq X$ affine:

$$\mathcal{F}|_U \cong \widehat{\alpha(U)},$$

compatible with restrictions, and $\mathcal{F}$ is unique up to isomorphism.

**Proof.** If $U \subseteq X$ is an affine open subset, consider $\mathcal{F}_U := \widehat{\alpha(U)}$. If $U \supseteq V$, we have $\alpha(U) \to \alpha(V)$, which induces

$$\alpha(U) \otimes_{O(U)} O(V) \to \alpha(V)$$

corresponding to the morphism of sheaves $\varphi_{U,V}: \mathcal{F}_U|_V \to \mathcal{F}|_V$.

If $U \supseteq V \supseteq W$, then $\varphi_{V,W} \circ \varphi_{U,V}|_W = \varphi_{U,W}$.

The assumptions also imply that $\varphi_{U,V}$ is an isomorphism if $V = D_U(f)$. In general, by covering $V$ by principal affine open subsets, then $\varphi_{U,V}$ is an isomorphism for all $V \subseteq U$.

This implies that given any two affine open $U_1, U_2$, we have a canonical isomorphism

$$\mathcal{F}_{U_1|_{U_1 \cap U_2}} \cong \mathcal{F}_{U_2|_{U_1 \cap U_2}}$$

since they are both isomorphic to $\mathcal{F}|_{U_1 \cap U_2}$. We can glue the $\mathcal{F}_U$ together to get $\mathcal{F}$. \qed

**Remark 1.6.14.** Given a morphism $f: X \to Y$, we get a similar statement if instead of using all affine open subsets of $X$, we only use those affine open subsets $U \subseteq X$ such that there is an affine open subset $V \subseteq Y$ such that $f(U) \subseteq V$.

**Corollary 1.6.15.** In our geometric setting, let
be morphisms of affine varieties where \( g \) is an open immersion, then

\[ \Omega_{\mathcal{O}(X)/\mathcal{O}(Y)} \cong \Omega_{\mathcal{O}(X)/\mathcal{O}(Z)}. \]

**Proof.** We have maps \( \mathcal{O}(Z) \to \mathcal{O}(Y) \to \mathcal{O}(X) \) which given an exact sequence by Proposition 1.6.9

\[ \Omega_{\mathcal{O}(Y)/\mathcal{O}(Z)} \otimes \mathcal{O}(X) \to \Omega_{\mathcal{O}(X)/\mathcal{O}(Z)} \to \Omega_{\mathcal{O}(X)/\mathcal{O}(Y)} \to 0 \]

It is enough to show that \( \Omega_{\mathcal{O}(Y)/\mathcal{O}(Z)} = 0 \). For any maximal ideal \( \mathfrak{m} \) in \( \mathcal{O}(Y) \), we need to show that

\[ \Omega_{\mathcal{O}(Y)/\mathcal{O}(Z)}|_{\mathcal{O}(U)} \cong \Omega_{\mathcal{O}(Y)/\mathcal{O}(Z)}|_{\mathcal{O}(Z)} \]

where the first equality follows from Proposition 1.6.7. By Remark 1.6.12,

\[ \Omega_{\mathcal{O}(Y)/\mathcal{O}(Z)}|_{\mathcal{O}(U)} \cong \Omega_{\mathcal{O}(Y)/\mathcal{O}(Z)}|_{\mathcal{O}(Z)} \]

since \( \mathcal{O}(Z)_{\mathfrak{m}} \to \mathcal{O}(Z) \) is an isomorphism, since \( g \) is an open immersion. \( \square \)

Suppose now \( f : X \to Y \) is a morphism of algebraic varieties. For every affine open subsets \( U \subseteq X, V \subseteq Y \) such that \( f(U) \subseteq V \), consider \( \Omega_{\mathcal{O}(U)/\mathcal{O}(V)} \). By Corollary 1.6.15, this is independent of the choice of \( V \).

If \( U' \subseteq U \to V \), we have a map

\[ \Omega_{\mathcal{O}(U)/\mathcal{O}(V)} \otimes \mathcal{O}(U') \to \Omega_{\mathcal{O}(U')/\mathcal{O}(V)} \]

which is an isomorphism if \( U' \) is a principal affine open subset. By Lemma 1.6.13, there exists a unique quasi-coherent sheaf \( \Omega_{X/Y} \) such that for \( U, V \) as above,

\[ \Omega_{X/Y}|_U \cong \Omega_{\mathcal{O}(U)/\mathcal{O}(V)}. \]

Then \( \Omega_{X/Y} \) is actually coherent.

**Definition 1.6.16.** For a morphism \( f : X \to Y \) of algebraic varieties, \( \Omega_{X/Y} \) is the **relative cotangent sheaf**. If \( Y \) is a point, we write \( \Omega_X \) for \( \Omega_{X/Y} \) and call it the **cotangent sheaf of** \( X \). We call \( T_X = \Omega_X^\vee \) the tangent sheaf.

**Remark 1.6.17.** For any \( x \in X \), we have that

\[ (\Omega_{X/Y})_x \cong \Omega_{\mathcal{O}(x)/\mathcal{O}(f(x))} \]

by Remark 1.6.12 and Proposition 1.6.7.

The two Propositions 1.6.9 and 1.6.10 globalize as follows

- If \( X \xrightarrow{f} Y \xrightarrow{g} Z \), we get an exact sequence of coherent sheaves on \( X \):

\[ f^*\Omega_{Y/Z} \longrightarrow \Omega_{X/Z} \longrightarrow \Omega_{X/Y} \longrightarrow 0 \]

- If \( f \) is a closed immersion with ideal \( \mathcal{I} \), we get an exact sequence of coherent sheaves on \( X \):

\[ \mathcal{I}/\mathcal{I}^2 \longrightarrow f^*\Omega_{Y/Z} \longrightarrow \Omega_{X/Z} \longrightarrow 0 \]
Definition 1.6.18. For a closed immersion $f : X \to Y$, $\mathcal{I}/\mathcal{I}^2$ is the conormal sheaf of $X$ in $Y$, and

$$(\mathcal{I}/\mathcal{I}^2)^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}/\mathcal{I}^2, \Omega_X)$$

is the normal sheaf of $X$ in $Y$, denoted $N_{X/Y}$.

Proposition 1.6.19. For every $x \in X$,

$$(\Omega_X)_x \cong (T_xX)^\vee.$$  

Proof. Recall that $T_xX = \text{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$, where $\mathfrak{m} \subseteq R = \mathcal{O}_{X,x}$ is the maximal ideal. Recall also that

$$(\Omega_X)_x = \Omega_{X,x}/\mathfrak{m} \cdot \Omega_{X,x},$$

so

$$(\Omega_X)^\vee_x = \text{Hom}_k(\Omega_{X,x}/\mathfrak{m}\Omega_{X,x}, k) = \text{Hom}_R(\Omega_{X,x}, k) = \text{Der}_k(R, k),$$

since $\Omega_{X,x} = \Omega_{R/k}$. We note note that

$$\text{Der}_k(R, k) = \text{Der}_k(R/\mathfrak{m}^2, k)$$

by the Leibniz rule, since $\mathfrak{m} \cdot k = 0$. Finally

$$R/\mathfrak{m}^2 = k + \mathfrak{m}/\mathfrak{m}^2$$

and it is easy to see that by restricting to $\mathfrak{m}/\mathfrak{m}^2$ we get that

$$\text{Der}_k(R/\mathfrak{m}^2, k) \cong \text{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k) = T_xX.$$

This completes the proof by dualizing. \qed

Proposition 1.6.20. A variety $X$ is smooth if and only if $\Omega_X$ is locally free. In this case, $\Omega_X$ has rank $n$ on an irreducible component of dimension $n$.

Proof. We may assume that $X$ is connected. If $X$ is smooth, it is irreducible, and if $n = \dim X$, then $\dim_k T_xX = n$, so by Proposition 1.6.19, $\dim_k(\Omega_X)_x = n$ for any $x \in X$, so $\Omega_X$ is locally free of rank $n$.

Suppose conversely that $\Omega_X$ is locally free of rank $n$. Recall that $X_{sm} \subseteq X$ is dense. Then every irreducible component of $X$ has dimension $n$, because we can find an open subset of that component which is smooth. Hence $\dim(\mathcal{O}_{X,x}) = n$ for any $x \in X$. Since $\dim_k(\Omega_X)_x = n$ for all $x \in X$, this shows by Proposition 1.6.19 that $X$ is smooth. \qed

Note that if $X$ is smooth, then $\Omega_X$ is locally free, and hence $T_X = \Omega_X^\vee$ is also locally free.

Definition 1.6.21. The sheaf of $p$-differentials on $X$ is defined by $\Omega_X^p = \bigwedge^p \Omega_X$. When $X$ is smooth, this is locally free.

Definition 1.6.22. If $X$ is irreducible of dimension $n$, $\omega_X = \Omega_X^n$ is the canonical line bundle of $X$.

Conjecture (Lipman-Zariski). A variety $X$ is smooth if and only if $T_X$ is locally free.
This is known in many cases (but not all of them).

**Example 1.6.23.** If \( X = \mathbb{P}^n \), we have a short exact sequence
\[
0 \longrightarrow \Omega_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus (n+1)} \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0
\]
This will be discussed during the problem session.

**Proposition 1.6.24.** If \( Y \subseteq X \) is a subvariety defined by \( \mathcal{I} \), with \( X \) and \( Y \) both smooth, then \( \mathcal{I}/\mathcal{I}^2 \) is locally free and we have a short exact sequence
\[
0 \longrightarrow T_Y \longrightarrow T_{X/Y} \longrightarrow N_{Y/X} \longrightarrow 0.
\]

*Proof.* We may assume \( X \) and \( Y \) are both irreducible. Recall that for any \( x \in X \), there is an affine open neighborhood \( U \) of \( x \), there exist \( f_1, \ldots, f_r \in \mathcal{O}(U) \) such that
\[
I_{Y \cap U/U} = (f_1, \ldots, f_r)
\]
where \( r = \text{codim}_X Y \). In this case,
\[
\mathcal{O}(U)/(f_1, \ldots, f_r)[t_1, \ldots, t_r] \longrightarrow \bigoplus_{m \geq 0} \frac{I_{Y \cap U/U}^m}{I_{Y \cap U/U}^{m+1}}
\]
\[
t_i \mapsto \overline{f}_i \in I/I^2
\]
is an isomorphism. In particular,
\[
I_{Y \cap U}/I_{Y \cap U/U}^2 \cong \bigoplus_{i=1}^r \mathcal{O}(U \cap Y)t_i,
\]
which is free. This shows that \( \mathcal{I}/\mathcal{I}^2 \) is locally free of rank equal to the codimension of \( Y \) in \( X \).

We have an exact sequence
\[
\mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega_X |_Y \xrightarrow{\varphi} \Omega_Y \rightarrow 0
\]
with \( \Omega_Y \) locally free of rank \( \dim Y \) and \( \Omega_X |_Y \) locally free of rank \( \dim X \). Then \( \ker(\varphi) \) is locally free of rank \( r \), and
\[
\mathcal{I}/\mathcal{I}^2 \rightarrow \ker(\varphi)
\]
is a surjective morphism between locally free sheaves of the same rank, so it is an isomorphism. Therefore the above sequence is actually a short exact sequence. Dualizing it, we get the result. \( \square \)

### 2. Normal varieties and divisors

In this section, we will discuss divisors on algebraic varieties, which allow to study morphisms to projective spaces. We first start with a few sections on normal varieties.
2.1. Normal varieties. We want to extend our definition of a normal variety to the case of varieties which may not be affine and possibly not irreducible.

**Proposition 2.1.1.** Suppose $A$ is a domain with fraction field $K$ and $S \subseteq A$ is a multiplicative system. If $A' \subseteq K$ is the integral closure of $A$ in $K$, then the integral closure of $S^{-1}A$ in $K$ is $S^{-1}A'$.

**Proof.** The proof is left as an exercise.

**Proposition 2.1.2.** Given a variety $X$, the following are equivalent:

1. for any affine open subset $U \subseteq X$ and every connected component $V$ of $U$, $\mathcal{O}(V)$ is a domain which is integrally closed (in its fraction field),
2. there exists an affine open cover of $X$ by $U_1, \ldots, U_n$ such that each $U_i$ is irreducible and $\mathcal{O}(U_i)$ is integrally closed,
3. for any irreducible closed subset $V \subseteq X$, $\mathcal{O}_{X,V}$ is an integrally closed domain,
4. for any $x \in X$, $\mathcal{O}_{X,x}$ is an integrally closed domain.

**Definition 2.1.3.** A variety $X$ is normal if the equivalent conditions in Proposition 2.1.2 hold.

Note that if $X$ is irreducible and affine, this agrees with the previous definition.

**Proof of Proposition 2.1.2.** We see immediately that (1) implies (2) and (3) implies (4). Moreover, (2) implies (3) by Proposition 2.1.1.

It remains to show that (4) implies (1). Since $\mathcal{O}_{X,x}$ is a domain, every point $x$ lies on a unique irreducible component, we may assume that $X$ is irreducible. If $A = \mathcal{O}(U)$ is a domain with integral closed $A'$, by assumption, we have that

$$A_m = A'_m$$

for any maximal ideal $m$ so $A = A'$.

**Review of DVRs.**

**Definition 2.1.4.** If $K$ is a field, a discrete valuation of $K$ is a surjective map

$$v: K \to \mathbb{Z} \cup \{\infty\}$$

such that

1. $v(a) = \infty$ if and only if $a = 0$,
2. $v(a + b) \geq \min\{v(a), v(b)\}$ for all $a, b$,
3. $v(ab) = v(a) + v(b)$.

**Example 2.1.5.** For $K = \mathbb{Q}$, we can let $v(p^n a) = n$ when $(a, p) = 1$ for a fixed prime $p$.

**Proposition 2.1.6.** Given a domain $R$ with fraction field $K$, the following are equivalent:

1. there exists a discrete valuation $v$ on $K$ such that $R = \{a \mid v(a) \geq 0\}$,
2. $R$ is a local PID, not a field,
3. $R$ is local with a principal maximal ideal, and Noetherian.
Definition 2.1.7. A domain $R$ is a **discrete valuation ring** (DVR) if the equivalent conditions in Proposition 2.1.6 hold.

Proof of Proposition 2.1.6. To show that (1) implies (2), $m = \{a \mid v(a) > 0\}$ is an ideal in $R$. If $a \in R \setminus m$, then $u^{-1} \in R$, so $m$ is the unique maximal ideal of $R$. If $I \neq 0$ is an ideal, choose $a \in I \setminus \{0\}$ such that $v(a)$ is minimal. If $b \in I$, $v(b) \geq v(a)$, then $\frac{b}{a} \in R$, so $I = (a)$.

Since (2) implies (3) is obvious, we only have to show that (3) implies (1). Suppose $m = (\pi)$ is the maximal ideal of $R$. If $a \in R$, by Krull’s Intersection Theorem, there exists a unique $j$ such that $a \in m^j \setminus m^{j+1}$. Then set $v(a) = j$. It is clear that $v(a + b) \geq \min\{v(a), v(b)\}$ and $v(a \cdot b) = v(a) + v(b)$, where the second assertion follows from $m = (\pi)$. This extends to $K$ by $v(a/b) = v(a) - v(b)$, and one shows that $R = \{a \mid v(a) \geq 0\}$.

Note that if $m = (\pi)$ is the maximal ideal in the local ring $R$, then every ideal in $R$ is $(0)$ or $(\pi^m)$ for $m \geq 0$. Therefore, $R$ has two prime ideals, $(0)$ and $m$, which shows that $\dim R = 1$.

Suppose $X$ is an algebraic variety and $V \subset X$ irreducible, closed, of dimension 1. Then $O_{X,V}$ is a DVR if and only if $X$ is smooth at $V$ and by Problem 1 from Problem Set 3 we have that $V \cap X_{sm} \neq \emptyset$.

Lemma 2.1.8. If $R$ is a Noetherian, integrally closed domain, $a \in R \setminus \{0\}$, $p \in \text{Ass}_R(R/(a))$, then $R_p$ is a DVR (in particular, $\text{codim}(p) = 1$).

Proof. Replace $R$ by $R_p$ to assume that $R$ is local and $p = m$ is the unique maximal ideal. By hypothesis, there exists $b \not\in (a)$ such that

$$m = \{u \in R \mid ub \in (a)\}.$$

For $\frac{b}{a} \in \text{Frac}(R)$, we have that $m \cdot \frac{b}{a} \subseteq R$.

If $m \cdot \frac{b}{a} \subseteq m$, by the determinant trick, we get that $\frac{b}{a}$ is integral over $R$. Since $R$ is integrally closed, $\frac{b}{a} \in R$. Then $m = R$, which contradicts $m$ being maximal.

Therefore, $m \cdot \frac{b}{a} = R$, which implies that $\frac{a}{b} \in m$. By the description of $m$ above, for any $u \in m$, $u^2 \frac{b}{a} \in R$, so $u \in \left(\frac{a}{b}\right)$. This shows that $m = \left(\frac{a}{b}\right)$. Therefore, $R$ is a DVR.

Lemma 2.1.9. Let $R$ be a ring. Then the following are equivalent:

1. $R$ is a DVR,
2. $R$ is a local, Noetherian domain with $\dim R = 1$, which is integrally closed.

Proof. Clearly, (1) implies (2) (note that since $R$ is a PID, $R$ is a UFD, so it is integrally closed).

To show that (2) implies (1), choose $a \in m \setminus \{0\}$. Then

$$\{0\} \not\in \text{Ass}_R(R/(a)) \neq \emptyset,$$

so $m \in \text{Ass}_R(R/(a))$, since $\dim R = 1$. The result then follows from Lemma 2.1.8.

Proposition 2.1.10. Let $A$ be a Noetherian domain. Then $A$ is integrally closed if and only if the following conditions hold:
(1) for any prime $p$ of codimension 1, $A_p$ is a DVR,

(2) $A = \bigcap_{\text{codim } p = 1} A_p$.

Moreover, (2) can be replaced by

(2') for any $a \in A \setminus \{0\}$ and any prime $p \in \text{Ass}_R(R/(a))$, codim$(p) = 1$.

Proof. Suppose first that (1) and (2) hold. Then $A_p$ is integrally closed for all $p$ of codimension 1, and $A = \bigcap_{\text{codim } p = 1} A_p$ implies that $A$ is integrally closed.

If $A$ is integrally closed, for any $p$ of codimension 1, $A_p$ is a DVR by Lemma 2.1.9, and (2') holds by Lemma 2.1.8.

The proof will be complete if we show that (2') implies (2). The ‘$\subseteq$’ inclusion is immediate. Suppose $\frac{b}{a} \in \bigcap_{\text{codim } p = 1} A_p$. Consider a minimal primary decomposition

$$a = q_1 \cap \cdots \cap q_r.$$  

Then each $q_i$ is primary and $\text{rad}(q_i) = p_i$ is prime. Then

$$\text{Ass}_R(R/(a)) = \{p_1, \ldots, p_r\}.$$  

By (2'), codim$(p_i) = 1$ for all $i$, so $\frac{b}{a} \in A_{p_i}$. Then for any $i$, there exists $s_i \in A \setminus p_i$ such that $s_i b \in (a) \subseteq q_i$, so $b \in q_i$ since $q_i$ is $p_i$-primary. Therefore, $b \in (a)$ and hence $\frac{b}{a} \in A$.  

\[ \square \]

2.2. Geometric properties of normal varieties.

Definition 2.2.1. An algebraic variety $X$ is smooth in codimension 1 if $\text{codim}_X (X_{\text{sing}}) \geq 2$.

If $Z \subseteq X$ is closed with irreducible components $Z_1, \ldots, Z_r$, $\text{codim}_X (Z) = \min \{\text{codim}_X (Z_i)\}$.

Note that $X$ is smooth in codimension 1 if and only if for any irreducible closed subset $V \subseteq X$ of codimension 1, we have that $V \cap X \neq \emptyset$. This is equivalent to $\mathcal{O}_{X,V}$ being a regular ring.

In particular, if $X$ is a normal variety, then $X$ is smooth in codimension 1 (since this holds for irreducible affine open subsets).

Proposition 2.2.2. Let $X$ be a normal variety. Then if $\mathcal{E}$ is a locally free sheaf on $X$ and $U \subseteq X$ is an open subset such that $\text{codim}_X (X \setminus U) \geq 2$, then the restriction map $\Gamma(X, \mathcal{E}) \to \Gamma(U, \mathcal{E})$ is an isomorphism.

Proof. It is enough to prove this when $X$ is affine and irreducible and $\mathcal{E} = \mathcal{O}_X$. Indeed, choose an open cover $X = U_1 \cup \cdots \cup U_n$ by affine irreducible subsets $U_i$ such that $\mathcal{E}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus r}$. Then we have the diagram
with exact rows (by the sheaf axiom) and the vertical maps being restriction maps. The special case implies that \( \beta, \gamma \) are isomorphisms, so \( \alpha \) is an isomorphism by Five Lemma.

Suppose that \( X \) is affine and \( A = \mathcal{O}(X) \). The map \( A = \mathcal{O}(X) \to \mathcal{O}(U) \) is injective since \( U \subseteq X \) is dense (otherwise, \( \text{codim}_X (X \setminus U) = 0 \)). If \( \varphi \in \mathcal{O}(U) \), then for any \( p \) in \( A \) of codimension 1, \( V(p) \cap U \neq \emptyset \), i.e. \( \varphi \in A_p \), this completes the proof. \( \square \)

**Corollary 2.2.3.** Suppose \( X \) is an irreducible normal variety and \( \varphi \in k(X) \) with domain \( U \). Then every irreducible component of \( X \setminus U \) has codimension 1.

**Proof.** The proof is left as an exercise. \( \square \)

**Notation.** Suppose \( X \) is an irreducible variety, smooth in codimension 1. If \( V \subseteq X \) is irreducible, closed, of codimension 1, then \( \mathcal{O}_{X,V} \) is a DVR. We write \( \text{ord}_V \) for the corresponding discrete valuation on \( k(X) \).

We say that \( \varphi \in k(X) \) has a pole along \( V \) if \( \text{ord}_V(\varphi) < 0 \). This is equivalent to saying that \( \varphi^{-1} \) is defined in an open subset \( U \) with \( U \cap V \neq \emptyset \) and \( \varphi^{-1}|_{U \cap V} = 0 \).

We say that \( \varphi \in k(X) \) has a pole of order \( m > 0 \) if \( \text{ord}_V(\varphi) = -m \), and has a zero of order \( m > 0 \) if \( \text{ord}_V(\varphi) = m \).

Note that if \( X \) is normal and \( \varphi \in k(X) \) with domain \( U \), then

\[
X \setminus U = \bigcup \{ V \subseteq X \mid V \text{ irreducible, codim}(V) = 1, \ \varphi \text{ has a pole along } V \}
\]

**Proposition 2.2.4.** Suppose that \( X \) is an irreducible variety, smooth in codimension 1.

1. If \( f : X \dashrightarrow Y \) is a rational map and \( Y \) is complete, \( U = \text{Dom}(f) \), then
   \[
   \text{codim}_X (X \setminus U) \geq 2.
   \]

2. More generally, if \( f : X \dashrightarrow Y \) is a rational map and \( g : Y \to Z \) is proper such that \( g \circ f \) is a morphism, \( U = \text{Dom}(f) \), then \( \text{codim}_X (X \setminus U) \geq 2 \).

**Proof.** It is enough to show (2). We may assume that \( f \) is dominant by replacing \( Y \) with a closed subvariety (hence \( Y \) is irreducible). By Chow’s Lemma, there is a birational map \( h : \tilde{Y} \to Y \) with \( \tilde{Y} \) irreducible such that \( g \circ h \) factors as in the diagram, with \( i \) a closed immersion, and \( p \) the projection:
It is enough to prove the conclusion for $h^{-1} \circ f$. Moreover, it is enough to prove this for $i \circ (h^{-1} \circ f) = (g \circ f, u)$ for some $u : X \rightarrow \mathbb{P}^n$.

It is enough to prove that $u$ is defined on the complement of a codim $\geq 2$ subset. Hence it is enough to consider rational map $X \rightarrow \mathbb{P}^n$.

There is an open subset $U \subseteq X$ and functions $\varphi_0, \ldots, \varphi_n \in \mathcal{O}(U)$ such that $f$ is defined on $U$, given by

$$x \mapsto [\varphi_0(x), \ldots, \varphi_n(x)].$$

We want to show that for any $V \subseteq X$ irreducible of codimension 1, $\text{Dom}(f) \cap V \neq \emptyset$. Let $j$ be such that $\text{ord}_V(\varphi_j) = \min \{ \text{ord}_V(\varphi_i) \mid 0 \leq i \leq n \}$. Then $\text{ord}_V(\varphi_i/\varphi_j) \geq 0$ for all $j$. Then there is $U \subseteq X$ open such that $U \cap V \neq \emptyset$ and $\varphi_i/\varphi_j \in \mathcal{O}(U)$. Then $f$ can be defined on $U$ by $[\varphi_0/\varphi_j, \ldots, \varphi_n/\varphi_j]$. □

**Theorem 2.2.5.** Let $A$ be a domain which is an algebra of finite type over a field $k$. If $K = \text{Frac}(A)$ and $L/K$ is a finite field extension, the integral closure $B$ of $A$ in $L$ is finite over $A$.

**Proof.** Since $A$ is Noetherian, it is enough to prove this when replacing $L$ by a finite field extension.

**Step 1.** Reduce to the case when $A$ is normal and $L/K$ is separable.

By Noether Normalization Theorem, there exists $R \subseteq A$ such that $R \cong k[x_1, \ldots, x_n]$ and $A/R$ is finite. After replacing $A$ by $R$, we may assume that $A = k[x_1, \ldots, x_n]$. By enlarging $L$, we may assume that $L/K$ is normal, $G = \text{Gal}(L/K)$, and $K' = L^G \subseteq L$. We then have $K \subseteq K' \subseteq L$ with $L/K$ separable and $K'/K$ purely inseparable. Let us show that the integral closure $A'$ of $A$ in $K'$ is finite over $A$. If $K' \neq K$, $p = \text{char}(K) > 0$ and for all $f \in K'$, there exists $e > 0$ such that

$$f^{pe} \in K = k(x_1, \ldots, x_n).$$

Then there exists a finite extension $k'/k$ such that

$$K' \subseteq k'(x_1^{\frac{1}{p^e}}, \ldots, x_n^{\frac{1}{p^e}})$$

for some $e$. Then $A'$ is contained in the integral closure of $k[x_1, \ldots, x_n]$ in $k'(x_1^{\frac{1}{p^e}}, \ldots, x_n^{\frac{1}{p^e}})$. This integral closure is

$$k'[x_1^{\frac{1}{p^e}}, \ldots, x_n^{\frac{1}{p^e}}].$$
which is finite over $k[x_1, \ldots, x_n]$. Therefore, $A'$ is finite over $A$, and we have reduced to the case where $A$ is normal and $L/K$ is separable.

**Step 2.** Suppose $A$ is normal and $L/K$ is separable. By enlarging $L$, we may assume that $L/K$ is Galois with $\text{Gal}(L/K) = G = \{\sigma_1, \ldots, \sigma_d\}$. Choose a basis $u_1, \ldots, u_d$ for $L/K$. We may assume that $u_1, \ldots, u_d \in B$ (multiply each $u_i$ by an element of $L$ to make the polynomial for $u_i$ monic).

Let

$$M = (\sigma_i(u_j)) \in M_d(B), \quad D = \det(M).$$

(1) If $D = 0$, then there exist $\lambda_1, \ldots, \lambda_d \in L$, not all 0, such that

$$\left( \sum_{i=1}^{d} \lambda_i \sigma_i \right)(u_j) = 0 \text{ for all } j.$$

Then we have that

$$\sum_{i=1}^{d} \lambda_i \sigma_i = 0,$$

which is a contradiction. Indeed, after reordering, we have that

$$\lambda_1 \sigma_1 + \cdots + \lambda_r \sigma_r = 0$$

with $\lambda_i \neq 0$ and $r$ minimal with this property. It is clear that $r \geq 2$. Then for $a, b \in L$,

$$\left( \sum_{i=1}^{r} \lambda_i \sigma_i \right)(ab) = 0,$$

and since this holds for any $b$

$$\sum_{i=1}^{r} \lambda_i \sigma_i(a) \sigma_i = 0.$$

If $a$ is such that $\sigma_1(a) \neq \sigma_2(a)$, then (2) − $\lambda_i \cdot (1)$ gives another relation for $\sigma_2, \ldots, \sigma_{r-1}$, contradicting the minimality of $r$.

(2) We may assume that $D \neq 0$. We then show that

$$B \subseteq \frac{1}{D^2} \sum_{i=1}^{d} A \cdot u_i.$$

Since the right hand side is finitely-generated over $A$, this will complete the proof.

Note that $D \in B$. Then $\sigma_i(D)$ is the determinant obtained from $M$ by permuting the rows, so $\sigma_i(D) = \pm D$, and this shows that

$$\sigma_i(D^2) = D^2 \text{ for all } i.$$

Hence $D^2 \in K$.

Given any $b \in B$,

$$b = \sum_{j=1}^{d} \alpha_j u_j, \quad \alpha_i \in K.$$
We want to show that $D^2 \alpha_i \in A$. We then have that 

$$B \ni \sigma_i(b) = \sum_{j=1}^{d} \alpha_j \sigma_i(u_j)$$

and this shows that 

$$M \cdot \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_d \end{pmatrix} \in B^n.$$ 

Multiplying by the adjoint of $M$, we see that $D \cdot \alpha_i \in B$, so $D^2 \alpha_i \in B \cap K = A$, since $A$ is normal.

This completes the proof.

Suppose $X$ is an irreducible algebraic variety. We want to find a normal variety that dominates $X$. For an affine open subset $U \subseteq X$, let $\mathcal{A}(U)$ be the integral closure of $\mathcal{O}_X(U)$ in $k(X)$. If $U \subseteq V$ are affine, the inclusion $\mathcal{O}_X(V) \hookrightarrow \mathcal{O}_X(U)$ gives an inclusion 

$$\mathcal{A}(V) \hookrightarrow \mathcal{A}(U)$$

If $U = D_V(f)$, then 

$$\mathcal{A}(V)_f \cong \mathcal{A}(U).$$

Then by Lemma 1.6.13, $\mathcal{A}$ can be extended to a quasicoherent $\mathcal{O}_X$-algebra. If $U \subseteq X$ is affine, $\mathcal{A}(U)$ is reduced and $\mathcal{A}(U)$ is finite over $\mathcal{O}_X(U)$, so $\mathcal{A}$ is a reduced coherent $\mathcal{O}_X$-algebra.

The normalization of $X$ is 

$$X^{\text{norm}} = \text{MaxSpec}(\mathcal{A}) \xrightarrow{\pi} X.$$ 

It is clear that 

(1) $X^{\text{norm}}$ is normal,  
(2) $\pi$ is finite,  
(3) $X^{\text{norm}}$ is irreducible,  
(4) $\pi$ is birational.

**Universal property of normalization.** Given a normal, irreducible variety $Z$ and a dominant map $f: Z \to X$, there is a unique $g: Z \to X^{\text{norm}}$ such that $\pi \circ g = f$, i.e. the diagram

$$
\begin{array}{ccc}
Z & \xrightarrow{g} & X^{\text{norm}} \\
\downarrow{f} & & \downarrow{\pi} \\
& & X \\
\end{array}
$$

commutes.

Proving this is left as a homework exercise.

**Remark 2.2.6.** If $X$ has irreducible components $X_1, \ldots, X_r$, the normalization of $X$ is 

$$\prod_{i=1}^{r} X_i^{\text{norm}} \to X.$$
Definition 2.2.7. A variety $X$ is locally factorial if for any $x \in X$, the ring $\mathcal{O}_{X,x}$ is a UFD.

Note that this implies that $\mathcal{O}_{X,V}$ is a UFD for all irreducible closed $V \subseteq X$. It is actually very rare that for an affine variety $X$, $\mathcal{O}(X)$ is a UFD, but it does happen quite often that the local rings are UFDs.

**Theorem 2.2.8.** If $X$ is a smooth variety, then $X$ is locally factorial (in particular, $X$ is normal).

The proof uses completions of local rings. For any $x \in X$:

$$\widehat{\mathcal{O}}_{X,x} = \lim_{q \geq 1} \mathcal{O}_{X,x}/m_{X,x}^q$$

This records, roughly, what happens in a very small neighborhood of a point. One way to think about it is that $\mathcal{O}_{X,x}/m_{X,x}^q$ records the first $q$ coefficients of the Taylor polynomial, so the whole inverse limit is similar to a Taylor polynomials.

**Example 2.2.9.** We have that

$$k[x_1, \ldots, x_n](x_1, \ldots, x_n) \cong k[[x_1, \ldots, x_n]].$$

Suppose that $X$ is an affine variety, $R = \mathcal{O}(X)$ and $Y \subseteq X$ is defined by $I$. Then we can set

$$\widehat{R} = \lim_{n} R/I^n.$$  

This is an algebraic analogue of a tubular neighborhood of $Y$ inside $X$.

**Proposition 2.2.10.** If $x \in X$ is a smooth point, $\dim(\mathcal{O}_{X,x}) = d$, then

$$\widehat{\mathcal{O}}_{X,x} \cong k[[t_1, \ldots, t_d]].$$

**Proof.** We show that if $m \subseteq \mathcal{O}_{X,x}$ is a maximal ideals, $u_1, \ldots, u_d \in m$ is a minimal system of generators, then

$$S = k[t_1, \ldots, t_d] \to \bigoplus_{i \geq 0} m^i/m^{i+1}$$

$$t_i \mapsto \overline{t_i} \in m/m^2$$

is an isomorphism. Let $n = (t_1, \ldots, t_d) \subseteq S$ be the maximal ideal and let

$$S/n^i \xrightarrow{\varphi_i} R/m^i$$

$$\overline{t_i} \mapsto u_i \mod m^i$$

We then have a commutative diagram

$$\begin{array}{ccccccc}
0 & \to & n^i/n^{i+1} & \to & S/n^{i+1} & \to & S/n^i & \to & 0 \\
& & \downarrow \approx & & \downarrow \varphi_{i+1} & & \downarrow \varphi_i & \\
0 & \to & m^i/m^{i+1} & \to & R/m^{i+1} & \to & R/m^i & \to & 0
\end{array}$$
Then by induction on $n$ and the Five Lemma, we can show that $\varphi_i$ is an isomorphism for all $i$. Then $\lim \varphi_i$ given an isomorphism

$$k[t_1, \ldots, t_d] \xrightarrow{\cong} \mathcal{O}_{X,x},$$

completing the proof. □

**Lemma 2.2.11.** Let $R$ be a domain.

1. If $R$ is a UFD, then for any $a, b \in R$, the ideal

   $$aR : bR = \{ h \in R \mid hb \in Ra \}$$

   is principal.

2. The converse holds if $R$ is Noetherian.

**Proof.** If

$$a = u \cdot \prod_{i=1}^{r} \pi_i^{m_i},$$

$$b = v \cdot \prod_{i=1}^{r} \pi_i^{n_i},$$

for units $u, v$ and irreducibles $\pi_i$, then

$$aR : bR = \left( \prod_{i=1}^{r} \pi_i^{\max\{0, m_i - n_i\}} \right).$$

Recall that a ring $R$ is a UFD if and only if

(i) every nonzero non-invertible element is a product of irreducible elements,

(ii) uniqueness up to reordering and rescaling by invertible elements.

Note also that (i) always holds for Noetherian rings. Moreover, if (i) holds then (ii) holds if and only if every irreducible element is prime.

Therefore, we just need to show that every irreducible element $\pi$ of $R$ is prime. If $\pi | ab$, then

$$b \in (\pi) : (a) = (h),$$

so $\pi \in (h)$, and hence $\pi = hh'$. Therefore, either $h$ is invertible, so $\pi | a$ or $h'$ is invertible, so $b \in (\pi)$, and hence $\pi | b$. □

**Lemma 2.2.12.** If $\hat{\mathcal{O}}_{X,x}$ is a UFD, then $\mathcal{O}_{X,x}$ is a UFD.

**Proof.** We have the map

$$\psi : \mathcal{O}_{X,x} \to \hat{\mathcal{O}}_{X,x}$$

$$a \mapsto (a \mod m^n)_{n \geq 1}$$

Note that $\hat{\mathcal{O}}_{X,x}$ is local ring with maximal ideal $m \cdot \hat{\mathcal{O}}_{X,x}$, and

$$\mathcal{O}_{X,x}/m \cong \hat{\mathcal{O}}_{X,x}/m \cdot \hat{\mathcal{O}}_{X,x}.$$  

Then $\psi$ is injective and flat.
Write $R = \mathcal{O}_{X,x}$ to simplify notation. By Lemma 2.2.11, it is enough to show that if $a, b \in R$, $J = aR : bR$ is principal.

We have the exact sequence

$$0 \rightarrow J \rightarrow R \xrightarrow{b} R/aR$$

and tensoring it with $\hat{R}$, we obtain

$$0 \rightarrow J\hat{R} \rightarrow \hat{R} \xrightarrow{b} \hat{R}/a\hat{R}$$

by flatness. Note that $J\hat{R} = a\hat{R} : b\hat{R}$ is principal since $R$ is a UFD. We finally see that

$$\dim_k J\hat{R}/Jm\hat{R} = 1$$

and

$$\overline{J/mJ} = J/Jm \cong J/Jm \otimes \hat{R} \cong J\hat{R}/Jm\hat{R}.$$ 

By Nakayama Lemma, this shows that $J$ is principal. \hfill \square

**Proof of Theorem 2.2.8.** By Proposition 2.2.10

$$\hat{O}_{X,x} \cong k[[t_1, \ldots, t_d]]$$

We know that $k[[t_1, \ldots, t_d]]$ is a UFD (see for example Zariski–Samuel). Then the result follows from Lemma 2.2.12. \hfill \square

2.3. **Divisors.** We will next study the following picture

geometric subvarieties of codimension 1 $\xleftrightarrow{\text{now}}$ line bundles

maps to projective spaces

2.4. **Weil divisors.**

**Definition 2.4.1.** Let $X$ be an irreducible variety, smooth in codimension 1. A *prime divisor* on $X$ is an irreducible closed subset $V \subset X$ of codimension 1. The *group of (Weil) divisors* is

$$\text{Div}(X) = \text{free abelian group on the set of prime divisors},$$

so a divisor $D \in \text{Div}(X)$ can be written as

$$D = \sum_{i=1}^{r} n_i V_i \quad n_i \in \mathbb{Z}, \quad V_i \text{ prime divisors.}$$

A divisor $D$ is *effective* if all the coefficients are nonnegative, $n_i \geq 0$. Write $D \leq E$ if $E - D$ is effective.
For $\varphi \in k(X)^*$, let
\[
\text{div}(\varphi) = \sum_{V \text{ prime divisor}} \text{ord}_V(\varphi)V \in \text{Div}(X).
\]
This is well-defined: suppose $\varphi$ is defined on $U$ and $\varphi$ is invertible on $U' \subseteq U$. Then $\text{ord}_V(\varphi) \neq 0$ implies that $V \subseteq X \setminus U'$ and there are only finitely many such $V$ of codimension 1.

Note that $\text{ord}_V(\varphi \psi) = \text{ord}_V(\varphi) + \text{ord}_V(\psi)$ for any $\varphi, \psi \neq 0$, so $\text{div} : k(X)^* \to \text{Div}(X)$ is a morphism of abelian groups.

**Definition 2.4.2.** A divisor $D \in \text{Div}(X)$ is principal if $D = \text{div}(\varphi)$ for some $\varphi \in k(X)^*$. The principal divisors form a subgroup $\text{PDiv}(X) = \{ \text{div}(\varphi) \mid \varphi \in k(X)^* \} \subseteq \text{Div}(X)$ and the quotient
\[
\text{Cl}(X) = \text{Div}(X) / \text{PDiv}(X)
\]
is called the class group of $X$. We write $[D] \in \text{Cl}(X)$ for the image of $D \in \text{Div}(X)$ in the class group of $X$.

**Remark 2.4.3.** Consider $X$ normal and $\varphi \in k(X)^*$. Then $\text{div}(\varphi) \geq 0$ if and only if $\varphi \in \mathcal{O}(X)$ and $\text{div}(\varphi) = 0$ if and only if $\varphi \in \mathcal{O}(X)^*$.

**Proposition 2.4.4.** Let $X$ be an affine irreducible normal variety. Then $\text{Cl}(X) = 0$ if and only if $\mathcal{O}(X)$ is a UFD.

**Lemma 2.4.5.** Let $A$ be a Noetherian domain. Then $A$ is a UFD if and only if any prime $p \subseteq A$ of codimension 1 is principal.

**Proof.** To show the ‘only if’ implication, choose $a \in p \setminus \{0\}$, and write
\[
a = u_1 \ldots u_r \text{ for } u_i \text{ irreducible}
\]
and since $p$ is prime, $u_i \in p$ for some $i$. Then
\[
(0) \subsetneq (u_i) \subseteq p
\]
and $(u_i)$ is prime since $A$ is a UFD, so $p = (u_i)$ since $p$ has codimension 1.

Conversely, note that since $A$ is Noetherian, it is enough to show that if $\pi$ is irreducible, then $(\pi)$ is prime. Let $p$ be a minimal prime containing $(\pi)$. Then the Principal Ideal Theorem shows that $\text{codim}(p) = 1$. By hypothesis, $p = (a)$ for some $a$, and $(\pi) \subseteq (a)$ shows that $\pi = a \cdot b$. Since $\pi$ is irreducible, $b$ is invertible, and hence $(\pi) = (a) = p$. This shows that $(\pi)$ is prime. \hfill \Box

**Proof of Proposition 2.4.4.** By definition, $\text{Cl}(X) = 0$ if and only if for any prime ideal $p$ of codimension 1 in $\mathcal{O}(X)$, $V(p)$ is principal, i.e. there exists $\varphi \in k(X)^*$ such that $\text{div}(\varphi) = V(p)$. This is equivalent to $\varphi \in \mathcal{O}(X)$ and $\varphi \mathcal{O}(X)_p = p\mathcal{O}(X)_p$ and $\varphi \not\in q$ for $q \neq p$ of codimension 1.

If $\mathcal{O}(X)$ is a UFD, given $p$, choose $\varphi$ such that $p = (\varphi)$ (by Lemma 2.4.5). Then the conditions above are clearly satisfied.
Conversely, suppose \( \text{Cl}(X) = 0 \) and let \( p \subseteq \mathcal{O}(X) \) be prime of codimension 1. Choose \( \varphi \) such that \( V(p) = \text{div}(\varphi) \), so \( \varphi \in p \), \( \text{ord}_V(p)(\varphi) = 1 \). If \( a \in p \), \( \text{div}(a/\varphi) \geq 0 \) by assumption on \( \varphi \). This means that \( \frac{a}{\varphi} \in \mathcal{O}(X) \), and hence \( p = (\varphi) \). Hence \( \mathcal{O}(X) \) is a UFD by Lemma 2.4.5.

By this Proposition, we know that for affine, irreducible, normal varieties \( \text{Cl}(X) = 0 \) if and only if \( \mathcal{O}(X) \) is a UFD. In general, the class group measures how far \( \mathcal{O}(X) \) is from being a UFD. Note that this is essentially the same as the class group for number fields, which measures how far the ring of integers is from being a UFD.

**Example 2.4.6.** If \( X = \mathbb{A}^n \), then Proposition 2.4.4 implies that \( \text{Cl}(X) = 0 \).

**Example 2.4.7.** Let \( X = \mathbb{P}^n \). Recall that if \( V \subseteq \mathbb{P}^n \) is irreducible, closed, of codimension 1, the prime ideal corresponding to \( V \) is generated by 1 element \( F \in S = k[x_0, \ldots, x_n] \), homogeneous of degree \( d > 0 \). Then we say that \( \deg(V) = d \). This lets us define a group homomorphism:

\[
\text{deg}: \text{Div}(\mathbb{P}^n) \to \mathbb{Z}
\]

\[
\sum_{i=1}^{r} n_i V_i \mapsto \sum_{i=1}^{r} n_i \deg(V_i)
\]

Note that 1 is the degree of a hyperplane, so this map is surjective.

We claim that if \( \varphi \in k(\mathbb{P}^n)^* \), then \( \deg(\text{div}(\varphi)) = 0 \). This will show that the degree map factors through the class group of \( \mathbb{P}^n \).

We can write \( \varphi = \frac{F}{G} \) for \( F, G \in S \) homogeneous, nonzero, of the same degree. Since \( S \) is a UFD, write

\[
F = c_F \cdot \prod_{i=1}^{r} F_i^{a_i}
\]

\[
G = c_G \cdot \prod_{j=1}^{s} G_j^{b_j}
\]

for \( a_i, b_j > 0 \) and \( F_i, G_j \) irreducible. Then

\[
\text{div}(\varphi) = \sum_{i=1}^{r} a_i V(F_i) - \sum_{j=1}^{s} b_j V(G_j)
\]

has degree

\[
\sum_{i=1}^{r} a_i \deg(F_i) - \sum_{j=1}^{s} b_j \deg(G_j) = \deg F - \deg G = 0.
\]

Hence we get a surjective map

\[
\text{deg}: \text{Cl}(\mathbb{P}^n) \to \mathbb{Z}.
\]

We claim that this map is also injective, and hence an isomorphism. Suppose \( D = \sum_{i=1}^{r} n_i V(F_i) \) has degree 0. Taking

\[
\varphi = \prod_{n_i > 0} F_i^{n_i} \frac{F_i^{n_i}}{F_i^{n_i-n_i}} \in k(X)^*,
\]

then
we see that \( \text{div}(\varphi) = D \), so \( D = 0 \) in \( \text{Cl}(\mathbb{P}^n) \).

**Definition 2.4.8.** Two divisors \( D \) and \( E \) are *linearly equivalent* if \( D - E \) is principal. We then write \( D \sim E \).

Let \( X \) be normal and irreducible. For a divisor \( D \) on \( X \), we will define a sheaf associated to it

\[ \mathcal{O}_X(D) \subseteq k(X) = \text{constant sheaf of rational functions}. \]

If \( U \subseteq X \) is open,

\[ \Gamma(U, \mathcal{O}_X(D)) = \{ 0 \} \cup \{ \varphi \in k(X)^* \mid \text{div}(\varphi)|_U + D|_U \geq 0 \}. \]

(Note that if \( U \subseteq X \) and \( E = \sum n_i V_i \) on \( X \), \( E|_U = \sum_{V_i \cap U \neq \emptyset} n_i (V_i \cap U) \) is a divisor on \( U \).)

It is clear that \( \mathcal{O}_X(D) \subseteq k(X) \) is a subsheaf, which is in fact a sub \( \mathcal{O}_X \)-module.

Note that:

1. if \( D = 0 \), \( \mathcal{O}_X(D) = \mathcal{O}_X \),
2. if \( D \geq E \), \( \mathcal{O}_X(E) \subseteq \mathcal{O}_X(D) \); in particular, if \( E \leq 0 \), then \( \mathcal{O}_X(E) \subseteq \mathcal{O}_X \).

**Proposition 2.4.9.** The sheaf \( \mathcal{O}_X(D) \) associated to a divisor \( D \) is coherent, and the stalk at \( X \) is \( k(X) \).

**Proof.** We first show that it is quasicoherent. Suppose \( U \subseteq X \) is an affine open subset, \( f \in \mathcal{O}_X(U) \). The map

\[ \Gamma(U, \mathcal{O}_X(D)) \to \Gamma(D_U(f), \mathcal{O}_X(D)) \]

is clearly injective, since \( \mathcal{O}_X(D) \) is a subsheaf of \( k(X) \) and \( k(X) \) is a domain. To show surjectivity, take \( \varphi \in \Gamma(D_U(f), \mathcal{O}_X(D)) \). Then

\[ (\text{div}(\varphi) + D)|_{D_U(f)} \geq 0. \]

We want to show that for some \( m \geq 0 \) such that

\[ (\text{div}(\varphi \cdot f^m) + D)|_U \geq 0. \]

Let \( D' = (D + \text{div}(\varphi))|_U \). Let \( Z_1, \ldots, Z_r \) be the prime divisors in \( U \) where \( D' \) has negative coefficient. Then \( Z_i \subseteq V(f) \), so \( \text{ord}_{Z_i}(f) \geq 1 \), and hence if \( m \gg 0 \), \( D' + (\text{div}(f^m))|_U \geq 0 \). Therefore,

\[ f^m \varphi \in \Gamma(U, \mathcal{O}_X(D)). \]

This proves that \( \mathcal{O}_X(D) \) is quasicoherent. It remains to show that it is coherent. Let \( U \subseteq X \) be affine. Let \( Y_1, \ldots, Y_r \) be the prime divisors that appear in \( D \) with positive coefficients. Choose

\[ g \in \prod_{i=1}^r I_U(Y_i). \]

If \( m \gg 0 \), \( \text{div}(g^m)|_U \geq D|_U \). Then

\[ \Gamma(U, \mathcal{O}_X(D)) \subseteq \{ \varphi \mid \text{div}(\varphi \cdot g^m) \geq 0 \} \cup \{ 0 \} = \frac{1}{g^m} \mathcal{O}_X(U) \]

which is clearly finitely generated over \( \mathcal{O}_X(U) \). This implies coherence.
If $D = \sum n_i V_i$ with $n_i \neq 0$, set

$$U = X \setminus \bigcup_i V_i.$$ 

Then $D|_U = 0$. Hence $O_X(D)|_U = O_U$. This implies the assertion about the stalk. \hfill \Box

**Proposition 2.4.10.** For divisors $D, E \in \text{Div}(X)$, $O_X(D) \cong O_X(E)$ if and only if $D \sim E$.

**Proof.** For the ‘if’ implication, suppose $D = E + \text{div}(\alpha)$ for some $\alpha$. Then $\varphi \in \Gamma(U, O_X(D))$ if and only if

$$(\text{div}(\varphi) + \text{div}(\alpha) + E)|_U \geq 0,$$

which is equivalent to $\varphi \alpha \in \Gamma(U, O_X(E))$. This gives an isomorphism $O_X(D) \to O_X(E)$ given on each open subset by multiplication by $\alpha$. The converse implication will be proved in the problem session. \hfill \Box

**Remark 2.4.11.** There is a canonical isomorphism

$$O_X(D) \otimes_{O_X} O_X(E) \to O_X(D + E)$$

induced by multiplication of rational functions.

Next, we describe the push-forward of Weil divisors.

**Definition 2.4.12.** If $f : X \to Y$ is a dominant of irreducible varieties with $\dim X = \dim Y$, then $k(Y) \hookrightarrow k(X)$ is finite, and we define the degree of $f$ as $\deg(f) = [k(X) : k(Y)]$, the degree of this extension.

**Definition 2.4.13.** Suppose $f : X \to Y$ is a finite surjective morphism. We define the push-forward as

$$f_* : \text{Div}(X) \to \text{Div}(Y)$$

$$\sum_{i=1}^{r} n_i V_i \mapsto \sum_{i=1}^{r} n_i \cdot \deg(V_i/f(V_i)) f(V_i)$$

**Proposition 2.4.14.** Let $f : X \to Y$ be a finite surjective morphism of varieties which are smooth in codimension 1. Then

$$f_*(\text{div}(\varphi)) = \text{div}(N_{k(X)/k(Y)}(\varphi)).$$

In particular, we get a map $f_* : \text{Cl}(X) \to \text{Cl}(Y)$.

**Proof.** We need to show that for any prime divisor $W \subseteq Y$

$$\sum_{V \subseteq X \text{ prime divisor such that } f(V) = W} \text{ord}_V(\varphi)[k(V) : k(W)] = \text{ord}_W(N_{k(X)/k(Y)}(\varphi)).$$

Replace $Y$ by $U$ affine such that $U \cap W \neq \emptyset$ and $X$ by $f^{-1}(U)$ to assume that $X$ and $Y$ are affine and $A = \mathcal{O}(Y)$, $B = \mathcal{O}(X)$. We then get a finite injective map

$$\varphi A \hookrightarrow B$$

and if $p \subseteq A$ is the ideal corresponding to $W$, then

$$A_p \hookrightarrow B_p$$
is finite and injective. Note that $A_p$ is a DVR. The maximal ideals $q_1, \ldots, q_s$ in $B_p$ are the localizations of the primes corresponding to the prime divisors $V$ such that $f(V) = W$, and by assumption $(B_p)_{q_i}$ is a DVR. We may assume $\varphi \in B$ (by writing it as a quotient of two functions in $B$). Then for $V_i = V(p_i)$

$$\text{ord}_{V_i}(\varphi) = \ell((B_p)_{q_i}/(\varphi)).$$

Then result then follows from Problem 2 from Problem Session 4. \qed

2.5. Cartier divisors. First, we discuss Cartier divisors on normal varieties. Let $X$ be a normal variety and $\text{Div}(X)$ be its group of divisors.

**Definition 2.5.1.** A divisor $D$ is *locally principal* if there is an open cover

$$X = \bigcup_{i \in I} U_i$$

such that for any $i$, $D|_{U_i}$ is a principal divisor, i.e. there exists $\varphi_i \in k(X)^*$ such that $D|_{U_i} = \text{div}(\varphi_i)|_{U_i}$.

A *Cartier divisor* is a locally principal divisor, and we write

$$\text{Cart}(X) \subseteq \text{Div}(X)$$

for the subgroup of Cartier divisors.

**Proposition 2.5.2.** A divisor $D$ on $X$ is locally principal if and only if $\mathcal{O}_X(D)$ is a line bundle.

**Proof.** If for $U \subseteq X$, $D|_U$ is principal, then $\mathcal{O}_X(D)|_U = \mathcal{O}_U(D|_U) \cong \mathcal{O}_U$, which shows that ‘only if’ implication. Conversely, if $\mathcal{O}_X(D)$ is a line bundle, we can cover $X$ by open $U_i$ such that

$$\mathcal{O}_{U_i}(D|_{U_i}) = \mathcal{O}_X(D)|_{U_i} \cong \mathcal{O}_{U_i}.$$  

Then by Proposition 2.4.10, $D|_{U_i}$ is principal. \qed

**Proposition 2.5.3.** If $D, E \in \text{Cart}(X)$, the map

$$\mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(E) \to \mathcal{O}_X(D + E)$$

is an isomorphism.

**Proof.** If $U \subseteq X$ is an open subset and $D|_U = \text{div}(\varphi)|_U$, then for an open subset $V \subseteq U$

$$\Gamma(V, \mathcal{O}_X(D)) = \{ \psi \mid \text{div}(\psi)|_V + \text{div}(\varphi)|_V \geq 0 \}.$$  

Note that $\text{div}(\psi)|_V + \text{div}(\varphi)|_V \geq 0$ if and only if $\varphi \psi \in \mathcal{O}_X(V)$. Therefore

$$\mathcal{O}_X(D)|_U = \frac{1}{\varphi} \mathcal{O}_U \subseteq k(X).$$

If $X = \bigcup U_i$ for affine open subsets $U_i \subseteq X$ such that

$$D|_{U_i} = \text{div}(\varphi_i)|_{U_i}, \quad E|_{U_i} = \text{div}(\psi_i)|_{U_i},$$

then

$$(D + E)|_{U_i} = \text{div}(\varphi_i \psi_i)|_{U_i}.$$
Therefore, on $U_i$, the morphism above is the map $$\frac{1}{\varphi_i}O_X(U_i) \otimes_{O_X(U_i)} \frac{1}{\psi_i}O_X(U_i) \rightarrow \frac{1}{\varphi_i\psi_i}O_X(U_i),$$ which is clearly an isomorphism. \qed

Therefore, we have a group homomorphism

$$\text{Cart}(X) \rightarrow \text{Pic}(X)$$

$$D \mapsto O_X(D)$$

with kernel $\text{PDiv}(X) \subseteq \text{Cart}(X)$, and hence we get an injective map

$$\text{Cart}(X) \twoheadrightarrow \text{Pic}(X).$$

We will see later that this is an isomorphism.

**Remark 2.5.4.** Arguing like in the proof of Proposition 2.4.4, we see that

$$\text{Div}(X) = \text{Cart}(X)$$

if and only if $X$ is locally factorial.

In particular, this is the case for smooth varieties.

**Example 2.5.5.** This implies that $\text{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$.

**Exercise.** Show that if $H \subseteq \mathbb{P}^n$ is a hyperplane, then $O_{\mathbb{P}^n}(H) \cong O_{\mathbb{P}^n}(1)$.

We now generalize the notion of Cartier divisors to all irreducible varieties. Suppose $X$ is irreducible and let $k(X)$ be the field of rational functions on $X$. We will write $k(X)^*$ for the constant sheaf (previously denoted by $\overline{k(X)}$).

We have the short exact sequence

$$0 \rightarrow O_X^* \hookrightarrow k(X)^* \twoheadrightarrow k(X)^*/O_X^* \rightarrow 0.$$

**Definition 2.5.6.** The set of **Cartier divisors** is

$$\text{Cart}(X) = \Gamma(X, k(X)^*/O_X^*).$$

Explicitly, a **Cartier divisor** is given by an open cover $X = \bigcup_i U_i$ together with $\varphi_i \in k(X)^*$ such that for any $i, j$

$$\frac{\varphi_i}{\varphi_j} \in O_X(U_i \cap U_j)^*.$$

Two Cartier divisors $D, E$ given by such data are equal if, refining the covers to assume they are the same, when $D$ is given by $(\varphi_i)_{i \in I}$ and $E$ is given by $(\psi_i)_{i \in I}$, we have that

$$\frac{\varphi_i}{\psi_i} \in O_X(U_i)^*.$$

Note that $D + E$ is given by $(\varphi_i \cdot \psi_i)_{i \in I}$.

We finally define the group of **principal Cartier divisors** as

$$\text{PCart}(X) = \text{im}(k(X)^* = \Gamma(X, k(X)^*) \rightarrow \Gamma(X, k(X)^*/O_X^*)),$$

i.e. a divisor is **principal** if it comes from a global section.
Given any Cartier divisor $D$ describe with respect to $X = \bigcup_i U_i$ by $(\varphi_i)_{i \in I}$, consider for each $i_i \varphi_i^{-1} O_{U_i} \subseteq k(X)$. Since for $\varphi_i \varphi_j \in O_X(U_i \cap U_j)^*$ we have

$$\frac{1}{\varphi_i} O_{U_i \cap U_j} = \frac{1}{\varphi_j} O_{U_i \cap U_j},$$

there is a unique subsheaf $O_X(D) \subseteq k(X)$ such that

$$O_X(D)|_{U_i} = \frac{1}{\varphi_i} O_{U_i}.$$

Note that by definition $O_X(D)$ is a line bundle. As in the normal case, we hence get a map

$$\text{Cart}(X) \to \text{Pic}(X)$$

$$D \mapsto O_X(D).$$

**Exercise.**

1. This is a group homomorphism: $O_X(D) \otimes O_X(E) \cong O_X(D + E)$.
2. $O_X(D) \cong O_X$ if and only if $D \in \text{PCart}(X)$.

Therefore, we get a map

$$\frac{\text{Cart}(X)}{\text{PCart}(X)} \hookrightarrow \text{Pic}(X).$$

**Proposition 2.5.7.** The map

$$\frac{\text{Cart}(X)}{\text{PCart}(X)} \hookrightarrow \text{Pic}(X).$$

is an isomorphism.

**Proof.** We need to show that if $\mathcal{L} \in \text{Pic}(X)$, then there is a Cartier divisor $D$ such that $O_X(D) \cong \mathcal{L}$.

Choose an open cover $X = \bigcup_{i \in I} U_i$ and isomorphisms $\alpha_i: \mathcal{L}|_{U_i} \to O_{U_i}$ with transition functions

$$\alpha_i|_{U_i \cap U_j} \circ \alpha_j^{-1}|_{U_i \cap U_j}: O_{U_i \cap U_j} \xrightarrow{\cong} O_{U_i \cap U_j}$$

given by multiplication with some $\alpha_{i,j} \in O_X(U_i \cap U_j)^*$ which satisfy:

1. $\alpha_{i,i} = 1$,
2. $\alpha_i, j \alpha_{j,k} = \alpha_{i,k}$ in $k(X)$ (since $U_i \cap U_j \cap U_0$ is dense in $X$, as $X$ is irreducible)

Define $\varphi_i = \alpha_{i,i_0} \in (X)^*$ for all $i$ and some $i_0$. Then

$$\frac{\varphi_i}{\varphi_j} = \frac{\alpha_{i,i_0}}{\alpha_{j,j_0}} = \alpha_{i,j} \in O_X(U_i \cap U_j)^*.$$

Therefore, the $\varphi_i$ define a divisor $D$. It is easy to see that $O_X(D) \cong \mathcal{L}$ (the local isomorphisms $O_X(D)|_{U_i} = \frac{1}{\varphi_i} O_{U_i} \cong O_{U_i} \cong \mathcal{L}|_{U_i}$ glue together).
We finally compare the two definitions of Cartier divisors. Suppose $X$ is smooth in codimension 1 and $D$ is a Cartier divisor on $X$ described by $X = \bigcup_{i \in I} U_i$ and $\varphi_i \in k(X)^*$. Consider $\text{div}(\varphi)|_{U_i}$. Since $\varphi_i \varphi_j \in \mathcal{O}_X^*(U_i \cap U_j)$,

$$\text{div}(\varphi_i)|_{U_i \cap U_j} = \text{div}(\varphi_j)|_{U_i \cap U_j}.$$ 

Therefore, there is a unique Weil divisor $\alpha(D)$ such that $\alpha(D)|_{U_i} = \text{div}(\varphi_i)|_{U_i}$.

We get a group homomorphism

$$\text{Cart}(X) \to \text{Div}(X).$$

If $X$ is normal, this map is injective, since $\text{div}(\varphi_i/\psi_i) = 0$ implies that $\varphi_i/\psi_i \in \mathcal{O}_X(U_i)^*$ by normality.

Moreover, the image consists of the locally principal divisors on $X$.

Therefore, on normal varieties, we can identify Cartier divisors with locally principal Weil divisors, as we did in Definition 2.5.1.

Next, we will define the pull-back of Cartier divisors. Let $X \to Y$ be a dominant morphism of irreducible varieties and $\nu: k(Y) \to k(X)$ be the corresponding map. We define the pull-back map

$$f^*: \text{Cart}(Y) \to \text{Cart}(X).$$

For $D$ described by an open cover $Y = \bigcup_i U_i$ with $\varphi \in k(U_i)^*$, we define $f^*(D)$ with respect to $X = \bigcup_{i \in I} f^{-1}(U_i)$ by $(\nu(\varphi_i))_{i \in I}$.

It is easy to see that

1. this definition is independent of the presentation of $D$,
2. $f^*$ is a group homomorphism preserving PCart:

$$\begin{array}{ccc}
\text{Cart}(Y) & \xrightarrow{f^*} & \text{Cart}(X) \\
\uparrow & & \uparrow \\
\text{PCart}(Y) & \longrightarrow & \text{PCart}(X)
\end{array}$$

and hence induces a commutative square

$$\begin{array}{ccc}
\text{Cart}(Y) & \xrightarrow{f^*} & \text{Cart}(X) \\
\downarrow \cong & & \downarrow \cong \\
\text{PCart}(Y) & \longrightarrow & \text{PCart}(X) \\
\downarrow \cong & & \downarrow \cong \\
\text{Pic}(Y) & \xrightarrow{f^*} & \text{Pic}(X).
\end{array}$$

**Fact 2.5.8.** Suppose $f: X \to Y$ be a finite surjective map of irreducible varieties smooth in codimension 1. For $D \in \text{Cart}(Y)$,

$$f_*(f^*(D)) = \deg(f) \cdot D$$

in $\text{Div}(Y)$. 
The proof of this will be a homework problem.

2.6. **Effective Cartier divisors.** We finally discuss effective Cartier divisors.

**Definition 2.6.1.** Let \( X \) be an irreducible variety and \( D \) be a Cartier divisor given with respect to \( X = \bigcup_{i \in I} U_i \) by \((\varphi_i)_{i \in I}\). Then \( D \) is effective if \( \varphi_i \in \mathcal{O}_X(U_i) \) for any \( i \in I \) (this is independent of the presentation of \( D \)).

It is clear that if \( X \) is smooth in codimension and \( D \) is an effective divisor, then the corresponding Weil divisor is effective. The converse holds if \( X \) is normal.

We give an equivalent description of effective Cartier divisors.

**Definition 2.6.2.** A coherent ideal \( \mathcal{I} \subseteq \mathcal{O}_X \) is locally principal if for any \( x \in X \), there exists an open affine neighborhood \( U \) of \( x \) such that \( \Gamma(U, \mathcal{I}) \subseteq \mathcal{O}_X(U) \) is generated by a non-zero element.

**Proposition 2.6.3.** There is a bijection between effective Cartier divisors on \( X \) and locally principal ideals in \( \mathcal{O}_X \) given by \( D \mapsto \mathcal{O}_X(-D) \).

**Proof.** Suppose that \( D \) is described by \( X = \bigcup_{i \in I} U_i \) and \((\varphi_i)_{i \in I}\). Then
\[
\mathcal{O}_X(-D)|_{U_i} = \mathcal{O}_X(U_i) \cdot \varphi_i \subseteq \mathcal{O}_X(U_i).
\]
Conversely, if \( \mathcal{I} \subseteq \mathcal{O}_X \) is a locally principal ideal, then there is an affine open cover \( X = \bigcup_{i \in I} U_i \) such that
\[
\Gamma(U_i, \mathcal{I}) = \beta_i \mathcal{O}_X(U_i)
\]
and \( \frac{\beta_i}{\beta_j} \in \mathcal{O}_X(U_i \cap U_j)^* \), so \((\beta_i)_{i \in I}\) defines an effective Cartier divisor.

It is finally easy to see that the two maps are inverse to each other. \( \square \)

**Definition 2.6.4.** If \( D \) is an effective Cartier divisor, we have an exact sequence
\[
0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X/\mathcal{O}_X(-D) \longrightarrow 0
\]
We define the **structure sheaf of** \( D \) as \( \mathcal{O}_D = \mathcal{O}_X/\mathcal{O}_X(-D) \).

We define the **support of** \( D \) as \( \text{supp}(D) = V(\mathcal{O}_X(-D)) \subseteq X \), a closed subset of \( X \).

**Example 2.6.5.** Suppose \( \mathcal{I} \) is an ideal sheaf on \( X \) and \( \pi: Y = \text{Bl}_\mathcal{I} X \to X \) is the blow-up of \( X \) along \( \mathcal{I} \). Since \( \mathcal{I} \cdot \mathcal{O}_Y \) is locally principal, there exists an effective Cartier divisor \( E \) such that \( \mathcal{I} \mathcal{O}_Y = \mathcal{O}_Y(-E) \). We then call \( E \) the **exceptional divisor.**

**Exercise.** If \( X \) is normal and \( D \) is an effective Cartier divisor, then \( \mathcal{O}_X(-D) \) is a radical ideal if and only if all coefficients of \( D \) are 1.

In general, if \( D \) is an effective Cartier divisor, then we define \( \text{supp}(D) = V(\mathcal{O}_X(-D)) \).
Effective Cartier divisors as zero-loci of sections of line bundles. Let $X$ be an irreducible variety, $\mathcal{L}$ be line bundle, $s \in \Gamma(X, \mathcal{L})$ and $s \neq 0$. By tensoring the map

$$\mathcal{O}_X \to \mathcal{L}$$

$$1 \mapsto s$$

with $\mathcal{L}^{-1}$, we get a map $\alpha: \mathcal{L}^{-1} \to \mathcal{O}_X$ whose image is a coherent ideal $\mathcal{I}$ in $\mathcal{O}_X$.

We claim that this is an injective morphism and $\mathcal{I}$ is locally principal.

Suppose $U \subseteq X$ is an open affine subset such that $\mathcal{L}|_U \cong \mathcal{O}_U$ and $s$ is sent to $f \in \mathcal{O}(U)$ via this isomorphism. This induces an isomorphism $\mathcal{L}^{-1}|_U \cong \mathcal{O}_U$ and $\alpha|_U$ gets identified with $\mathcal{O}_U \xrightarrow{f} \mathcal{O}_U$ which is clearly injective, and $\mathcal{I}|_U$ is generated by $f$.

Therefore, there is a Cartier divisor $Z(s)$ such that $\mathcal{I} = \mathcal{O}(-Z(s))$, called the zero locus of $s$.

Note that $x \in X$ lies in $Z(s)$ if and only if $s(x) \in \mathcal{L}(x)$ is 0.

**Proposition 2.6.6.**

\[ (1) \] By construction, $\mathcal{O}(-Z(s)) \cong \mathcal{L}^{-1}$, i.e. $\mathcal{O}(Z(s)) \cong \mathcal{L}$.

\[ (2) \] If $s' \in \Gamma(X, \mathcal{L}) \setminus \{0\}$, then $Z(s) = Z(s')$ if and only if $s = gs'$ for some $g \in \mathcal{O}(X)^*$.

\[ (3) \] If $D$ is an effective divisor such that $\mathcal{O}(D) \cong \mathcal{L}$, then there is a section $s \in \Gamma(X, \mathcal{L}) \setminus \{0\}$ such that $Z(s) = D$.

**Proof.** Parts (1) and (2) are immediate, so we just need to show (3). Suppose $X = \bigcup U_i$ is an open cover such that $D$ is described by $(\varphi_i)_{i \in I}$ for $\varphi_i \in \mathcal{O}_X(U_i)$. Then

$$\mathcal{O}(D)|_{U_i} = \frac{1}{\varphi_i} \mathcal{O}_{U_i} \subseteq k(X),$$

so $1 \in \Gamma(X, \mathcal{O}_X(D)))$. Checking that the zero-locus of 1 is $D$ is left as an exercise. Then we map this section to $\Gamma(X, \mathcal{L})$ via the isomorphism. \qed

**Remark 2.6.7.** Suppose $X$ is complete. We will see later that $\Gamma(X, \mathcal{L})$ is a finite-dimensional vector space over $k$. Therefore,

$$\left\{ \begin{array}{c} D \text{ effective Cartier} \\ \text{divisor such that } \mathcal{O}(D) \cong \mathcal{L} \end{array} \right\} \cong \left\{ \begin{array}{c} \text{projective space parametrizing} \\ \text{lines in } \Gamma(X, \mathcal{L}) \end{array} \right\}.$$  

This is called the linear system corresponding to $\mathcal{L}$ and denoted $|\mathcal{L}|$.

### 3. Cohomology

#### 3.1. Derived functors.

Fix the category $\mathcal{C}$ to be $\mathcal{O}_X$-modules for some ringed space $(X, \mathcal{O}_X)$. In general, we could let $\mathcal{C}$ be any abelian category but we will stick to $\mathcal{C} = \mathcal{O}_X$-mod to simplify the exposition, since this is the only case we will be interested in.

**Definition 3.1.1.** A complex of objects in $\mathcal{C}$

$$A^* : \cdots \to A^m \xrightarrow{d^m} A^{m+1} \to \cdots$$

is a collection $(A^m)_{m \in \mathbb{Z}^+}$ with maps $d^m : A^m \to A^{m+1}$ such that $d^m \circ d^{m-1} = 0$. 


A morphism of complexes $u: A^\bullet \to B^\bullet$ is a sequence of maps $u^m: A^m \to B^m$ for all $m \in \mathbb{Z}^+$ such that $d \circ u^m = u^{m+1} \circ d$ for all $m \in \mathbb{Z}^+$.

Since the morphisms can be composed component-wise, complexes in $\mathcal{C}$ form a category. This category has kernels and cokernels, described componentwise, which make it into an abelian category.

**Definition 3.1.2.** If $A^\bullet$ is a complex, define for $i \in \mathbb{Z}$ the $i$th cohomology functor by letting

$$
\mathcal{H}^i(A^\bullet) = \frac{\ker(A^i \to A^{i+1})}{\text{im}(A^{i-1} \to A^i)} \in \mathcal{C}
$$

and for $u: A^\bullet \to B^\bullet$, $\mathcal{H}^i(u)$ to be the natural map

$$
\mathcal{H}^i(A^\bullet) \to \mathcal{H}^i(B^\bullet).
$$

**Proposition 3.1.3 (Long exact sequence in cohomology).** Given an exact sequence of complexes

$$
0 \longrightarrow A^\bullet \overset{u}{\longrightarrow} B^\bullet \overset{v}{\longrightarrow} C^\bullet \longrightarrow 0
$$

there is a connecting map $\delta$ that makes the sequence

$$
\cdots \longrightarrow \mathcal{H}^i(A^\bullet) \overset{\mathcal{H}^i(u)}{\longrightarrow} \mathcal{H}^i(B^\bullet) \overset{\mathcal{H}^i(v)}{\longrightarrow} \mathcal{H}^i(C^\bullet) \overset{\delta}{\longrightarrow} \mathcal{H}^{i+1}(A^\bullet) \longrightarrow \cdots
$$

exact. Moreover, this is functorial with respect to morphisms of exact sequences of complexes.

**Sketch of proof.** We first define $\delta$. Given $s \in \Gamma(U, \mathcal{H}^i(C^\bullet))$ for an open neighborhood $U$ of $x \in X$, we can find a lift $s'(x) \in \Gamma(U_x, \ker(C^i \to C^{i+1}))$ of $s|_{U(x)}$ where $U_x$ is an open neighborhood of $x$. After passing to the smaller $U_x$, we may assume that $s'(x) = v(s''(x))$ for some $s''(x) \in \Gamma(U_x, B^i)$, there exists $t(x) \in \Gamma(U_x, A^{i+1})$ such that $u(t(x)) = d(s''(x))$. It is easy to see that

$$
t(x) \in \Gamma(U_x, \ker(A^{i+1} \to A^{i+2})).
$$

The images $\overline{t(x)} \in \Gamma(U(x), \mathcal{H}^{i+1}(A^\bullet))$ glue together, giving $\delta(s)$.

To check exactness, pass to stalks and just deal with modules over a ring. This is left as an exercise. ☐

**Definition 3.1.4.** Two morphisms of complexes $u, v: A^\bullet \to B^\bullet$ are homotopic ($u \approx v$), if there are map $\theta^i: A^i \to B^{i-1}$

$$
\begin{array}{ccccccccc}
A^{i-1} & \overset{d}{\longrightarrow} & A^i & \overset{d}{\longrightarrow} & A^{i+1} \\
\downarrow & & \downarrow & & \downarrow \\
B^{i-1} & \overset{d}{\longrightarrow} & B^i & \overset{d}{\longrightarrow} & B^{i+1}
\end{array}
$$

such that $u^i - v^i = d \circ \theta^i + \theta^{i+1} \circ d$ for all $i$.

Note that if $u \approx v$ then $\mathcal{H}^i(u) = \mathcal{H}^i(v)$ for all $i$.

**Definition 3.1.5.** Let $\mathcal{A}$ be an abelian category. Then $Q \in \text{Ob}\mathcal{A}$ is injective if $\text{Hom}_\mathcal{A}(-, Q)$ is exact.
Exercise. If $Q_i$ are injective objects, then $\prod_{i \in I} Q_i$ is injective.

**Definition 3.1.6.** We say that $\mathcal{A}$ has *enough injectives* if for any $A \in \text{Ob}(\mathcal{A})$ there is an injective map $$A \to Q$$ with $Q$ injective.

**Remark 3.1.7.** Review Sheet 4 proves that the category of $R$-modules has enough injectives. We use this to show that the category of $\mathcal{O}_X$-modules also has enough injectives.

**Proposition 3.1.8.** The category $\mathcal{O}_X$-$\text{mod}$ has enough injectives.

**Proof.** Suppose $x \in X$ and $A$ is an $\mathcal{O}_{X,x}$-module. Define an $\mathcal{O}_X$-module $A(x)$ by $$\Gamma(U, A(x)) = \begin{cases} A & \text{if } x \in U, \\ 0 & \text{otherwise} \end{cases}$$ (with $\mathcal{O}(U)$ acting via $\mathcal{O}(U) \to \mathcal{O}_{X,x}$ for $x \in U$). If $\mathcal{F}$ is an $\mathcal{O}_X$-mod, $$\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, A(x)) \cong \text{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, A).$$ (proving this is left as an exercise). Therefore, if $A$ is an injective $\mathcal{O}_{X,x}$-module, $A(x)$ is an injective $\mathcal{O}_X$-module.

Given an $\mathcal{O}_X$-module $\mathcal{M}$, consider for each $x \in X$, and injective morphism $$\mathcal{M}_x \hookrightarrow I^{(x)}$$ where $I^{(x)}$ is an injective $\mathcal{O}_{X,x}$-module. Then consider $$\mathcal{M} \hookrightarrow \prod_{x \in X} (\mathcal{M}_x)(x) \hookrightarrow \prod_{x \in X} (I^{(x)})(x).$$ This gives an embedding of $\mathcal{M}$ in an injective $\mathcal{O}_X$-module. \qed

**Definition 3.1.9.** A resolution of $\mathcal{M} \in \text{Ob}(\mathcal{C})$ is a complex $A^\bullet$ with $A^i = 0$ for $i < 0$ and with a morphism of complexes $\mathcal{M} \to A^\bullet$ inducing an isomorphism in cohomology. Equivalently, $A^\bullet$ is a resolution if $$0 \longrightarrow \mathcal{M} \longrightarrow A^0 \longrightarrow A^1 \longrightarrow \cdots$$ is exact.

An injective resolution of $\mathcal{M}$ is a resolution $A^\bullet$ with all $A^i$ injective.

**Proposition 3.1.10.**

1. Given any $\mathcal{M} \in \text{Ob}(\mathcal{C})$, $\mathcal{M}$ has an injective resolution.
2. Suppose we have $$\begin{array}{cccccc} 0 & \longrightarrow & \mathcal{M} & \longrightarrow & A^0 & \longrightarrow & A^1 & \longrightarrow & \cdots \\ & & & & \downarrow{\alpha} & & & & \\ 0 & \longrightarrow & \mathcal{N} & \longrightarrow & B^0 & \longrightarrow & B^1 & \longrightarrow & \cdots \end{array}$$
such that the top row is a resolution of $\mathcal{M}$ and the bottom row is a complex with all $B^i$ injective, there is a morphism $u: A^\bullet \to B^\bullet$ such that

$$\mathcal{M} \xrightarrow{\alpha} \xrightarrow{u} A^\bullet \xrightarrow{\alpha} \xrightarrow{u} B^\bullet$$

commutes.

(3) If $v: A^\bullet \to B^\bullet$ is another such morphism, then $u \approx v$.

**Proof.** For (1), by Proposition 3.1.8, there is an embedding $\mathcal{M} \hookrightarrow I^0$ with $I^0$ injective. Then apply Proposition 3.1.8 again to get and embedding

$I^0/\mathcal{M} \hookrightarrow I^1$ with $I^1$ injective.

This gives an exact sequence

$$0 \to \mathcal{M} \to I^0 \to I^1.$$

Repeating this, we obtain an injective resolution of $\mathcal{M}$.

For (2), we first get $u^0$:

$$0 \to \mathcal{M} \to A^0 \to A^1 \to \cdots$$

$$\downarrow \alpha \quad \quad \downarrow u^0$$

$$0 \to \mathcal{N} \to B^0 \to B^1 \to \cdots$$

since the map $\mathcal{M} \hookrightarrow A^0$ in injective and $B^0$ is an injective object. Then we have

$$A^0/\mathcal{M} \xleftarrow{u^0} A^1$$

$$\downarrow u^0 \quad \downarrow u^1$$

$$\text{coker}(\mathcal{N} \to B^0) \xrightarrow{u^1} B^1$$

where we get $u^1$ since $B^1$ is an injective object. Continuing this way, we get the chain map $u$.

For (3), suppose we have

$$0 \to \mathcal{M} \to A^0 \to A^1 \to \cdots$$

$$\downarrow \alpha \quad \quad \downarrow v^0 \quad \downarrow v^1 \quad \downarrow u^1$$

$$0 \to \mathcal{N} \to B^0 \to B^1 \to \cdots$$

The map $u_0 - v_0$ induces a map

$$A^0/\mathcal{M} \xrightarrow{\theta^1} A^1$$

$$\downarrow \quad \quad \quad \downarrow \theta^1$$

$$B^0 \quad \quad \quad B^1$$
and since $B^0$ is injective we get a map $\theta^1$ as in the diagram above such that $\theta^1 \circ d = u^0 - v^0$. Then $u^1 - v^1 - d \circ \theta^1$ vanishes on $\text{im}(A^0 \to A^1)$ construction. Hence, it induces a map

$$
coker(A^0 \to A^1) \xrightarrow{\theta^2} A^2 \xrightarrow{\kappa} B^1
$$

such that $\theta^2 \circ d = u^1 - v^1$. Continuing this way, we get the desired homotopy showing $u \approx v$. □

**Proposition 3.1.11 (Horseshoe Lemma).** Given an exact sequence

$$
0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0
$$

and an injective resolution $(I')^\bullet$, $(I'')^\bullet$ for $I'$, $I''$ respectively, there is a commutative diagram of complexes

$$
\begin{array}{cccccc}
0 & \longrightarrow & F' & \longrightarrow & F & \longrightarrow & F'' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (I')^\bullet & \longrightarrow & (I')^\bullet \oplus (I'')^\bullet & \longrightarrow & (I'')^\bullet & \longrightarrow & 0.
\end{array}
$$

In particular,

$$
F \rightarrow (I')^\bullet \oplus (I'')^\bullet
$$

is an injective resolution of $F$.

**Sketch of proof.** We will construct the first maps $\beta = (\beta_1, \beta_2)$:

$$
\begin{array}{cccccc}
0 & \longrightarrow & F' & \longrightarrow & F & \longrightarrow & F'' & \longrightarrow & 0 \\
\downarrow & \alpha & & \downarrow & \beta & & \downarrow & \gamma & \\
0 & \longrightarrow & (I')^0 & \longrightarrow & (I')^0 \oplus (I'')^0 & \longrightarrow & (I'')^0 & \longrightarrow & 0.
\end{array}
$$

where $\beta_2$ is the composition $F \to F'' \to (F'')^0$ and $\beta$ is the unique map making the diagram commutative, which exists by injectivity of $(I')^0$. The injectivity of $\beta$ follows.

By the Snake Lemma, we then get an exact sequence

$$
0 \longrightarrow \text{coker } \alpha \longrightarrow \text{coker } \beta \longrightarrow \text{coker } \gamma \longrightarrow 0.
$$
Repeat the above argument to show the result. □

**Right derived Functors.** Let $F: C \to D$ be a left exact functor for two additive categories. We will usually consider $C$ and $D$ to be categories of $\mathcal{O}_X$-modules and $\mathcal{O}_Y$-modules.

**Examples 3.1.12.**

1. Let $C$ be the category of $\mathcal{O}_X$-modules, where $\mathcal{O}_X$ is a sheaf of $R$-algebras. Then
   $$F = \Gamma(X, -): C \to R\text{-mod}$$
   is a left exact functor.

2. Let $f: (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Then
   $$f_*: \mathcal{O}_X\text{-mod} \to \mathcal{O}_Y\text{-mod}$$
   is a left exact functor.
   For example, if $Y$ is a point, $\mathcal{O}_Y = R$, so we recover example 1.

3. Consider $(X, \mathcal{O}_X)$ where $\mathcal{O}_X$ is a sheaf of $R$-algebras and let $F$ be an $\mathcal{O}_X$-module. Then
   $$\text{Hom}_{\mathcal{O}_X}(F, -): \mathcal{O}_X\text{-mod} \to R\text{-mod}$$
   is a left exact functor.

4. Consider $(X, \mathcal{O}_X)$ and let $F$ be an $\mathcal{O}_X$-module. Then
   $$\text{Hom}_{\mathcal{O}_X}(F, -): \mathcal{O}_X\text{-mod} \to \mathcal{O}_X\text{-mod}$$
   is a left exact functor.

The idea is that in general $F$ is not exact, and we want to measure the failure of right exactness.

**Definition 3.1.13.** A $\delta$-functor is given by a sequence of functors $(F^i)_{i \geq 0}$ and for any short exact sequence

$$0 \longrightarrow \mathcal{M}' \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}'' \longrightarrow 0$$

a connecting homomorphism

$$F^i(\mathcal{M}'') \overset{\delta}{\longrightarrow} F^{i+1}(\mathcal{M}')$$

which is functorial with respect to morphisms of short exact sequences, and for every short exact sequence as above, we have a long exact sequence:

$$0 \longrightarrow F^0(\mathcal{M}') \longrightarrow F^0(\mathcal{M}) \longrightarrow F^0(\mathcal{M}'') \longrightarrow \cdots$$

A morphism of $\delta$-functors $(F^i)_{i \geq 0} \to (G^i)_{i \geq 0}$ is a collection of functorial transformations $F^i \to G^i$ for all $i$ such that any short exact sequence

$$0 \longrightarrow \mathcal{M}' \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}'' \longrightarrow 0$$

the diagram
\[
\begin{array}{c}
F^i(M''') \xrightarrow{\delta} F^{i+1}(M') \\
\downarrow \quad \downarrow \\
G^i(M''') \xrightarrow{\delta} G^{i+1}(M')
\end{array}
\]

commutes. (This implies that we get a morphism between the long exact sequences.)

**Theorem 3.1.14.** Given a left exact functor \(F: C \to D\), there exists a \(\delta\)-functor \((R^iF)_{i \geq 0}\) such that

1. \(R^0F \cong F\),
2. \(R^iF(I) = 0\) for all injective objects \(I\) of \(C\), \(i \geq 1\).

Moreover, such a \(\delta\)-functor is unique up to canonical isomorphism of \(\delta\)-functors. In fact, given any \(\delta\)-functor \((G^i)_{i \geq 0}\) and a natural transformation \(F \xrightarrow{T} G^0\), there is a unique morphism of \(\delta\)-functors \((R^iF)_{i \geq 0} \to (G^i)_{i \geq 0}\) which extends \(T\) for \(i = 0\).

**Proof.** We begin by showing existence. Choose for each object \(M\) an injective resolution \(M \to I^\bullet\) and define

\[R^iF(M) := H^i(F(I^\bullet)).\]

Given \(u: M_1 \to M_2\), by Proposition 3.1.10, choose \(\tilde{u}: I_1^\bullet \to I_2^\bullet\) that makes the diagram commute and define

\[(R^iF)(u) = H^i(F(\tilde{u})).\]

If \(\tilde{u}'\) is another such map then \(\tilde{u} \approx \tilde{u}'\), so \(R^iF(u) = R^iF(u')\).

It is easy to see that this is functorial (using independence of the choice of \(\tilde{u}\)).

This also implies that if \((I'^\bullet)\) is another injective resolution of \(M\), we have an isomorphism

\[R^iF(M) \cong H^i(F((I'^\bullet))).\]

In particular, if \(M\) is injective, we can choose an injective resolution \(0 \to M \to M \to 0\) of \(M\), and hence

\[R^iF(M) = 0\] for \(i \geq 1\).

If

\[0 \longrightarrow M \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots\]

is an injective resolution of \(M\) and \(F\) is left exact, then

\[0 \longrightarrow F(M) \longrightarrow F(I^0) \longrightarrow F(I^1) \longrightarrow \cdots\]
is exact, so we get a functorial isomorphism \( R^0 F \cong F \).

We claim that we can define a connecting homomorphism in a functorial way. Given a short exact sequence

\[
0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0
\]

and choosing resolutions \( M' \to (I')^* \), \( M \to (I)^* \), \( M'' \to (I'')^* \), Proposition 3.1.11 gives a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & F' & \longrightarrow & F & \longrightarrow & F'' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (I')^* & \longrightarrow & (I')^* \oplus (I'')^* & \longrightarrow & (I'')^* & \longrightarrow & 0.
\end{array}
\]

Since \( F \) is additive we have a short exact sequence of complexes:

\[
0 \longrightarrow F((I')^*) \longrightarrow F((I')^* \oplus (I'')^*) \longrightarrow F((I'')^*) \longrightarrow 0.
\]

By Proposition 3.1.3, we get a long exact sequence

\[
\begin{array}{ccccccccc}
R^i F(M') & \longrightarrow & H^i(F((I')^* \oplus (I'')^*)) & \longrightarrow & R^i F(M'') & \delta & R^{i+1} F(M')
\end{array}
\]

This proves the existence.

To finish, it suffices to show that a \( \delta \)-functor satisfying properties (1) and (2) satisfies the universal property.

Given any object \( M \) in \( C \), consider the injective resolution

\[
0 \longrightarrow M \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots
\]

and truncate it to get a short exact sequence

\[
0 \longrightarrow M \longrightarrow I^0 \longrightarrow N \longrightarrow 0
\]

where \( N = I^0 / M \). We then get a diagram with exact rows

\[
\begin{array}{cccccc}
0 & \longrightarrow & F M & \longrightarrow & F(I^0) & \longrightarrow & F(N) & \longrightarrow & R^1 F(M) & \longrightarrow & R^1 F(I^0) = 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G^0 M & \longrightarrow & G^0(I^0) & \longrightarrow & G^0(N) & \longrightarrow & G^1(M)
\end{array}
\]
where we have a unique map $R^i \mathcal{F}(\mathcal{M}) \to G^i(\mathcal{M})$ such that the diagram commutes. It is easy to see this is a natural transformation.

For $i \geq 1$, we proceed by induction on $i$. Having constructed $R^i \mathcal{F}(\mathcal{N}) \to G^i(\mathcal{N})$, we have the diagram

$$
\begin{array}{ccc}
R^i \mathcal{F}(\mathcal{N}) & \xrightarrow{=} & R^{i+1} \mathcal{F}(\mathcal{M}) \\
\downarrow & & \downarrow \\
G^i(\mathcal{N}) & \longrightarrow & G^{i+1}(\mathcal{M})
\end{array}
$$

and hence there is a unique map $R^{i+1} \mathcal{F}(\mathcal{M}) \to G^{i+1}(\mathcal{M})$ making this diagram commute. One can then check that this is a morphism of $\delta$-functors, completing the proof. \hfill \Box

**Definition 3.1.15.** The functor $R^i \mathcal{F}$ given by Theorem 3.1.14 is called the $i$th derived functor of $\mathcal{F}$.

In practice, it is better to compute $R^i \mathcal{F}$ using a resolution by $\mathcal{F}$-acyclic objects, rather than injective objects.

**Definition 3.1.16.** An object $\mathcal{A}$ of $\mathcal{C}$ is $\mathcal{F}$-acyclic if $R^i \mathcal{F}(\mathcal{A}) = 0$ for $i \geq 1$.

For example, injective objects are $\mathcal{F}$-acyclic.

**Proposition 3.1.17.** If $\mathcal{M}$ is an object of $\mathcal{C}$ and we have a resolution $\mathcal{M} \to \mathcal{A}^\bullet$ with the objects $\mathcal{A}^p$ being $\mathcal{F}$-acyclic for all $p$, there is a canonical isomorphism

$$R^i \mathcal{F}(\mathcal{M}) \cong \mathcal{H}^i(\mathcal{F}(\mathcal{A}^\bullet)).$$

**Proof.** The isomorphism for $i = 0$ follows by left-exactness of $\mathcal{F}$. We have an exact sequence

$$0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{A}^0 \longrightarrow \mathcal{N} = \text{coker}(\mathcal{M} \to \mathcal{A}^0) \longrightarrow 0,$$

which gives

$$\mathcal{F}(\mathcal{A}^0) \longrightarrow \mathcal{F}(\mathcal{N}) \longrightarrow R^1 \mathcal{F}(\mathcal{M}) \longrightarrow R^1 \mathcal{F}(\mathcal{A}^0) = 0.$$

Thus

$$R^1 \mathcal{F}(\mathcal{M}) = \text{coker}(\mathcal{F}(\mathcal{A}^0) \to \mathcal{F}(\mathcal{N})) \cong \mathcal{H}^1(\mathcal{F}(\mathcal{A}^\bullet)).$$

because we have

$$
\begin{array}{ccc}
0 & \longrightarrow & \mathcal{N} & \longrightarrow & \mathcal{A}^1 & \longrightarrow & \mathcal{A}^2 \\
\uparrow & & & & & & \downarrow \\
& & \mathcal{A}^0
\end{array}
$$
so we get
\[ F(A^0) \rightarrow F(N) \hookrightarrow F(A^1) \]
and
\[ H^1(F(A^\bullet)) = F(N)/\text{im}(F(A^0) \rightarrow F(N)) = \text{coker}(F(A^0) \rightarrow F(N)). \]
Also, \( R^iF(N) \cong R^{i+1}F(M) \) for all \( i \geq 1 \), since \( N \) has an \( F \)-acyclic resolution
\[ 0 \rightarrow N \rightarrow A^1 \rightarrow A^2 \rightarrow \cdots . \]
Hence, if we know the assertion for \( i \) and \( N \), we get it for \( i+1 \) and \( M \). This completes the proof by induction. \( \square \)

3.2. Cohomology of sheaves. Let \((X, \mathcal{O}_X)\) be a ringed space and \( \mathcal{O}_X \) be a sheaf of \( R \)-algebras. The right derived functors of \( F = \Gamma(X, -) \) are the sheaf cohomology, written
\[ H^i(X, -) = R^i\Gamma(X, -). \]
We have that
\[ \begin{align*}
\bullet & \quad H^0(X, \mathcal{F}) \cong \Gamma(X, \mathcal{F}), \\
\bullet & \quad \text{for any short exact sequence} \ 0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0, \ \text{we have a long exact sequence in cohomology} \\
& \quad \cdots \rightarrow H^i(\mathcal{F}') \rightarrow H^i(\mathcal{F}) \rightarrow H^i(\mathcal{F}'') \rightarrow H^{i+1}(\mathcal{F}') \rightarrow \cdots
\end{align*} \]

**Definition 3.2.1.** An \( \mathcal{O}_X \)-module \( \mathcal{F} \) on \( X \) is flasque (or flabby) if for any \( U \subseteq X \) open, the restriction map
\[ \Gamma(X, \mathcal{F}) \rightarrow \Gamma(U, \mathcal{F}) \]
is surjective.

**Remark 3.2.2.** Every \( \mathcal{O}_X \)-module has a canonical flasque resolution. For \( \mathcal{M} \), define \( \mathcal{E}^0(\mathcal{M}) \) by \( U \mapsto \prod_{x \in U} \mathcal{M}_x \) with restriction maps given by projection onto the corresponding component. Clearly, \( \mathcal{E} \) is flasque and we have an injective morphism \( \mathcal{M} \rightarrow \mathcal{E}^0 \) given by
\[ \Gamma(U, \mathcal{M}) \ni s \mapsto (s_x)_{x \in U}. \]
Then we define recursively for \( i \geq 2 \)
\[ \mathcal{E}^i(\mathcal{M}) = \mathcal{E}^0(\text{coker}(\mathcal{E}^{i-2}(\mathcal{M}) \rightarrow \mathcal{E}^i(\mathcal{M}))) \]
with \( \mathcal{E}^0(\mathcal{M}) = \mathcal{M} \).

**Proposition 3.2.3.** If we have a short exact sequence
\[ 0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0 \]
with \( \mathcal{F}' \) flasque, then
\[ 0 \rightarrow \mathcal{F}'(X) \rightarrow \mathcal{F}(X) \rightarrow \mathcal{F}''(X) \rightarrow 0 \]
is exact.

**Proof.** The proof is left as a homework exercise. \( \square \)

**Remark 3.2.4.** If \( \mathcal{F} \) is flasque, then \( \mathcal{F}|_U \) is flasque for all \( U \subseteq X \) open.
Corollary 3.2.5. If we have a short exact sequence

\[ 0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0 \]

with \( \mathcal{F}' \) flasque, then \( \mathcal{F} \) is flasque if and only if \( \mathcal{F}'' \) is flasque.

Proof. Proposition 3.2.3 and Remark 3.2.4 show that we have a commutative diagram with exact arrows

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathcal{F}'(X) & \longrightarrow & \mathcal{F}(X) & \longrightarrow & \mathcal{F}''(X) & \longrightarrow & 0 \\
\downarrow & & \downarrow \beta & & \downarrow \gamma & & \\
0 & \longrightarrow & \mathcal{F}'(U) & \longrightarrow & \mathcal{F}(U) & \longrightarrow & \mathcal{F}''(U) & \longrightarrow & 0
\end{array}
\]

so by the Snake Lemma \( \text{coker } \beta \cong \text{coker } \gamma \).

\[ \square \]

Proposition 3.2.6. If \( \mathcal{I} \) is an injective \( \mathcal{O}_X \)-module, then \( \mathcal{I} \) is flasque.

Proof. The proof is left as a homework exercise.

\[ \square \]

Proposition 3.2.7. Every flasque \( \mathcal{O}_X \)-module is \( \Gamma \)-acyclic. In particular, if \( \mathcal{M} \to \mathcal{A}^\bullet \) is a flasque resolution, then

\[ H^i(X, \mathcal{M}) \cong H^i(\Gamma(X, \mathcal{A}^\bullet)). \]

Proof. If \( A \) is flasque, consider

\[ 0 \longrightarrow A \longrightarrow \mathcal{I} \longrightarrow \mathcal{B} \longrightarrow 0 \]

for an injective object \( \mathcal{I} \). By Proposition 3.2.3, we have a short exact sequence

\[ 0 \longrightarrow \Gamma(X, A) \longrightarrow \Gamma(X, \mathcal{I}) \longrightarrow \Gamma(X, \mathcal{B}) \longrightarrow 0 \]

Hence \( H^1(X, \mathcal{A}^\bullet) = 0 \). For \( i \geq 2 \), the long exact sequence in cohomology shows that

\[ H^i(X, \mathcal{A}) \cong H^{i-1}(X, \mathcal{B}). \]

Since \( \mathcal{I} \) is injective, it is flasque by Proposition 3.2.6, and \( A \) is flasque, so by Corollary 3.2.5 \( \mathcal{B} \) is flasque. This completes the proof by induction.

\[ \square \]

We summarize what we have done so far and make a few comments. Suppose \((X, \mathcal{O}_X)\) is a ringed space where \( \mathcal{O}_X \) is a sheaf of \( R \)-algebras. We then have a left exact functor \( \Gamma(X, -) : \mathcal{O}_X \text{-mod} \to R \text{-mod} \) and its right derived functors are the sheaf cohomology groups \( H^i(X, -) \).

Note that if \( R \) is an \( S \)-algebra with \( \varphi : R \to S \), then we have a diagram
What if we change $\mathcal{O}_X$-modules to abelian groups? We have a diagram

$$
\xymatrix{
\mathcal{O}_X\text{-mod} \ar[r]^{\mathcal{F} = \Gamma(X,-)} \ar[d]_{\mathcal{G} = \Gamma(X,-)} & R\text{-mod} \\
S\text{-mod} 
}
$$

and $R^iG = \varphi \circ R^i\mathcal{F}$ by construction.

The natural transformation $\Gamma(X,-) \to \Gamma(U,-)$ extends to a morphism of $\delta$-functors

$$(H^i(X,-) \to H^i(U,-))_{i \geq 0}.$$

This describe this explicitly, note that if $A \to I^\bullet$ is an injective resolution, then we have a commuting square

$$
\xymatrix{
\mathcal{H}^i(\Gamma(X,I^\bullet)) \ar[r]^{\cong} \ar[d] & H^i(X,F) \\
\mathcal{H}^i(\Gamma(U,I^\bullet)) \ar[r]^{\cong} & H^i(U,F)
}
$$

which is functorial with respect to inclusion of open subsets.

### 3.3. Higher direct images

Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Then the functor

$$
f_* : \mathcal{O}_X\text{-mod} \to \mathcal{O}_Y\text{-mod}
$$

is left exact. Its derived functors are the *higher direct image functors*, $R^i f_*$. Then

- $R^0 f_* \cong f_*$,
- if $0 \to F' \to F \to F'' \to 0$ is a short exact sequence of $\mathcal{O}_X$-modules, then we have a long exact sequence
\[ \cdots \longrightarrow R^i f_*(\mathcal{F}') \longrightarrow R^i f_*(\mathcal{F}) \longrightarrow R^i f_*(\mathcal{F}'') \rightarrow R^{i+1} f_*(\mathcal{F}') \longrightarrow \cdots \]

in cohomology.

**Definition 3.3.1.** If \( f \) is as above, \( U \subseteq Y \) is open and \( \mathcal{F} \) is and \( \mathcal{O}_X \)-module, take \( H^i(f^{-1}(U), \mathcal{F}) \).

If \( V \subseteq U \), we have natural maps
\[ H^i(f^{-1}(U), \mathcal{F}) \to H^i(f^{-1}(V), \mathcal{F}) \]
which satisfy the usual compatibility condition. Note that \( H^i(f^{-1}(U), \mathcal{F}) \) is an \( \mathcal{O}_X(f^{-1}(U)) \)-module, so it is an \( \mathcal{O}_Y(U) \)-module via \( \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U)) \). We therefore get a presheaf of \( \mathcal{O}_Y \)-modules denoted \( \tilde{R}^i f_*(\mathcal{F}) \).

**Proposition 3.3.2.** We have a functorial isomorphism
\[ \tilde{R}^i f_*(\mathcal{F})^+ \cong R^i f_*(\mathcal{F}). \]

**Proof.** We show that \( (\tilde{R}^i f_*(\mathcal{F})^+)_{i \geq 0} \) satisfy the universal property, so we actually have an isomorphism of \( \delta \)-functors.

When \( i = 0 \), we have \( \tilde{R}^0 f_*(-)^+ = f_*(-) \).

If \( \mathcal{I} \) is an injective \( \mathcal{O}_X \)-module, then \( \mathcal{I} \) is flasque (Proposition 3.2.6), so
\[ H^i(f^{-1}(U), \mathcal{I}) = 0 \text{ for all } U \text{ open, } i \geq 1, \]
and hence
\[ \tilde{R}^i f_*(\mathcal{I})^+ = 0 \text{ for } i \geq 1. \]

Finally, if \( 0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0 \) is a short exact sequence, for any open set \( U \), we get a long exact sequence
\[ \cdots \longrightarrow H^i(f^{-1}(U), \mathcal{F}') \longrightarrow H^i(f^{-1}(U), \mathcal{F}) \longrightarrow \cdots \]
which by definition gives
\[ (\ast) \quad \cdots \longrightarrow \tilde{R}^i f_*(\mathcal{F}') \longrightarrow \tilde{R}^i f_*(\mathcal{F}) \longrightarrow \cdots \]
and taking the sheafification gives a sequence
\[ \cdots \longrightarrow \tilde{R}^i f_*(\mathcal{F}')^+ \longrightarrow \tilde{R}^i f_*(\mathcal{F})^+ \longrightarrow \cdots \]
By passing to stalks and using exactness of sequence \((\ast)\), we get that this sequence is exact.

Altogether \(\left\{\tilde{R}^if_*(-)^/+\right\}_{i\geq 0}\) satisfies the universal property of \(\left\{R^if_*(-)\right\}_{i\geq 0}\), so there is an isomorphism of \(\delta\)-functors between them.

**Corollary 3.3.3.** If \(f\) is as above and \(\mathcal{M}\) is a flasque \(\mathcal{O}_X\)-module, \(R^if_*(\mathcal{M}) = 0\) for \(i \geq 1\). Therefore, for any \(\mathcal{O}_X\)-module \(\mathcal{A}\), if \(\mathcal{A} \to \mathcal{M}^\bullet\) is a flasque resolution, then

\[R^if_*(\mathcal{A}) \cong \mathcal{H}^i(f_*(\mathcal{M}^\bullet)).\]

**Proof.** If \(U \subseteq Y\) is open, \(\mathcal{M}|_{f^{-1}(U)}\) is flasque, so \(H^i(f^{-1}(U), \mathcal{M}) = 0\) for all \(i \geq 1\), so \(\tilde{R}^if_*(\mathcal{M}) = 0\) for all \(i \geq 1\), and hence by Proposition 3.3.2, \(R^if_*(\mathcal{M}) = 0\) for all \(i \geq 1\). □

**Proposition 3.3.4.** If \(f: X \to Y\) is a morphism of algebraic varieties and \(\mathcal{M}\) is a quasicoherent sheaf on \(X\), then \(R^if_*(\mathcal{M})\) is quasicoherent for all \(i \geq 1\).

Moreover, if \(U \subseteq Y\) is an affine open subset, then

\[\Gamma(U, R^if_*(\mathcal{F})) \cong H^i(f^{-1}(U), \mathcal{F}).\]

Before we prove this result, we need another proposition.

**Proposition 3.3.5.** If \(X\) is an affine algebraic variety and \(I\) is an injective \(\mathcal{O}(X)\)-module, then \(\tilde{I}\) is flasque.

We will assume this result for now and delay the proof until later.

**Corollary 3.3.6.** If \(X\) is an algebraic variety, then for any quasicoherent sheaf \(\mathcal{F}\) on \(X\), there is a quasicoherent flasque sheaf \(\mathcal{E}\) with an injective map \(\mathcal{F} \hookrightarrow \mathcal{E}\).

**Proof.** Let \(X = \bigcup_{i=1}^r U_i\) be an affine open cover. Note that \(\mathcal{F}|_{U_i}\) is still quasicoherent. Let \(Q_i\) be an injective \(\mathcal{O}(U_i)\)-module such that there is an inclusion

\[\mathcal{F}(U_i) \hookrightarrow Q_i.\]

Then \(\mathcal{F}|_{U_i} \hookrightarrow \tilde{Q}_i\) and \(\tilde{Q}_i\) is flasque on \(U_i\) by Proposition 3.3.5. We then have

\[\mathcal{F} \hookrightarrow \bigoplus_{i=1}^r (\alpha_i)_*(\mathcal{F}|_{U_i}) \hookrightarrow \bigoplus_{i=1}^r (\alpha_i)_*(\tilde{Q}_i)\]

and the last sheaf is quasicoherent and flasque. □

We can finally prove Proposition 3.3.4.

**Proof of Proposition 3.3.4.** By Proposition 3.3.5, there is a resolution

\[\mathcal{M} \to \mathcal{Q}^\bullet\]

with all \(\mathcal{Q}^i\) quasicoherent and flasque. Then

\[R^if_*(\mathcal{M}) \cong \mathcal{H}^i(f_*(\mathcal{Q}^\bullet))\]
and since \( f_*(\mathcal{Q}^\bullet) \) is quasicoherent (as a pushforward of a quasicoherent sheaf), this shows \( R^i f_*(\mathcal{M}) \) is quasicoherent.

If \( U \subseteq Y \) is open an affine, then we have
\[
\Gamma(U, R^i f_*(\mathcal{F})) \cong \Gamma(U, \mathcal{H}^i(f_*(\mathcal{Q})))
\cong \mathcal{H}^i(\Gamma(U, f_*(\mathcal{Q}^\bullet))) \quad \text{since } \Gamma(U, -) \text{ exact on quasicoherent sheaves on affine varieties}
\cong \mathcal{H}^i(\Gamma(U, f_*(\mathcal{Q}^\bullet)))
\cong \mathcal{H}^i(\Gamma(f^{-1}(U), \mathcal{Q}^\bullet))
\cong H^i(f^{-1}(U), \mathcal{M}).
\]

This completes the proof. \(\square\)

3.4. Cohomology of quasicoherent sheaves on affine varieties.

**Theorem 3.4.1** (Serre). If \( X \) is an algebraic variety, the following are equivalent

1. \( X \) is affine,
2. \( H^i(X, \mathcal{F}) = 0 \) for any \( \mathcal{F} \) quasicoherent and \( i \geq 1 \),
3. \( H^1(X, \mathcal{I}) = 0 \) for all coherent ideals sheaves \( \mathcal{I} \subseteq \mathcal{O}_X \).

**Proof.** We first show that (1) implies (2). If \( \mathcal{F} \) is quasicoherent, there is a flasque resolution \( \mathcal{F} \to \mathcal{Q}^\bullet \) such that \( \mathcal{Q}^i \) is quasicoherent for all \( i \) by Corollary 3.3.6. Then
\[
H^i(X, \mathcal{F}) \cong \mathcal{H}^i(\Gamma(X, \mathcal{Q}^\bullet)) = 0,
\]

since \( \Gamma(X, -) \) is exact in the category of quasicoherent sheaves on affine varieties.

Note that (2) implies (3) is immediate, so it remains to show that (3) implies (1). For any \( x \in X \), choose an affine open neighborhood \( U \) of \( x \). Let \( Z = \{x\} \cup (X \setminus U) \), which is closed in \( X \), and let \( \mathcal{I}_Z \) be the corresponding radical ideal sheaf. We then have an exact sequence
\[
0 \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Z \longrightarrow 0
\]

and the long exact sequence in cohomology gives
\[
\Gamma(X, \mathcal{O}_X) \longrightarrow \Gamma(Z, \mathcal{O}_Z) \longrightarrow H^1(X, \mathcal{I}_Z) = 0.
\]

Therefore, there exists \( f \in \mathcal{O}_X(X) \) such that \( f(x) \neq 0 \) and \( f|_{X \setminus U} = 0 \). Then \( x \in D_X(f) \subseteq U \), so \( D_X(f) = D_U(f|_U) \), which is affine since \( U \) is affine.

Since \( X \) is quasicompact, there exist \( f_1, \ldots, f_r \in \mathcal{O}_X(X) \) such that \( X = \bigcup_{i=1}^r D_X(f_i) \) and each \( D_X(f_i) \) is affine.

If we show that \( \mathcal{O}_X(X) = (f_1, \ldots, f_r) \), then (by a result from a homework on Math 631) \( X \) is affine. To show this, consider the map
\[
\mathcal{O}_X^{\oplus r} \xrightarrow{\xi} \mathcal{O}_X \quad e_i \mapsto f_i.
\]
This is surjective, since on $D_X(f_i)$, $f_i$ generates $O_{D_X(f_i)}$. Let $\mathcal{F}$ be the kernel of $\varphi$. We have a short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow O_X^{\oplus r} \longrightarrow O_X \longrightarrow 0.$$  

It is enough to show that $H^1(X, \mathcal{F}) = 0$. Indeed, this implies that

$$\Gamma(X, O_X^{\oplus r}) \rightarrow \Gamma(X, O_X)$$

is surjective, so $\Gamma(X, O_X) = (f_1, \ldots, f_r)$.

Let $\mathcal{E}_i \subseteq O_X^{\oplus r}$ be generated by $e_1, \ldots, e_i$. Then $\mathcal{E}_{i+1}/\mathcal{E}_i \cong O_X$. Consider

$$0 \subseteq \mathcal{F} \cap \mathcal{E}_i \subseteq \mathcal{F} \cap \mathcal{E}_2 \subseteq \cdots \subseteq \mathcal{F} \cap \mathcal{E}_r = \mathcal{F},$$

which gives a short exact sequence

$$0 \longrightarrow \mathcal{F} \cap \mathcal{E}_i \longrightarrow \mathcal{F} \cap \mathcal{E}_{i+1} \longrightarrow \frac{\mathcal{F} \cap \mathcal{E}_{i+1}}{\mathcal{F} \cap \mathcal{E}_i} \longrightarrow 0.$$  

Note that

$$\frac{\mathcal{F} \cap \mathcal{E}_{i+1}}{\mathcal{F} \cap \mathcal{E}_i} \hookrightarrow \mathcal{E}_{i+1}/\mathcal{E}_i \cong O_X$$

is a coherent ideal, so

$$H^1\left(X, \frac{\mathcal{F} \cap \mathcal{E}_{i+1}}{\mathcal{F} \cap \mathcal{E}_i}\right) = 0$$

by assumption. The long exact sequence in cohomology shows that

$$H^1(\mathcal{F} \cap \mathcal{E}_i) = 0 \text{ implies } H^i(\mathcal{F} \cap \mathcal{E}_{i+1}) = 0.$$  

Since $\mathcal{F} \cap \mathcal{E}_0 = 0$, by induction on $i$, we have that

$$H^1(X, \mathcal{F} \cap \mathcal{E}_i) = 0 \text{ for all } i.$$  

Taking $i = r$, this completes the proof. \qed

We finally give a sketch of the proof of Proposition 3.3.5.

**Sketch of proof of Proposition 3.3.5.** Let $A = O(X)$ and $Q$ be an injective $A$-module. We want to show that $\tilde{Q}$ is flasque.

**Step 1.** Show that if $U = D_X(f)$ then

$$\underbrace{\Gamma(X, \tilde{Q})}_{\mathcal{Q}} \rightarrow \underbrace{\Gamma(U, \tilde{Q})}_{\mathcal{Q}_f}$$

is surjective.

Consider $\text{Ann}(f) \subseteq \text{Ann}(f^2) \subseteq \cdots$. As $A$ is Noetherian, there is an $r$ such that $\text{Ann}(f^r) = \text{Ann}(f^{r+1}) = \cdots$. Consider $u \in Q_f$, $u = \frac{a}{f^r}$. Define a morphism

$$(f^{r+s}) \overset{\mathcal{E}}{\rightarrow} Q$$

$$f^{r+s}b \mapsto f^rb_a.$$  

This is well-defined since $f^{r+s}b = f^{r+s}b'$ implies that $f^r b = f^r b'$. 

Since \( Q \) is injective, this can be extended to a map \( \psi : A \to Q \) and let \( v = \psi(1) \). Then \( f^{r+s} v = f^r a \), and hence \( \frac{a}{f^s} = \frac{v}{f^r} \) in \( Q_f \).

For the other steps, see the official notes.

3.5. Soft sheaves on paracompact spaces.

**Definition 3.5.1.** A topological space \( X \) is *paracompact* if the following conditions hold:

- Hausdorff,
- every open cover has a locally finite refinement.

It is easy to see that a closed subset of a paracompact space is paracompact.

**Examples 3.5.2.**

1. Topological manifolds (which are assumed to be Hausdorff and have a countable basis of open subsets)
2. Simplicial complexes
3. CW complexes

The following result is always useful: if \( X = \bigcup U_i \) is a locally finite open cover, then there is an open cover \( X = \bigcup V_i \) such that \( V_i \subseteq U_i \).

A special case shows that if \( F \subseteq U \) where \( F \) is an open subset and \( U \) is a closed subset (so \( V \cup X \setminus F \) is an open cover), then there is an open set \( W \) such that \( F \subseteq W \subseteq \overline{W} \subseteq V \). In other words, a paracompact space is *normal*.

**Definition 3.5.3.** Let \( X \) be a topological space. A sheaf \( F \) is *soft* if for any closed subset \( Z \subseteq X \), \( \Gamma(X, F) \to \Gamma(Z, F) \) is surjective.

We recall a result from the problem session. If \( X \) is paracompact and \( Z \subseteq X \) is closed, then for any \( s \in F(Z) \), there is an open subset \( U \) containing \( Z \) and \( s_U \in F(U) \) such that \( s_U|_Z = s \). In particular, if \( F \) is flasque, then it is soft.

**Lemma 3.5.4.** Suppose \( X \) is paracompact. If

\[
0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0
\]

is a short exact sequence with \( F' \) soft, then we have an exact sequence

\[
0 \longrightarrow F'(X) \longrightarrow F(X) \longrightarrow F''(X) \longrightarrow 0
\]

**Proof.** We omit the proof here since this is similar to Problem 1 on Homework 6, but it can be found in the official notes.

**Corollary 3.5.5.** If \( X \) is paracompact and

\[
0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0
\]
is a short exact sequence with $\mathcal{F}'$ soft, then $\mathcal{F}$ is soft if and only if $\mathcal{F}''$ is soft.

**Proof.** If $Z \subseteq X$ is closed, we have a commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & \mathcal{F}'(X) \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{F}''(X) \longrightarrow 0 \\
\downarrow{\alpha} & & \downarrow{\beta} & \downarrow{\beta} & \\
0 & \longrightarrow & \mathcal{F}'(Z) \longrightarrow \mathcal{F}(Z) \longrightarrow \mathcal{F}''(Z) & \longrightarrow & 0
\end{array}
$$

with exact rows by Lemma 3.5.4 (note that $\mathcal{F}'|Z$ is also soft and $Z$ is also paracompact so the lemma applies). As $\alpha$ is surjective by hypothesis, the Snake Lemma shows that $\text{coker } \beta \cong \text{coker } \gamma$. \hfill \Box

**Proposition 3.5.6.** If $X$ is paracompact and $\mathcal{F}$ is soft, then $H^i(X, \mathcal{F}) = 0$ for $i \geq 1$. In particular, if $\mathcal{M}$ is any $\mathcal{O}_X$-module and $\mathcal{M} \to \mathcal{F}^\ast$ is a resolution by soft $\mathcal{O}_X$-modules, then $H^i(X, \mathcal{M}) \cong H^i(\Gamma(X, \mathcal{F}^\ast))$.

**Proof.** Consider an embedding $\mathcal{F} \hookrightarrow \mathcal{A}$ into a flasque sheaf $\mathcal{A}$, and let

$$
0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow 0
$$

be the corresponding short exact sequence. Then the long exact sequence in cohomology gives the exact sequences

$$
0 \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{A}) \longrightarrow \Gamma(X, \mathcal{B}) \\
\xrightarrow{H^1(X, \mathcal{F})} \longrightarrow H^1(X, \mathcal{A}) = 0,
$$

as $\mathcal{A}$ is flasque.

$$
0 = H^i(X, \mathcal{A}) \longrightarrow H^i(X, \mathcal{B}) \longrightarrow H^{i+1}(X, \mathcal{F}) \longrightarrow H^{i+1}(X, \mathcal{A}) = 0.
$$

This shows that $H^i(X, \mathcal{B}) \cong H^{i+1}(X, \mathcal{F})$ for $i \geq 1$. Since $\Gamma(X, \mathcal{A}) \to \Gamma(X, \mathcal{B})$ is surjective by Lemma 3.5.4, $H^1(X, \mathcal{F}) = 0$.

By Corollary 3.5.5, since $\mathcal{F}$ is soft and $\mathcal{A}$ is soft (since flasque), $\mathcal{B}$ is also soft. By induction, we see that $H^i(X, \mathcal{B}) = 0$, so $H^{i+1}(X, \mathcal{F}) = 0$, which completes the proof. \hfill \Box

### 3.6. De Rham cohomology and sheaf cohomology.

Let $X$ be a smooth manifold (in particular, since it is Hausdorff and has a countable basis, it is paracompact). Let

$$
\mathcal{C}_X^\infty = \text{sheaf of smooth functions form } X \to \mathbb{R},
$$

$$
\mathcal{E}_X^p = \text{sheaf of smooth } p\text{-differential forms on } X.
$$

Note that $\mathcal{E}_X^0 = \mathcal{C}_X^\infty$.

Let $d: \mathcal{E}_X^p \to \mathcal{E}_X^{p+1}$ be the exterior differential.
Recall that if $U \subseteq X$ is open with coordinates $x_1, \ldots, x_n$, any $\omega \in \Gamma(U, \mathcal{E}_X^p)$ can be written as

$$\omega = \sum_{|I| = p} f_I dx_I$$

where $I$ is an ordered $p$-tuple $i_1 < \cdots < i_p$ and we write $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_p}$. Then

$$d\omega = \sum_{|I| = p} \left( \sum_{i=1}^{n} \frac{\partial f_I}{\partial x_i} dx_i \right) \wedge dx_I.$$  

Note that $d \circ d$, and hence we get the de Rham complex:

$$0 \longrightarrow \mathcal{E}_X^0(X) \longrightarrow \mathcal{E}_X^1(X) \longrightarrow \cdots \longrightarrow \mathcal{E}_X^n(X) \longrightarrow 0$$

where $n = \dim(X)$. The de Rham cohomology groups are then defined as

$$H^p_{\text{dR}}(X) = H^p(\mathcal{E}_X^\bullet(X)),$$

which are $\mathbb{R}$-vector spaces.

**Theorem 3.6.1.** We have a canonical isomorphism

$$H^p_{\text{dR}}(X) \cong H^p(X, \mathcal{R}),$$

where $\mathcal{R}$ is the constant sheaf.

**Lemma 3.6.2.** Every $\mathcal{C}_X^\infty$-module $F$ (for example, $\mathcal{E}_X^p$) is soft.

**Proof.** Let $Z \subseteq X$ be closed and let $s \in F(Z)$. We know that there is an open subset $U \supseteq Z$ and $s_U \in F(U)$ such that $s_U|_Z = s$.

Choose open subset $V_1, V_2$ such that

$$Z \subseteq V_1 \subseteq V_1 \subseteq V_2 \subseteq V_2 \subseteq U.$$  

By the smooth version of Urysohn’s Lemma, there is a function $f \in \mathcal{C}_X^\infty(X)$ such that

$$f = \begin{cases} 1 & \text{on } V_1, \\ 0 & \text{on } X \setminus V_2. \end{cases}$$

Consider $f s_U$ on $U$ and 0 on $X \setminus V_2$. They agree on $U \setminus V_2$, so there exists $t \in F(X)$ such that

$$t|_U = f s_U.$$  

Then $t|_{V_1} = s_U|_{V_1}$, and hence $t|_Z = s$. □

Consider the complex

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{E}_X^0(X) \longrightarrow \mathcal{E}_X^1(X) \longrightarrow \cdots \longrightarrow \mathcal{E}_X^n(X) \longrightarrow 0.$$  

The following lemma shows that this complex gives a resolution for $\mathbb{R}$ when $X = \mathbb{R}^n$.

**Lemma 3.6.3** (Poincaré Lemma). For every $n \geq 0$, the complex

$$\mathcal{E}_{\mathbb{R}^n}(\mathbb{R}^n) : 0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{E}_{\mathbb{R}^n}^0(\mathbb{R}^n) \longrightarrow \mathcal{E}_{\mathbb{R}^n}^1(\mathbb{R}^n) \longrightarrow \cdots \longrightarrow \mathcal{E}_{\mathbb{R}^n}^n(\mathbb{R}^n) \longrightarrow 0.$$
is exact.

**Proof.** We use induction on \( n \geq 0 \). The \( n = 0 \) case is trivial. For \( n \geq 0 \), consider the maps

\[
\begin{array}{ccc}
\mathbb{R}^{n-1} & \xrightarrow{i} & \mathbb{R}^n \\
(x_1, \ldots, x_n) & \xrightarrow{\pi} & (x_2, \ldots, x_n) \\
(x_2, \ldots, x_n) & \xrightarrow{} & (0, x_2, \ldots, x_n)
\end{array}
\]

This gives maps

\[
\mathcal{E}^\bullet_{\mathbb{R}^{n-1}}(\mathbb{R}^{n-1}) \xrightarrow{\pi^*} \mathcal{E}^\bullet_{\mathbb{R}^n}(\mathbb{R}^n) \xrightarrow{i^*} \mathcal{E}^\bullet_{\mathbb{R}^{n-1}}(\mathbb{R}^{n-1})
\]

whose composition is the identity. To complete the proof by induction, it suffices to show that \( \pi^* \circ i^* \approx 1_{\mathcal{E}^\bullet_{\mathbb{R}^n}(\mathbb{R}^n)} \). To define a differential

\[
\mathcal{E}^p(\mathbb{R}^n) \xrightarrow{d} \mathcal{E}^{p+1}(\mathbb{R}^n)
\]

we use integration:

\[
fdx_I \mapsto \begin{cases} 
0 & \text{if } 1 \not\in I, \\
\int_0^{x_1} f(t, x_2, \ldots, x_n)dt \ dx_I' & \text{if } I = \{1\} \cup I'.
\end{cases}
\]

The fact that this gives a homotopy as above is left as an exercise. For example, if \( 1 \not\in I \), then

\[
(\theta^{p+1} \circ d + d \circ \theta^p)(fdx_I) = \theta^{p+1} \left( \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i \wedge dx_I \right) = \left( \int_0^{x_1} \frac{\partial f}{\partial x_1}(t, x_2, \ldots, x_n)dt \right) dx_I = (f(x_1, \ldots, x_n) - f(0, x_2, \ldots, x_n))dx_I \quad \text{by the FTC}
\]

The other case is a similar computation. \( \square \)

**Proof of Theorem 3.6.1.** We have the following complex of sheaves

\[
0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{E}^0_X \longrightarrow \cdots \longrightarrow \mathcal{E}^n_X \longrightarrow 0.
\]
We know that if we take sections on $U$ diffeomorphic to $\mathbb{R}^n$, then by Poincaré Lemma 3.6.3, we get an exact complex. Since every point has a basis of neighborhoods diffeomorphic to $\mathbb{R}^n$, the above complex of sheaves is exact. By Lemma 3.6.2, $\mathbb{R} \to \mathcal{E}_X^\bullet$ is a soft resolution of $\mathbb{R}$, and the result follows from Proposition 3.5.6. □

We now move on to general topological spaces instead.

**Theorem 3.6.4.** If $X$ is a locally contractible topological space, which is paracompact, then for every commutative ring $R$ and $R$-module $A$, we have an isomorphism

$$H^p(X, A) \cong \underline{H}^p_{\text{sing}}(X, A).$$

Recall that $X$ is a locally contractible space if every point in $X$ has a basis of open neighborhoods which are contractible.

**Remark 3.6.5.** By a recent result, one can drop the paracompactness hypothesis.

**Proof of Theorem 3.6.4.** For every $p$, let $\Delta_p$ be the standard $p$-dimension simplex. Then a $p$-simplex in $X$ is a continuous map $\Delta_p \to X$, and we set

$$C_p(X) = \text{free abelian group generated by } p\text{-simplices in } X$$

and we have the standard map $\partial: C_p(X) \to C_{p-1}(X)$ such that $\partial^2 = 0$. We let

$$C^p(X, A) = \text{Hom}_\mathbb{Z}(C_p(X), A).$$

Then

$$H^p_{\text{sing}}(X, A) = \mathcal{H}^p(C^\bullet(X, A)).$$

Note that any map $f: X \to Y$ gives a chain map $C^\bullet(Y, A) \to C^\bullet(X, A)$.

For every $p$, let $\mathcal{C}_X^p$ be the presheaf that assigns to $X \supseteq U$, $C^p(U, A)$, and for $V \subseteq U$, the natural map $C^p(U, A) \to C^p(V, A)$ gives the restriction maps.

We get a complex

$$0 \longrightarrow A \longrightarrow \mathcal{C}_X^0 \longrightarrow \mathcal{C}_X^1 \longrightarrow \cdots \longrightarrow \mathcal{C}_X^n \longrightarrow 0.$$

Note that $\mathcal{C}_X^p$ is not a sheaf: functions that agree on intersections can be glued, but far from uniquely. Hence let $\mathcal{S}_X^p = (\mathcal{C}_X^p)^+$. Since $X$ is locally contractible, each point has a basis of neighborhoods $U$ such that the corresponding complex of sections of $\mathcal{C}_X^\bullet$ on $U$ is exact. We then get an exact sequence

$$0 \longrightarrow A \longrightarrow \mathcal{S}_X^0 \longrightarrow \mathcal{S}_X^1 \longrightarrow \cdots \longrightarrow \mathcal{S}_X^n \longrightarrow 0.$$

If every open subset of $X$ is paracompact (for example, if $X$ is a topological manifold), all $\mathcal{S}_X^p$ are flasque, since all maps $\mathcal{C}_X^p(U) \to \mathcal{S}_X^p(U)$ are surjective (this is by Problem 2 on Problem Set 6). In general, $\mathcal{S}_X^p$ are just soft, and this case is dealt with in the notes.
This shows that
\[ H^p(X, A) \cong \mathcal{H}^p(S^\bullet_X(X)). \]
Consider
\[ 0 \longrightarrow V^\bullet(X) \longrightarrow C^\bullet_X(X) \longrightarrow S^\bullet_X(X) \longrightarrow 0. \]
It is enough to show that \( \mathcal{H}^p(V^\bullet(X)) = 0 \) for all \( p \). Recall that
\[ V^\bullet(X) = \lim_{\longrightarrow} V^\bullet(U)(X) \]
where
\[ V^\bullet(U)(X) = \left\{ \sigma \in C^p(X, A) \mid \sigma \text{ vanishes on } p \text{-simplices in } X \text{ whose image is contained in some element of } U \right\}. \]
It is enough to show that \( \mathcal{H}^p(V^\bullet(U)(X)) = 0 \) for each \( U \).

A known fact from singular cohomology is that if \( C^\bullet_U \) is the free abelian group generated by the \( p \)-simplices in \( X \) whose image is contained in some element of \( U \), then
\[ C^\bullet_U(X) \hookrightarrow C^\bullet(X) \]
is a homotopy equivalence. (This is proved using baricentric subdivision.)
This is still a homotopy equivalence after applying \( \text{Hom}_{\mathbb{Z}}(-, A) \). Then the exact sequence
\[ 0 \longrightarrow V^\bullet(U)(X) \longrightarrow C^\bullet(X, A) \longrightarrow \text{Hom}(C^\bullet_U(X), A) \longrightarrow 0 \]
shows that \( \mathcal{H}^p(V^\bullet(U)(X)) = 0 \) for all \( p \).

3.7. Introduction to spectral sequences. Let \( K^\bullet \) be a complex in a category \( \mathcal{C} \) (for example, the category of \( \mathcal{O}_X \)-modules for a ringed space \( (X, \mathcal{O}_X) \)). Consider a decreasing filtration \( F^\bullet K^\bullet = (F^p K^\bullet)_{p \in \mathbb{Z}} \), i.e. a chain of subcomplexes:
\[ K^\bullet \supseteq \cdots \supseteq F^p K^\bullet \supseteq F^{p+1} K^\bullet \supseteq \cdots. \]
This gives a filtration on the cohomology of \( K^\bullet \) given by
\[ F^p \mathcal{H}^n(K^\bullet) = \text{im}(\mathcal{H}^n(F^p K^\bullet) \rightarrow \mathcal{H}^n(K^\bullet)). \]
We want a description of the quotients
\[ \text{gr}_p \mathcal{H}^n(K^\bullet) = \frac{F^p \mathcal{H}^n(K^\bullet)}{F^{p+1} \mathcal{H}^n(K^\bullet)} \]
in terms of some data coming from the successive quotients of \( K^\bullet \).
This data is encoded by the spectral sequence. For \( r \geq 0 \), we denote by \( (E^{p,q}_r)_{p,q \in \mathbb{Z}} \) the \( r \)th page of the spectral sequence, where \( p \) is related to the filtration level and \( p + q \) records the place in the complex. To define it, for \( r \in \mathbb{Z} \) let
\[ Z^{p,q}_r = \left\{ u \in F^p K^{p+q} \mid d(u) \in F^{p+r} K^{p+q+1} \right\} \]
and set
\[ E^{p,q}_r = \frac{Z^{p,q}_r}{Z^{p+1,q-1}_{r-1} + d(Z^{p-r+1,q+r-2}_{r-1})}. \]
Note that \( Z_{r-1}^{p,q} = Z_0^{p,q} = F_pK^{p,q} \). Also,
\[
E_0^{p,q} = \frac{F_pK^{p,q}}{F_{p+1}K^{p,q}}
\]
and there is a map
\[
E_0^{p,q} \to E_0^{p,q+1}
\]
induced by \( d \). Similarly,
\[
E_1^{p,q} = \frac{\{u \in F_pK^{p,q} \mid du \in F_{p+1}K^{p,q+1}\}}{F_{p+1}K^{p,q} + d(F_p(K^{p,q-1}))} = \mathcal{H}(E_0^{p-1,q} \to E_0^{p,q} \to E_0^{p,q+1}).
\]
In general, \( d \) induces a map \( d_r : E_r^{p,q} \to E_r^{p+r,q-r+1} \).

**Proposition 3.7.1.** For each \( r \geq 0 \), there is a canonical isomorphism
\[
E_{r+1}^{p,q} = \mathcal{H}(E_r^{p-r,q+r-1} \to E_r^{p,q} \to E_r^{p+r,q-r+1}).
\]

*Proof.* The proof is left as an exercise. \( \square \)

**Definition 3.7.2.** A filtration \( F \cdot K^\cdot \) on \( K^\cdot \) is pointwise finite if for any \( n \) we have
\[
F_pK^n = 0 \text{ for } p \gg 0,
F_pK^n = K^n \text{ for } p \ll 0.
\]

**Proposition 3.7.3.** If the filtration on \( K^\cdot \) is pointwise finite, then for any \( p,q \in \mathbb{Z} \), \( E_r^{p,q} \) is eventually constant. We denote this value by \( E_\infty^{p,q} \). Moreover, for all \( p,q \) we have that
\[
E_\infty^{p,q} \cong \text{gr}_p \mathcal{H}^{p,q}(K^\cdot).
\]

*Proof.* Fix \( p,q \). We have that
\[
Z_r^{p,q} = F_pK^{p,q} \cap \text{ker}(d).
\]
Consider the sequence
\[
E_r^{p-r,q+r-1} \to E_r^{p,q} \to E_r^{p+r,q-r+1}
\]
whose cohomology gives \( E_{r+1}^{p,q} \) by Proposition 3.7.1. For \( r \gg 0 \), \( Z_r^{p+r,q-r+1} = 0 \), since \( F_pK^n = 0 \) for \( p \gg 0 \). Similarly, \( E_r^{p-r,q+r-1} \) for \( r \gg 0 \), since \( Z_r^{p-r,q+r-1} = Z_{r-1}^{p-r+1,q+r-2} \). Therefore, taking the cohomology of the above sequence gives simply
\[
E_r^{p,q} = E_r^{p,q} \text{ for } r \gg 0.
\]
Moreover,
\[
dZ_{r-1}^{p-r+1,q+r-2} = d(K^{p-q-1} \cap d(F_pK^{p+q}) = F_pK^{p+q} \cap \text{im}(d).
\]
It is easy to check that
\[
\text{gr}_p \mathcal{H}^{p,q}(K^\cdot) \cong \frac{F_pK^{p,q} \cap \text{ker}(d)}{F_{p+1}K^{p,q} \cap \text{ker}(d) + (F_pK^{p,q} \cap \text{im}(d))},
\]
which completes the proof. \( \square \)

**Definition 3.7.4.** If the conclusion of Proposition 3.7.3, we write
\[
E_r^{p,q} \Rightarrow_p \mathcal{H}^{p,q}(K^\cdot)
\]
and say that the spectral sequence *converges* with respect to \( p \) to the cohomology of \( K^\cdot \).
Definition 3.7.5. The spectral sequence *collapses at level* $r_0$ if $d_r = 0$ for $r \geq r_0$. In this case, $E_\infty = E_{r_0}$.

Suppose there is an $a$ such that $E_{r_0}^{p,q} = 0$ unless $p = a$ and $r_0 \geq 1$. Then $E_\infty^{p,q} = 0$ if $p \neq a$ and $E_\infty^{p,q} = E_{r_0}^{p,q}$ for all $p,q$. This shows that

$$\mathcal{H}^n(K^\bullet) \cong E_{r_0}^{a,n-a}$$

This way, we recover the cohomology of the complex from a spectral sequence.

Similarly, if there is a $b$ such that $E_{r_0}^{p,q} = 0$ unless $q = b$, then $E_\infty^{p,q} = 0$ unless $q = b$, and

$$E_\infty^{p,b} = \begin{cases} E_{r_0}^{p,b} & \text{if } r_0 \geq 2, \\ E_2^{p,b} & \text{if } r_0 = 1. \end{cases}$$

In this case,

$$\mathcal{H}^n(K^\bullet) \cong E_{r_0}^{n-b,b}.$$

We now describe the spectral sequence of a double complex.

Definition 3.7.6. A *double complex* $A^{\bullet,\bullet}$ is a collection $(A^{p,q})_{p,q \in \mathbb{Z}}$ of objects together with morphisms

$$d_1: A^{p,q} \to A^{p+1,q}$$
$$d_2: A^{p,q} \to A^{p,q+1}$$

such that $0 = d_1 \circ d_1 = d_2 \circ d_2$ and $d_1 \circ d_2 = d_2 \circ d_1$.

In particular, both $A^{p,\bullet}$ and $A^{\bullet,q}$ are complexes all fixed $p$ and $q$.

Definition 3.7.7. The *total complex* of $A^{\bullet,\bullet}$ is $K^\bullet = \text{Tot}(A^{\bullet,\bullet})$ is defined by $K^n = \bigoplus_{i+j=n} A^{i,j}$ together with maps

$$d: K^n \to K^{n+1}$$

such that $d|_{A^{i,j}} = d_1 + (-1)^i d_2$.

It is easy to see that $d \circ d = 0$ so the total complex is indeed a complex.

We consider two filtration on $K^\bullet$:

$$F_p' K^n = \bigoplus_{i+j=n, i \geq p} A^{i,j},$$
$$F_p'' K^n = \bigoplus_{i+j=n, j \geq p} A^{i,j}.$$

We will always assume that for any $n$, there are only finitely many $p$ such that $A^{p,n-p} \neq 0$. Hence both filtrations are pointwise finite. This happens, for example, when $A^{p,q} = 0$ unless $p,q \geq 0$, i.e. for a *first quadrant double complex*.

In this case, Proposition 3.7.3 shows that the two spectral sequences associated to these two filtrations converge to $\mathcal{H}^n(K^\bullet)$. 
Let us first consider the spectral sequence $^rE_{p,q}^r$ with respect to the $F'$ filtration. We have that $^rE_0^{p,q} = A^{p,q}$ and the induced map $d_0: E_0^{p,q} \to E_0^{p,q+1}$ is given by $(-1)^q d_2$. This gives

$$^rE_1^{p,q} = H^q(A^{p,*}).$$

The map induced by taking $\mathcal{H}^q$ of $A^{p,*} \to A^{p+1,*}$ gives

$$^rE_{1,q} \to ^rE_{1,q+1}. $$

We then define

$$^rE_2^{p,q} = H^p(A^{q,*}) = H(H^q(A^{p-1,*}) \to H^q(A^{p,*}) \to H^q(A^{p+1,*})).$$

Similarly, for the $F''$ filtration we get

$$^rE_2^{p,q} = H^p(A^{q,p}),$$

$$^rE_1^{p,q} = H^q(A^{p-*}),$$

$$^rE_0^{p,q} = A^{q,p},$$

3.8. **The Grothendieck spectral sequence for a composition of 2 functors.** Consider two left exact functors

$$C_1 \xrightarrow{G} C_2 \xrightarrow{F} C_3.$$ 

**Theorem 3.8.1.** Suppose for any injective object $I$ of $C_1$, $G(I)$ is $F$-acyclic. Then for any object $A$ of $C_1$, there is a spectral sequence $E_2^{p,q} = R^pF(R^qG(A))$ and

$$E_2^{p,q} \Rightarrow R^{p+q}(G \circ F)(A).$$

**Example 3.8.2 (Leray Spectral Sequence).** A composition of morphisms of algebraic varieties

$$X \xrightarrow{g} Y \xrightarrow{f} Z$$

induces

$$\mathcal{O}_{X\text{-mod}} \xrightarrow{g_*} \mathcal{O}_{Y\text{-mod}} \xrightarrow{f_*} \mathcal{O}_{Z\text{-mod}}$$

If $I \in \mathcal{O}_{X\text{-mod}}$ is injective, it is flasque, and hence $g_*I$ is flasque, i.e. $f_*$-acyclic. Then by Theorem 3.8.1, we get a spectral sequence

$$E_2^{p,q} = R^p f_*(R^q g_*(G)) \Rightarrow R^{p+q}((f \circ g)_*F).$$

In particular, if $Z$ is a point, we see that for a morphism $g: X \to Y$ and and $\mathcal{O}_{X\text{-module}} F$, we gave a spectral sequence

$$E_2^{p,q} = H^p(Y, R^q g_*(F)) \Rightarrow H^{p+q}(X,F).$$
Example 3.8.3. Suppose $g$ is affine (for example, if it is finite or a closed immersion). If $\mathcal{F}$ is quasicoherent on $X$, then $R^q f_* \mathcal{F} = 0$ for $q \geq 1$. If $U \subseteq Y$ is affine, then

$$\Gamma(U, R^q f_* \mathcal{F}) \cong H^q(f^{-1}(U), \mathcal{F}) = 0 \text{ for } q \geq 1.$$  

The Leray Spectral Sequence shows that

$$H^n(X, \mathcal{F}) = H^n(Y, g_*(\mathcal{F}))$$

for any quasicoherent sheaf $\mathcal{F}$ if $g$ is affine.

Definition 3.8.4. Given a complex $C^\bullet$ bounded from below ($C^p = 0$ if $p \ll 0$), a Cartan–Eilenberg resolution of $C^\bullet$ is a double complex $A^{\bullet,\bullet}$ together with a morphism of complexes $C^\bullet \to A^{\bullet,\bullet}$ such that

1. there is a $p_0$ such that $A^{p,q} = 0$ if $p \leq p_0$ for any $q$; $A^{p,q=0}$ if $q < 0$ for any $p$,
2. for any $p$, $C^p \to A^{p,0}$ is an injective resolution,
3. for any $p$, $\ker(d_p) \to \ker(d_{p,1})$ in an injective resolution,
4. for any $p$, $\text{im}(d_p) \to \text{im}(d_{p,1})$ is an injective resolution,
5. for any $p$, $H^p(C^\bullet) \to H^p(A^{\bullet,0}) \to H^p(A^{\bullet,1}) \to \cdots$ is an injective resolution.

Lemma 3.8.5. In any category with enough injectives, any complex bounded from below has a Cartan–Eilenberg resolution.

Proof. Fix $p_0$ such that $C^p = 0$ for $p \leq p_0$. For any $p$, we have two short exact sequences

\begin{align*}
(1) \quad 0 & \longrightarrow \text{im}(d^{p-1}) \longrightarrow \ker(d^p) \longrightarrow H^p(C^\bullet) \longrightarrow 0 \\
(2) \quad 0 & \longrightarrow \ker(d^p) \longrightarrow C^p \longrightarrow \text{im}(d^p) \longrightarrow 0
\end{align*}

For any $p$, choose injective resolutions

$$H^p(C^\bullet) \to U^{p,\bullet}$$

$$\text{im}(d^{p-1}) \to V^{p,\bullet}$$

such that $U^{p,\bullet} = V^{p,\bullet} = 0$ if $p \leq p_0$. By Horseshoe Lemma 3.1.11 applied to the exact sequence (1), we get an injective resolution $\ker(d^p) \to W^{p,\bullet}$ such that the diagram

\begin{align*}
0 & \longrightarrow \text{im}(d^{p-1}) \longrightarrow \ker(d^p) \longrightarrow H^p(C^\bullet) \longrightarrow 0 \\
\downarrow & \downarrow \downarrow \downarrow \\
0 & \longrightarrow V^{p,\bullet} \longrightarrow W^{p,\bullet} \longrightarrow U^{p,\bullet} \longrightarrow 0
\end{align*}

commutes. By Horseshoes Lemma 3.1.11 applied to the exact sequence (2), we get an injective resolution $C^p \to A^{p,\bullet}$ such that the diagram

\begin{align*}
0 & \longrightarrow \ker(d^p) \longrightarrow C^p \longrightarrow \text{im}(d^p) \longrightarrow 0 \\
\downarrow & \downarrow \downarrow \downarrow \\
0 & \longrightarrow W^{p,\bullet} \longrightarrow A^{p,\bullet} \longrightarrow V^{p+1,\bullet} \longrightarrow 0
\end{align*}
commutes. Putting these two together, we get morphisms of complexes
\[ A^p_\cdot \to V^{p+1}_\cdot \hookrightarrow W^{p_1}_\cdot \hookrightarrow A^{p_1}_\cdot, \]
making \( A^{\bullet} \cdot \) a double complex. \( \square \)

Suppose now that we have a complex \( C^\bullet \) bounded from below of objects in a category \( C \) and \( C^\bullet \to A^{\bullet} \cdot \) is a Cartan–Eilenberg resolution. Let \( \mathcal{G} : C \to C' \) be a left exact functor. The goal is to describe the spectral sequence associated to the double complex \( \mathcal{G}(A^{\bullet} \cdot) \).

The first spectral sequence associated to this double complex is
\[ E_2^{p,q} = \mathcal{H}^q(\mathcal{G}(A^{p} \cdot)) = R^q \mathcal{G}(C^p) \]
with
\[ E_1^{p,q} \to E_1^{p+1,q} \]
induced by the map \( C^p \to C^{p+1} \). This gives
\[ E_2^{p,q} = \mathcal{H}^p(R^q \mathcal{G}(C^\bullet)) \]
and
\[ E_1^{p,q} \Rightarrow \mathcal{H}^{p+q}(\text{Tot}(\mathcal{G}(A^{\bullet} \cdot))). \]

We show that the second exact sequence associate to the double complex \( \mathcal{G}(A^{\bullet} \cdot) \) is
\[ E_2^{p,q} = \mathcal{H}(\mathcal{G}(A^{p} \cdot)) \]
Recall that for every \( p \) we have the two exact sequences
\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \ker(d_1^{i,p}) & \longrightarrow & A^{i,p} & \longrightarrow & \im(d_1^{i,p}) & \longrightarrow & 0 \\
0 & \longrightarrow & \im(d_1^{i-1,p}) & \longrightarrow & \ker(d_1^{i,p}) & \longrightarrow & \mathcal{H}^q(A^{\bullet,q}) & \longrightarrow & 0
\end{array}
\]
which splits since \( \ker(d_1^{i,p}) \), \( \im(d_1^{i-1,p}) \) are injective. Hence the sequences stay exact after applying \( \mathcal{G} \), and hence
\[ E_2^{p,q} = \mathcal{G}(\mathcal{H}(A^{\bullet,p})). \]

Since we know \( \mathcal{H}^q(C^\bullet) \to \mathcal{H}^q(A^{\bullet} \cdot) \) is an injective resolution, we conclude that
\[ E_2^{p,q} = R^p \mathcal{G}(\mathcal{H}(C^\bullet)) \Rightarrow \mathcal{H}^{p+q}(\text{Tot}(\mathcal{G}(A^{\bullet} \cdot))). \]

Suppose that, in addition, all \( C^p \) are \( \mathcal{G} \)-acyclic. Then \( E_2^{p,q} = \mathcal{H}^q(R^p \mathcal{G}(C^\bullet)) \) shows that \( E_1^{p,q} = 0 \) if \( q \neq 0 \), and
\[ \mathcal{H}^n(\text{Tot}(\mathcal{G}(A^{\bullet} \cdot))) \cong \mathcal{H}^n(\mathcal{G}(C^\bullet)). \]

Therefore, we have a spectral sequence
\[ E_2^{p,q} = R^p \mathcal{G}(\mathcal{H}(C^\bullet)) \Rightarrow \mathcal{H}^n(\mathcal{G}(C^\bullet)). \]

Suppose now we are in the setting of Grothendieck spectral sequence 3.8.1:
\[ \mathcal{C}_1 \xrightarrow{\mathcal{F}} \mathcal{C}_2 \xrightarrow{\mathcal{G}} \mathcal{C}_3 \]
with \( \mathcal{F} \) and \( \mathcal{G} \) left exact, and \( \mathcal{F} \) mapping injective objects to \( \mathcal{G} \)-acyclic objects. For \( \mathcal{A} \in \text{Ob}(\mathcal{C}_1) \), let \( \mathcal{A} \to \mathcal{I}^\bullet \) be an injective resolution.
Consider the complex $\mathcal{F}(\mathcal{I}^\bullet)$. By assumption, all the terms are $\mathcal{G}$-acyclic, and hence the above discussion shows that the Grothendieck spectral sequence 3.8.1 becomes:

$$E_2^{pq} = R^pG(\mathcal{H}^q(\mathcal{F}(\mathcal{I}^\bullet))) = R^pG(R^q\mathcal{F}(A)) \Rightarrow \mathcal{H}^{p+q}(\mathcal{G}(\mathcal{F}(\mathcal{I}^\bullet))) = R^{p+q}(\mathcal{G} \circ \mathcal{F})(A).$$

3.9. Čech cohomology. Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{U} = (U_i)_{i \in I}$ be a finite open cover. Moreover, let $\mathcal{F}$ be an $\mathcal{O}_X$-module (or just a sheaf of abelian groups).

For $J \subseteq I$, write $U_J = \bigcap_{i \in J} U_i$, and by convention $U_\emptyset = X$. Choose an order on $I$. For $p \geq 0$, let

$$C^p(U, \mathcal{F}) = \bigoplus_{J \subseteq I} |J| = p + 1 \mathcal{F}(U_J).$$

For example,

$$C^0(U, \mathcal{F}) = \bigoplus_i \mathcal{F}(U_i), \quad C^1(U, \mathcal{F}) = \bigoplus_{i < j} \mathcal{F}(U_i \cap U_j).$$

Define the map

$$d : C^p(U, \mathcal{F}) \to C^{p+1}(U, \mathcal{F})$$

\[(s_J)_J \mapsto (s_{J'})_{J'}\]

where for $J' = \{j_0 < \cdots < j_{p+1}\}$ we set

$$s_{J'} = \sum_{q=0}^{p+1} (-1)^q s_{J' \setminus \{j_q\}}|U_{j'}.\]

Exercise. Show that $d \circ d = 0$.

Definition 3.9.1. The Čech complex associated to $\mathcal{F}$ and the cover $\mathcal{U}$ is $C^\bullet(U, \mathcal{F})$. The Čech cohomology is the cohomology of this complex

$$\check{H}^p(U, \mathcal{F}) = H^p(C^\bullet(U, \mathcal{F})).$$

Note that by the sheaf axiom, $\check{H}^0(U, \mathcal{F}) = \mathcal{F}(X)$.

Theorem 3.9.2. If $X$ is an algebraic variety, $\mathcal{U}$ is a finite affine open cover, and $\mathcal{F}$ is a quasicoherent $\mathcal{O}_X$-module, then there is a functorial isomorphism

$$\check{H}^p(U, \mathcal{F}) \cong H^p(X, \mathcal{F}).$$

Before we prove this theorem, we prove two lemmas. Let us first sheafify the above construction. For $J \subseteq I$, let $\alpha_J : U_J \to X$. Then set $\mathcal{F}_J = \mathcal{F}|_{U_J}$ and

$$C^p = C^p(U, \mathcal{F}) = \bigoplus_{|J| = p+1} (\alpha_J)_* \mathcal{F}_J \quad \text{for } p \geq -1.$$
with $C^{-1} = \mathcal{F}$ by convention. Then

$$\Gamma(U, C^\bullet) = \bigoplus_{|J| = p+1} \mathcal{F}(U \cap U_J)$$

and we have maps $d^p : C^p \to C^{p+1}$ defined by the same formulas as above. We then get a complex

$$0 \to C^{-1} = \mathcal{F} \to C^0 \to C^1 \to \cdots$$

and applying $\Gamma(X, -)$ recovers the previous complex $C^\bullet(U, \mathcal{F})$:

$$0 \to \mathcal{F}(X) \to C^\bullet(U, \mathcal{F}) \to \cdots$$

**Lemma 3.9.3.** The complex

$$0 \to C^{-1} \to C^0 \to C^1 \to \cdots$$

is an exact complex of sheaves.

**Proof.** We show that for any $x \in X$, the corresponding sequence of stalks at $x$ is exact. By choosing $i_0 \in I$ such that $x \in U_{i_0}$, it is enough to show that for all open subsets $U \subseteq U_{i_0}$, the sequence

$$0 \to C^{-1}(U) \to C^0(U) \to C^1(U) \to \cdots$$

is exact. In other words, we need to show that the identity map and the zero map on this complex are homotopic, $id \approx 0$. Define

$$\theta^p : C^p(U) = \bigoplus_{|J| = p+1} \mathcal{F}(U \cap U_J) \to C^{p-1}(U)$$

where

$$s_{J'} = \begin{cases} 0 & \text{if } i_0 \notin J' \\ (-1)^s s_{J' \setminus \{i_0\}} & \text{if } i_0 \in J' \text{ and } J' \text{ contains exactly } s \text{ elements } j' \text{ with } j' < i_0. \end{cases}$$

Note that $U \cap U_{J \setminus \{i_0\}} = U \cap U_J$.

**Exercise.** The maps $(\theta^p)_{p \geq 0}$ give a homotopy between id and 0.

**Lemma 3.9.4.** If $U$ is an open affine subset of $X$, $j : U \hookrightarrow X$ is the inclusion map, and $\mathcal{F}$ is a quasicoherent sheaf on $U$, then $H^p(X, j_*(\mathcal{F})) = 0$ for $p \geq 1$.

**Proof.** Since $j$ is an affine map, this is a consequence of the Leray Spectral Sequence (Example 3.8.2):

$$H^p(X, j_*(\mathcal{F})) \cong H^p(U, \mathcal{F}) = 0$$

for $p \geq 1$ by vanishing of cohomology of quasicoherent sheaves on affine varieties (Theorem 3.4.1).
Proof of Theorem 3.9.2. By Lemma 3.9.3 we have a resolution of $\mathcal{F}$ given by $\mathcal{F} \to \mathcal{C}^\bullet$. By Lemma 3.9.4, for any $\emptyset \neq J \subseteq I$, writing $\alpha_J: U_J \to X$ for the inclusion map, $(\alpha_J)_*\mathcal{F}_J$ is $\Gamma(X, -)$-acyclic. Therefore,

$$H^p(X, \mathcal{F}) = H^p(\Gamma(X, \mathcal{C}^\bullet)) \quad \text{by definition of sheaf cohomology}$$

$$= \check{H}^p(U, \mathcal{F}) \quad \text{by definition of Čech cohomology}$$

completing the proof.  

Corollary 3.9.5.  

(1) If any algebraic variety $X$, there is a $d$ such that $H^i(X, \mathcal{F}) = 0$ for all $i \geq d$ and any quasicoherent sheaf $\mathcal{F}$ on $X$.

(2) Suppose $X = \text{MaxProj}(S)$ and $n = \dim X$.

- If $\mathcal{F}$ is quasicoherent on $X$, then $H^i(X, \mathcal{F}) = 0$ for $i > n$.
- If $\mathcal{F}$ is coherent on $X$, $\dim(\text{supp}(\mathcal{F})) = r$, then $H^i(X, \mathcal{F}) = 0$ for $i > r$.

Proof. For (1), take $d$ such that there is a cover of $X$ by $d$ affine open subsets and use Čech cohomology 3.9.2.

For (2), if $Z \subseteq X$ is a closed subvariety of dimension $r$, there are affine open subsets $U_1, \ldots, U_{r+1}$ in $X$ such that $Z \subseteq \bigcup_{i=1}^{r} U_i$. In fact, we can take $U_i = D_X^+(h_i)$, $h_i \in S_1$. For $Y = \text{MaxSpec}(S_0)$, we have

$$\begin{array}{ccc}
P^N & \xrightarrow{q} & \mathbb{P}^N \\
\downarrow f & \searrow & \downarrow p \\
X & \xleftarrow{i} & \mathbb{P}^N \times Y \\
\downarrow & & \downarrow \\
Y & & Y
\end{array}$$

If $\dim(f(Z)) \leq r$, then $f(Z) \cap \bigcap_{i=1}^{r+1} V(h_i) = \emptyset$ for suitable

$$h_i \in k[x_0, \ldots, x_n]_1 \subseteq (S_0[x_0, \ldots, x_n])_1 \to S_1.$$  

This shows that $H^i(X, \mathcal{F}) = 0$ for $i > r$ using Čech cohomology to compute the sheaf cohomology (Theorem 3.9.2).  

Remark 3.9.6 (Grothendieck). If $X$ is an algebraic variety of dimension $n$, then for any sheaf of abelian groups $\mathcal{F}$ on $X$, $H^i(X, \mathcal{F}) = 0$ for $i > n$.

3.10. Coherent sheaves on projective varieties. Let $X = \text{MaxProj}(S)$ where $S = \bigoplus_{i \geq 0} S_i$ is an $\mathbb{N}$-graded reduced $k$-algebra such that $S_0$ is finitely-generated over $k$ and $S$ is generated by $S_1$ as an $S_0$-algebra and $S_1$ is a finitely-generated $S_0$-module. In other words, there is a surjective map

$$S_0[x_0, \ldots, x_n] \twoheadrightarrow S$$

and hence we have a diagram
Unsurprisingly, to define coherent sheaves on projective varieties, we need to introduce a grading.

**Definition 3.10.1.** A graded $S$-module is an $S$-module $M$ with a decomposition

$$M = \bigoplus_{i \in \mathbb{Z}} M_i$$

such that $S_i \cdot M_j \subseteq M_{i+j}$ for any $i,j$. Elements of $M_i$ are called homogeneous of degree $i$.

A morphism of graded modules $f: M \to N$ is a morphism of modules such that $f(M_i) \subseteq N_i$ for any $i$.

Since we can compose these morphisms, this gives a category of graded $S$-modules.

If $M$ is a graded $S$-module, a graded submodule $N \subseteq M$ is a submodule generated by homogeneous elements. Equivalently, the decomposition $M = \bigoplus M_i$ induces a decomposition $N = \bigoplus_i (N \cap M_i)$, so $N$ is a graded module such that $N \hookrightarrow M$ is a graded module. Then the quotient

$$M/N = \bigoplus_{i \in \mathbb{Z}} (M_i/N \cap M_i)$$

is a graded module such that $M \twoheadrightarrow M/N$ is a graded morphism.

It is clear that if $f: M \to N$ is a morphism of graded modules, then $\ker(f) \subseteq M$ and $\operatorname{im}(f) \subseteq N$ are graded submodules. Using quotients, we construct $\operatorname{coker}(f)$.

Altogether, this shows that the category of graded $S$-modules is an abelian category.

Recall that on $X$ we have a basis for the topology given by

$$D_X^+(f) \text{ with } f \text{ homogeneous }, \deg(f) > 0.$$  

Each of these is affine and

$$\Gamma(D_X^+(f), \mathcal{O}_X) = S(f).$$

Recall that, by definition, $S(f) = (S_f)_0$, the 0-graded piece of $S_f$. We will similarly write $M(f)_0$ for the 0-graded piece of $M_f$.

Suppose now $M$ is a graded $S$-module. Given $D_X^+(f)$, consider $(M_f)_0$. Note that if $D_X^+(g) \subseteq D_X^+(f)$, we get canonical map $(M_f)_0 \to (M_g)_0$. Indeed, $V(g) \subseteq V(f)$, so $g \subseteq \sqrt{(f)}$, so by universal property of localization, we get a map

$$M_f \twoheadrightarrow M_g.$$
which is graded, and hence gives a map $(M_f)_0 \to (M_g)_0$.

**Lemma 3.10.2.** Given $f$ and $f_1, \ldots, f_r \in S$ homogeneous of positive degree such that $D_X^+(f) = \bigcup_{i=1}^r D_X^+(f_i)$, then the following sequence

$$0 \longrightarrow M(f) \longrightarrow \bigoplus_{i=1}^r M(f_i) \longrightarrow \bigoplus_{i<j} M(f_i f_j)$$

**Proof.** This is similar to the corresponding assertion for affine varieties, so it is left as an exercise. (See the official notes for a solution.) \qed

We conclude that there is an $\mathcal{O}_X$-module $\tilde{M}$ such that

$$\Gamma(D_X^+(f), \tilde{M}) = M(f).$$

(It remains to check the compatibility of the restriction maps but this is clear.)

**Examples 3.10.3.**

1. Trivially, $\tilde{S} = \mathcal{O}_X$.
2. Given $m \in \mathbb{Z}$ and a graded module $M$, let $M(m)$ be $M$ as an $S$-module, but $M(m)_i = M_{m+i}$. Then $M(m)$ is a graded module. Set

$$\tilde{S}(m) = \mathcal{O}_X(m).$$

We claim that this is a line bundle. Since $S$ is generated as an $S_0$-algebra by $S_1$, we can cover $X$ by open subsets of the form $D_X^+(f)$ for $D_X(f) = 1$. Now, note that

$$\Gamma(D^+(X), \mathcal{O}_X(m)) = (S_f)_m \xrightarrow{\cong} (S_f)_0,$$

$$u \mapsto f^{-m} u.$$

3. If $X = \mathbb{P}^n$, we recover the old $\mathcal{O}_{\mathbb{P}^n}(m)$. Indeed, the above isomorphism for $U_i = D_{\mathbb{P}^n}(x_i)$ becomes

$$\varphi_i: \mathcal{O}_{\mathbb{P}^n}(m)|_{U_i} \to \mathcal{O}_{\mathbb{P}^n}|_{U_i}$$

given by multiplication by $\frac{x_j}{x_i}$. Then the transition functions are given by $\varphi_i \circ \varphi_j^{-1} = \left(\frac{x_j}{x_i}\right)^m$

$$\mathcal{O}_{\mathbb{P}^n}(m)|_{U_i \cap U_j} \xrightarrow{\varphi_i \circ \varphi_j^{-1} = \left(\frac{x_j}{x_i}\right)^m} \mathcal{O}_{\mathbb{P}^n}|_{U_i \cap U_j}$$

This agrees with the previously computed transition functions.

**Notation.** If $F$ is an $\mathcal{O}_X$-module, let

$$F(m) = F \otimes_{\mathcal{O}_X} \mathcal{O}_X(m).$$
We claim that \( \tilde{M}(m) = \hat{M}(m) \). Indeed, for \( \deg(f) = 1 \), we clearly have an isomorphism
\[
\Gamma(D_X(f), \hat{M}(m)) = (M_f)_0 \otimes_{(S_f)_0} (S_f)_m \cong (M_f)_m = \Gamma(D_X(f), \tilde{M}(m)),
\]
\[
\frac{u}{f^r} \otimes \frac{a}{f^s} \mapsto \frac{au}{f^{r+s}}.
\]

For example, if \( M = S(n) \), we have that
\[
\mathcal{O}_X(m) \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) \cong \mathcal{O}_X(m+n).
\]

Finally, the assignment \( M \mapsto \tilde{M} \) is functorial. If \( \varphi : M \to N \) is a morphism of graded modules, we get \( (M_f)_0 \to (N_f)_0 \) for every homogeneous \( f \) of positive degree. These induce morphisms of \( \mathcal{O}_X \)-modules \( \tilde{M} \to \tilde{N} \).

**Properties.**

1. This is an exact functor (since localization is exact).
2. It commutes with arbitrary direct limits.

**Proposition 3.10.4.** The \( \mathcal{O}_X \)-module \( \tilde{M} \) is quasicoherent. If \( M \) is finitely generated, \( \tilde{M} \) is coherent.

**Proof.** Choose generators \( (u_i)_{i \in I} \) homogeneous of \( \deg u_i = a_i \) (finite if \( M \) is finitely-generated). Consider the map
\[
\bigoplus_{i \in I} S(-a_i) \xrightarrow{\varphi} M, \\
 e_i \mapsto u_i.
\]
Repeat this for \( \ker(\varphi) \) to get an exact sequence
\[
\bigoplus_{j \in J} S(-b_j) \longrightarrow \bigoplus_{i \in I} S(-a_i) \longrightarrow M \longrightarrow 0.
\]
Applying \( \sim \), we get an exact sequence
\[
\bigoplus_{j \in J} \mathcal{O}_X(-b_j) \longrightarrow \bigoplus_{i \in I} \mathcal{O}_X(-a_i) \longrightarrow \tilde{M} \longrightarrow 0.
\]
Then \( \tilde{M} \) is quasicoherent (or coherent if \( M \) is finitely-generated) as a cokernel of a morphism of quasicoherent (coherent) sheaves. \( \square \)

**Question.** When is \( \tilde{M} = 0 \)?

It is enough to see when \( (M_f)_0 \) for \( f \in S_1 \) (since \( \tilde{M} \) is quasicoherent). Since \( (M_f)_0 \cong (M_f)_m \) for any \( m \in \mathbb{Z} \) by mapping \( u \mapsto f^m u \), we see that this is equivalent to \( M_f = 0 \) for all \( f \in S_1 \).

Since \( S \) is generated over \( S_0 \) by \( S_1 \), we have \( S_+ = \bigoplus_{i>0} S_i = (S_1) \).
**Answer.** This shows that $\tilde{M} = 0$ if and only if for any $u \in M$, $(S_+)^N \cdot u = 0$ for some $N$.

If $M$ is finitely-generated, this is equivalent to $(S_+^N) \cdot M = 0$ for some $N$.

**Exercise.** This is equivalent to $M_i = 0$ for $i \gg 0$.

We now define a functor in the opposite direction. For a quasicoherent sheaf $\mathcal{F}$ on $X$, let

$$\Gamma_*(\mathcal{F}) = \bigoplus_{m \in \mathbb{Z}} \Gamma(X, \mathcal{F}(m))$$

as a graded abelian group.

**Remarks 3.10.5.**

1. For any $M$, we have a morphism

$$M \xrightarrow{\Phi_M} \Gamma(X, \tilde{M}) = \bigoplus_{i \in \mathbb{Z}} \Gamma(X, \tilde{M}(i)).$$

Indeed, take $u \in M_i$. For $f$ homogeneous of positive degree, consider

$$\frac{u}{1} \in (M_f)_i = \Gamma(D_X^+(f), \tilde{M}(i)).$$

These glue to give $\Phi_M(u) \in \Gamma(X, \tilde{M}(i))$.

For $M = S$, this gives a map

$$\Phi_S: S \to \bigoplus_{i \in \mathbb{Z}} \Gamma(X, \mathcal{O}_X(i)).$$

2. For any quasicoherent sheaf $\mathcal{F}$ on $X$, there is a map

$$\Gamma(X, \mathcal{O}_X(i)) \otimes \Gamma(X, \mathcal{F}(j)) \to \Gamma(X, \mathcal{F}(i + j))$$

given by tensor product of sections. For $\mathcal{F} = \mathcal{O}_X$, this makes $\Gamma_*(\mathcal{O}_X)$ a graded ring such that $\Phi_S$ is a graded homomorphism.

For any $\mathcal{F}$, this makes $\Gamma_*(\mathcal{F})$ a graded module over $\Gamma_*(\mathcal{O}_X)$, and hence a graded module over $S$ via $\Phi_S$.

3. We get a functor

$$\text{Qcoh}(X) \to \text{graded } S\text{-modules}.$$  

Indeed, for a map $\mathcal{F} \to \mathcal{G}$, we get maps $\mathcal{F}(m) \to \mathcal{G}(m)$ for all $m$, which give a map $\Gamma_*(\mathcal{F}) \to \Gamma_*(\mathcal{G})$.

**Proposition 3.10.6.** For every $\mathcal{F}$, we have a canonical isomorphism

$$\psi_\mathcal{F}: \tilde{\Gamma}_*(\mathcal{F}) \to \mathcal{F}.$$  

Note that if $f \in S_+$, $D_X^+(f) = X \setminus V(f)$. If $f \in S_m$, we get a section in $\mathcal{O}_X(m)$, locally given by $\frac{1}{f}$. Then $V(f)$ is the zero locus of this section.

The following lemma is a generalization of this.

**Lemma 3.10.7.** Suppose $X$ is an algebraic variety, $\mathcal{F} \in \text{Qcoh}(X)$, $s \in \Gamma(X, \mathcal{L})$, and $U = X \setminus V(s)$.  

(1) If \( t \in \Gamma(X, \mathcal{F}) \) is a section such that such that \( t|_U = 0 \), then there is an \( N \) such that
\[ s^N \cdot t \in \Gamma(X, \mathcal{F} \otimes \mathcal{L}^N) \]
is 0.

(2) For any \( t \in \Gamma(U, \mathcal{F}) \), there is a \( q \) such that \( s^q|_U \cdot t \) is the restriction of a section in \( \Gamma(X, \mathcal{F} \otimes \mathcal{L}^q) \).

**Proof.** Exercise. (Cover \( X \) by affine open subsets \( U_i \) such that \( \mathcal{L}|_{U_i} \cong \mathcal{O}_{U_i} \). In this case, we know both assertion for \( U_i \subseteq U \cap U_i \), so we just need to glue these.) \( \square \)

**Proof of Proposition 3.10.6.** We need compatible maps for each \( f \in S \), homogeneous of degree \( d > 0 \),
\[ (\Gamma_*(\mathcal{F})_f)_0 \to \Gamma(D^+_X(f), \mathcal{F}). \]

An element of \( (\Gamma_*(\mathcal{F})_f)_0 \) can be written as
\[ \frac{s}{f^m} \]
for \( s \in \Gamma(X, \mathcal{F}(md)) \), since \( \frac{1}{f^m} \in \Gamma(D^+_X(f), \mathcal{O}_X(-md)) \). We define
\[ (\Gamma_*(\mathcal{F})_f)_0 \to \Gamma(D^+_X(f), \mathcal{F}), \]
\[ \frac{s}{f^m} \mapsto \frac{1}{f^m}s|_{D^+_X(f)}. \]

These glue to give \( \psi_\mathcal{F} \). Let us show that \( \psi_\mathcal{F} \) is an isomorphism. It is enough to show it is an isomorphism on each \( D^+_X(f) \).

For injectivity, suppose \( \frac{s}{f^m} \mapsto 0 \), so \( s|_{D^+_X(f)} = 0 \), since \( \frac{1}{f^m} \neq 0 \) on \( D^+_X(f) \). Now, \( D^+_X(f) = X \setminus V(f) \) and Lemma 3.10.7 shows that there is an \( N \) such that \( f^N \cdot s = 0 \) in \( \Gamma(X, \mathcal{F}(md + Nd)) \). Then
\[ \frac{s}{f^m} = \frac{f^N s}{f^m + N} = 0 \]
in \( \Gamma_*(\mathcal{F}) \).

For surjectivity, given \( t \in \Gamma(D^+_X(f), \mathcal{F}) \), Lemma 3.10.7 shows that there is a \( q \) such that \( f^q|_{D^+_X(f)} \cdot t \) extends to \( t' \in \Gamma(X, \mathcal{F}(qd)). \) Then \( t = \psi(t'/f^q) \), showing surjectivity. \( \square \)

**Corollary 3.10.8.** If \( \mathcal{F} \in \text{Coh}(X) \), there exists a finitely-generated \( S \)-module \( M \) such that \( \tilde{M} = \mathcal{F} \).

**Proof.** By Proposition 3.10.6, there is an \( S \)-module \( N \) such that \( \mathcal{F} \cong \tilde{N} \). Choose a finite cover \( X = \bigcup_{i=1}^r D^+_X(f_i) \) such that
\[ (N_{f_i})_0 = \Gamma(D^+_X(f_i), \mathcal{F}) \]
is finitely-generated over \( (S_{f_i})_0 \).

Choose generators for each of these and let \( M \subseteq N \) be the graded \( S \)-module generated by the numerators of these generators. Then
\[ (N_{f_i})_0 \subseteq \Gamma(D^+_X(f_i), \tilde{M}) \subseteq \Gamma(D^+_X(f_i), \tilde{N}) \]
for all \( i \), and hence \( \tilde{M} = \tilde{N} \). \( \square \)
**Remark 3.10.9.** Given any $M$, we have $\Phi_M: M \to \Gamma_*(\widetilde{M})$. This gives a morphism

$$\widetilde{\Phi}_M: \widetilde{M} \to \Gamma_*(\widetilde{M})$$

and by Proposition 3.10.6, we have an isomorphism

$$\Psi_{\widetilde{M}}: \Gamma_*(\widetilde{M}) \cong \widetilde{M}.$$  

It is easy to check that $\psi_{\widetilde{M}} \circ \widetilde{\Phi}_M = 1$, and hence $\widetilde{\Phi}_M$ is an isomorphism.

In particular, if $M$ is finitely generated, then for $i \gg 0$, the map

$$M_i \to \Gamma(X, \widetilde{M}(i))$$

is an isomorphism.

**Exercise.** Suppose $S \twoheadrightarrow T$ is a surjective graded homomorphism, inducing

$$i: \text{MaxProj}(T) = X \hookrightarrow \text{MaxProj}(S) = Y.$$  

If $N$ is a graded $S$-module, then $M = N \otimes_S T = N/I\!N$ where $I = \ker(S \to T)$ is a graded $T$-module. Then

$$i^*\widetilde{N} \cong \widetilde{M}.$$  

In particular, $i^*\mathcal{O}_Y(m) \cong \mathcal{O}_X(m)$.

**Remark 3.10.10.** The construction of $\mathcal{O}(1)$ globalizes. Suppose $T$ is any variety and $S = \bigoplus_{m \geq 0} S_m$ is a graded, reduced, quasicoherent $\mathcal{O}_T$-algebra such that $S_0$ and $S_1$ are coherent and $S$ is generated by $S_1$ over $S_0$. Then we get a map $\pi$ such that for any affine open $U$ the diagram

$$\begin{array}{ccc}
X = \text{MaxProj}(S) & \xleftarrow{\pi} & \text{MaxProj}(S(U)) \\
\downarrow \pi & & \downarrow \\
T & \xleftarrow{} & U
\end{array}$$

commutes. For each $U$, we have $\mathcal{O}_{\pi^{-1}(U)}(1)$ and these glue to give $\mathcal{O}_X(1)$.

We get a canonical morphism

$$S_i \to \pi_*\mathcal{O}_X(i).$$

**Example 3.10.11.** Suppose $T$ is irreducible and $\mathcal{I}$ is a coherent ideal on $T$. The blow-up along $\mathcal{I}$ was defined as

$$\widetilde{T} = \text{MaxProj} \left( \bigoplus_{m \geq 0} \mathcal{I}^m \right)$$

$$\downarrow \pi$$

$$T$$

Then $\mathcal{I} \cdot \mathcal{O}_{\widetilde{T}} = \mathcal{O}_{\widetilde{T}}(-E)$ for an effective Cartier divisor $E$.

**Exercise.** Check that $\mathcal{O}_{\widetilde{T}}(1) = \mathcal{O}_{\widetilde{T}}(-E)$. 
To do this, write this down explicitly locally.

Let $X$ be any variety and $F$ be a quasicoherent sheaf. There is a canonical morphism $\Gamma(X, F) \otimes_k \mathcal{O}_X \to F$ given on an open subset $U$ by

$$\Gamma(X, F) \otimes_k \mathcal{O}_X(U) \to F(U)$$

$$\sum_{i=1}^r s_i \otimes f_i \mapsto f_i \cdot s_i|_U$$

where $f_i \in \mathcal{O}_X(U)$.

**Definition 3.10.12.** The quasicoherent sheaf $F$ is **globally generated** if this map is surjective, i.e. for $x \in X$, $F_x$ is generated over $\mathcal{O}_{X,x}$ by

$$\{s_x \mid s \in \Gamma(X, F)\}.$$

**Definition 3.10.13.** A line bundle $L \in \text{Pic}(X)$ is **ample** if for any $F \in \text{Coh}(X)$, $F \otimes L^m$ is globally generated for $m \gg 0$.

For example, if $X$ is affine, every quasicoherent sheaf is globally generated. In particular, every line bundle is ample.

**Proposition 3.10.14.** If $X = \text{MaxProj}(S)$, $\mathcal{O}_X(1)$ is ample.

**Proof.** Let $F \in \text{Coh}(X)$. Then there is a finitely-generated $S$-module $M$ such that $F \cong \widetilde{M}$. Let $u_1, \ldots, u_n \in M$ be homogeneous generators with $d_i = \text{deg}(u_i)$.

We show that if $d \geq \max d_i$, then $F \otimes \mathcal{O}(d)$ is globally generated.

Let $S_+ = \bigoplus_{i>0} S_i$. Let $T \subseteq M$ be the submodule generated by $S_+^{d-d_i} \cdot u_i$ for $1 \leq i \leq n$. Then $T$ is finitely-generated over $S$ and it is generated by elements of degree $d$. Therefore, there is a surjective map

$$S(-d)^{\otimes q} \to T,$$

which gives a map $\mathcal{O}_X(-d)^{\otimes q} \to \widetilde{T}$. Twisting by $\mathcal{O}_X(d)$, we get

$$\mathcal{O}_X^{\otimes q} \to \widetilde{T}(d).$$

Hence $\widetilde{T}(d)$ is globally generated. For any $i$, $S_+^{d-d_i} \cdot u_i \subseteq T$, so there is an $N$ such that $S_+^N \cdot (M/T) = 0$. Therefore, $\widetilde{T} = \widetilde{M}$. \qed

3.11. **Cohomology of coherent sheaves on projective varieties.** Let $X = \text{MaxProj}(S)$.

**Theorem 3.11.1 (Serre).** If $F \in \text{Coh}(X)$,

1. $H^i(X, F)$ is a finitely-generated $S_0$-module for all $i$,
2. there exists $m_0(F)$ such that $H^i(X, F(m)) = 0$ for $m \geq m_0(F)$, $i \geq 1$.

Assume for now that this results holds if $S = S_0[x_0, \ldots, x_n]$ (i.e. $X$ is the product of an affine variety with $\mathbb{P}^n$) and $F = \mathcal{O}(j)$ for some $j$. We will deal with this case later.
Proof. Choose a graded surjection

\[ S_0[x_0, \ldots, x_n] \twoheadrightarrow S \]

inducing

\[ j : X \hookrightarrow Y = Z \times \mathbb{P}^n \text{ for } Z = \text{MaxSpec}(S_0). \]

If \( \mathcal{F} \in \text{Coh}(X) \), \( H^i(X, \mathcal{F} \otimes \mathcal{O}_X(m)) \cong H^i(Y, j_*(\mathcal{F} \otimes j^*\mathcal{O}_Y(m))) \), since we have noted before that \( \mathcal{O}_X(m) = j^*\mathcal{O}_Y(m) \). By the projection formula,

\[ j_\ast \mathcal{F} \otimes j^*\mathcal{O}_Y(m) = j_*(\mathcal{F}) \otimes \mathcal{O}_Y(m). \]

Hence, we may assume that \( S = S_0[x_0, \ldots, x_n] \).

For \( \mathcal{F} \in \text{Coh}(X) \), since \( \mathcal{O}_X(1) \) is ample by Proposition 3.10.14, there is a surjective map

\[ \mathcal{O}_X^r \twoheadrightarrow \mathcal{F} \otimes \mathcal{O}(m) \]

(exercise: check why this is true). Suppose \( \mathcal{F} \cong \widetilde{M} \) for a finitely-generated \( M \). There is a surjection

\[ \bigoplus_{i=1}^r S(-q_i) \twoheadrightarrow M. \]

We then have a short exact sequence

\[ 0 \longrightarrow \mathcal{G} \longrightarrow \bigoplus_{i=1}^r \mathcal{O}_X(-q_i) \longrightarrow \mathcal{F} = \widetilde{M} \longrightarrow 0. \]

After tensoring with \( \mathcal{O}_X(m) \) (which preserves exactness since \( \mathcal{O}_X(m) \) is locally free) and taking the long exact sequence in cohomology, we get the exact sequence

\[ (\ast) \quad \bigoplus_{j=1}^r H^i(X, \mathcal{O}_X(m - q_j)) \longrightarrow H^i(X, \mathcal{F}(m)) \longrightarrow H^{i+1}(X, \mathcal{G}(m)). \]

We argue by decreasing induction. We know both assertions for \( H^i \) if \( i > \dim \), since all cohomology groups vanish. For the inductive step, suppose we know \( H^{i+1}(X, \mathcal{M}) \) is finitely generated over \( S_0 \) and \( H^{i+1}(X, \mathcal{M}(m)) = 0 \) for \( m \gg 0 \) if \( \mathcal{M} \) is coherent. The exact sequence \((\ast)\) for \( m = 0 \) together with the inductive hypothesis and what we assume about \( \mathcal{O}(j) \), we conclude that \( H^i(X, \mathcal{F}) \) is finitely-generated over \( S_0 \).

Finally, for \( m \gg 0 \), the left term is 0 for \( i \geq 1 \) by what we assume about \( \mathcal{O}(j) \), and the right term is 0 by the inductive hypothesis, and hence \( H^i(X, \mathcal{F}(m)) = 0 \) for all \( m \gg 0 \). \( \square \)

We still have to deal with the case when \( S = S_0[x_0, \ldots, x_n] \) and \( \mathcal{F} = \mathcal{O}(j) \) for some \( j \). We prove a stronger result that allows to compute the cohomology explicitly in this case.

**Theorem 3.11.2.** Let \( X = \text{MaxProj}(S) \) where \( S = A[x_0, \ldots, x_n] \). Then

1. the canonical map \( S \to \bigoplus_{m \in \mathbb{Z}} \Gamma(X, \mathcal{O}(m)) \) is an isomorphism,
2. \( H^i(X, \mathcal{O}(m)) = 0 \) for all \( 1 \leq i \leq n - 1 \) and all \( m \).
(3) $H^n(X, \mathcal{O}(-n - 1)) \cong A$ and we have for every $m$ a canonical perfect pairing of finitely-generated free $A$-modules

$$\Gamma(X, \mathcal{O}(m)) \times H^n(X, \mathcal{O}(-m - n - 1)) \to H^n(X, \mathcal{O}_X(-n - 1)) \cong A$$

for all $m \in \mathbb{Z}$.

Note that this simplifies that every $\mathcal{O}(j)$ on $X$ satisfies the conclusion of Serre’s Theorem 3.11.1.

**Idea of Proof.** We was to compute the cohomology groups using Cech cohomology (Theorem 3.9.2) with respect to the cover by $D^+(X)(x_i)$ for $0 \leq i \leq n$. For $J \subseteq \{0, \ldots, x_n\}$, write $x_J = \prod_{i \in J} x_i$ and $U_J = \bigcap_{i \in J} U_i = D^+(x_J)$. We have that

$$C^p = \bigoplus_{J \subseteq \{0, \ldots, n\} \atop |J| = p + 1} (S_{x_J})_m.$$ 

This will allow us to compute the cohomology. \qed