VARIOUS TYPES OF SHIMURA VARIETY AND THEIR MODULI

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Abstract. This is the note for a talk given by the author in the SHIMURA seminar in the summer, 2020, at the University of Michigan. We introduce the Siegel, PEL-type, Hodge-type and abelian type Shimura varieties together with their moduli interpretations in order, following [Mil06] and [Lan10].

1. Siegel modular variety: $(\mathrm{GSp}_{2n}, \mathcal{H}_n^{\pm})$

In this section, we give an explicit example for *Siegel modular variaties*

1.1. **The group** $\mathrm{GSp}(\psi)$ **and** $\mathrm{Sp}(\psi)$**.** Let (V,ψ) be a symplectic space of dimension $2n$ over $\mathbb{Q}$; namely, V is a vector space of dimension $2n$ over a $\mathbb{Q}$, and ψ is a $\mathbb{Q}$-linear non-degenerate alternate form on V.

Denote $\mathrm{GSp}(\psi)$ to be the subgroup of $\mathrm{GL}(V)$ such that for any $\mathbb{Q}$-algebra R we have

$$\mathrm{GSp}(\psi)(R) := \{g \in \mathrm{GL}(V)(R) \mid \psi(gu, gv) = \nu(g)\cdot\psi(u,v) \ \mathit{for\ some}\ \nu(g) \in R^{\times}\}.$$

This defines a homomorphism ν of $\mathrm{GSp}(\psi)$ to the multiplicative group $\mathbb{G}_m$, and we denote $\mathrm{Sp}(\psi)$ to be the kernel. Here the subgroup $\mathrm{Sp}(\psi)$ is in fact the derived subgroup of $\mathrm{GSp}(\psi)$.

Example 1.1. An example of (V,ψ) can be given by the bilinear form defined by the following matrix on the vector space $\mathbb{Q}^{2n}$

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

The groups under the assumption of ψ above are given by

$$\mathrm{GSp}(\psi)(R) = \mathrm{GSp}_{2n} = \{g \in \mathrm{GL}(V)(R) \mid g^t J_n g = r\cdot J_n \ \mathit{for\ some}\ r \in R^{\times}\}; \tag{1.2}$$

$$\mathrm{Sp}(\psi)(R) = \mathrm{Sp}_{2n}(R) = \{g \in \mathrm{GL}(V)(R) \mid g^t J_n g = J_n\}. \tag{1.3}$$

Note that the similar description holds for more general forms of ψ, by replacing J_n by the defining matrix under a basis.

1.2. **Hodge structure.** Recall that there exists a bijection of the following two collections of data

$$\begin{array}{ccc} \{\mathit{complex\ structures}\ J\ \mathit{on}\ V(\mathbb{R})\} & \Longleftrightarrow & \{\mathit{Hodge\ structures}\ h_J : \mathbb{S} \to \mathrm{GL}(V)_{\mathbb{R}} \mid h_J\ \mathit{of\ type}\ (-1,0)\ \&\ (0,-1)\}. \\ \uparrow & & \uparrow \\ X^+ := \{\ J \mid {}^{\psi(Ju,Jv)=\psi(u,v);}_{\psi_J(\cdot,\cdot):=\psi(\cdot,J\cdot)\ positive-definite}\} & \Longleftrightarrow & \{\ h_J \mid 2\pi i\cdot\psi\ \mathit{is\ a\ polarization\ of}\ (V,h_J)\} \end{array}$$

Similarly we could define X^- to be those J with ψ_J replaced by the negative-definite condition (correspondingly ψ is a polarization of (V, h_{-j})). The union $X(\psi) := X^+ \coprod X^-$ admits an action of $\mathrm{GSp}(\psi)$ via

$$J \mapsto g\cdot J,$$

where $\mathrm{Sp}(\psi)$ leaves X^+ and X^- invariant separately.

Example 1.4. Assume (V,ψ) is of the form in the Example 1.1. Consider the homomorphism $h_0 : \mathbb{S} \to \mathrm{GL}(V)$ whose $\mathbb{R}$-valued points are given as follows:

$$\begin{aligned} \mathbb{C}^{\times} &\longrightarrow \mathrm{GL}(V)(\mathbb{R}); \\ z = x + iy &\longmapsto \begin{pmatrix} xI_n & -yI_n \\ yI_n & xI_n \end{pmatrix}. \end{aligned}$$

The corresponding complex structure on V is given by the matrix

$$J := h_0(i) = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

In particular, it is easy to check that J satisfies the condition

$$\psi(J\cdot, J\cdot) = \psi(\cdot,\cdot);$$
$$\psi_J(\cdot,\cdot) \ positive\ definite.$$

Thus the Hodge structure $h_0 : \mathbb{S} \to \mathrm{GL}(V)$ factors through GSp_{2n}. Moreover, we want to mention that the stabilizer of J in GSp_{2n} is equal to the unitary group as follows

$$\{\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in \mathrm{Sp}_{2n}(\mathbb{R})\} \cong \{g = A + iB \in \mathrm{M}_n(\mathbb{C}) \mid \overline{g}^t \cdot g = I_n\} = U_n.$$

1.3. **Siegel Shimura datum.** The action of $\mathrm{GSp}(\psi)$ on $X(\psi)$ is transitive. Moreover, it could be shown that the subgroup $\mathrm{Sp}(\psi)$ is transitively acting on X^+ and X^- separately. This allows us to define the pair $(\mathrm{GSp}(\psi), X(\psi))$ and $(\mathrm{Sp}(\psi), X^+)$, consisting of a transitive action of a reductive group on a Hermitian symmetric domain. In fact, we have the following:

Proposition 1.5. *Let (V,ψ) be a symplectic space of dimension $2n$ over $\mathbb{Q}$.*

(i) The pair $(\mathrm{GSp}(\psi), X(\psi))$ is a Shimura datum in the sense of [Mil06, Section 5]*, satisfying the axioms $SV1 - SV6$.*
(ii) The pair $(\mathrm{Sp}(\psi), X^+)$ is a connected Shimura datum in the sense of [Mil06, Section 4].

Example 1.6. In the case of $\mathrm{GSp}(\psi) = \mathrm{GSp}_{2n}$, the quotient GSp_{2n}/U_n is isomorphic to the *Siegel upper half plane*

$$\mathcal{H}_n^{\pm} = \{Z \in \mathrm{Sym}_n(\mathbb{C}) \mid \mathrm{Im} Z \text{ is positive/negative definite}\}.$$

When $n = 1$, $\mathcal{H}_1^+$ is the usual upper half plane in $\mathbb{C}$, and the Shimura data we get are $(\mathrm{GL}_2, \mathcal{H}^{\pm})$ and $(\mathrm{SL}_2, \mathcal{H})$ associated to modular curves.

1.4. **Moduli of polarized abelian varieties.** Recall that for a (connected) Shimura datum (G, X), the associated Shimura varieties, as an inverse system of algebraic varieties over $\mathbb{C}$, is defined by

$$\mathrm{Sh}_K(G,X) := G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K,$$

where K is among compact open subgroups of $G(\mathbb{A})$.

Now we translate points in $\mathrm{Sh}_K(G,X)$ for $(G,X) = (\mathrm{GSp}(\psi), X(\psi))$ into isomorphism classes of Hodge structures with polarizations and level structures.

Construction 1.7. Consider the set $\mathcal{H}_K$ of triples $((W,h), s, \eta K)$, where

- (W,h) is a rational Hodge structure of type $(-1,0),\ (0,-1)$;
- s or $-s$ is a polarization for (W,h);
- ηK is a K-orbit of $\mathbb{A}_f$-linear isomorphisms $V(\mathbb{A}_f) \to W(\mathbb{A}_f)$ under which ψ corresponds to an $\mathbb{A}_f^\times$-multiple of s.

We define the isomorphism $((W,h), s, \eta K) \to ((W',h'), s', \eta' K)$ to be an isomorphism of rational Hodge structure

$$b : (W,h) \longrightarrow (W',h'),$$

such that we have the equality $b \circ \eta K = \eta' K$, and the map b sends s to a $\mathbb{Q}^\times$-multiple of s'.

The above construction makes it clear the Shimura variety parametrizes the triples above.

Proposition 1.8. *There exist a natural bijection*

$$\mathcal{H}_K/\sim \longrightarrow \mathrm{Sh}_K(\mathrm{GSp}(\psi), X(\psi));$$
$$((W,h), s, \eta K) \longmapsto [{}^a h, a \circ \eta],$$

where $a : V \to W$ is a $\mathbb{Q}$-linear isomorphism of vector space sending ψ to a $\mathbb{Q}^\times$-multiple of s.

In fact, the above admits a more geometric description in terms of abelian varieties. Recall the following classical result about abelian varieties:

Fact 1.9. *There exists a natural equivalence of categories*

$$\{Abelian\ varieties/\mathbb{C}\}/isogenies \longrightarrow \{rational\ polarizable\ Hodge\ structures\ of\ type\ (-1,0),\ (0,-1)\};$$
$$A \longmapsto \mathrm{H}_1(A,\mathbb{Q}).$$

Under the equivalence, the polarization of an abelian variety corresponds to the polarization of the Hodge structure on $\mathrm{H}_1(A,\mathbb{Q})$.

So by translating the construction of $\mathcal{H}_K$ into abelian varieties, we get:

Proposition 1.10. *There exists a natural bijection between* $\mathrm{Sh}_K(\mathrm{GSp}(\psi), X(\psi))$ *and the set* $\mathcal{M}_K$ *of isomorphism classes of triples* $(A, \lambda, \eta K)$*, where*

- A *is an abelian variety over* $\mathbb{C}$*;*
- *A* $\mathbb{Q}^\times$*-multiple of* λ *is a polarization on* A*;* [1]
- ηK *is a* K*-orbit of the level structure* $V(\mathbb{A}_f) \to V_f(A) = T_f(A) \otimes \mathbb{A}_f$*, sending* ψ *onto an* $\mathbb{A}_f^\times$*-multiple of the Weil pairing* e_λ.

The isomorphism of triples $(A, \lambda, \eta K) \to (A', \lambda', \eta' K)$ *is defined by an isogeny* $f : A \to A'$*, sending* λ *onto a* $\mathbb{Q}^\times$*-multiple of* λ'*, and* ηK *onto* $\eta' K$.

Here we implicitly use the identity that $\mathrm{H}_1(A,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{A}_f$ is canonically isomorphic to the adelic Tate module $V_f(A) = T_f(A) \otimes_{\widehat{\mathbb{Z}}} \mathbb{A}_f$.

2. PEL-type Shimura variety

In the last section, we see that the Shimura variety $\mathrm{Sh}_K(\mathrm{GSp}(\psi), X(\psi))$ for a symplectic vector space (V, ψ) produces the moduli space $\mathcal{M}_K$ the abelian varieties with polarizations and level structures. This is in fact an extremely powerful tool, as it allows us to study the geometric structure of the Shimura varieties using tools from algebaric geometry: with the help of GIT, it could be showed that the moduli space $\mathcal{M}_K$ can be defined over the ring of integers. In particular, by taking the base change to the positive characterstic, we could study the local structure of $\mathrm{Sh}_K(\mathrm{GSp}(\psi), X(\psi))$ using the moduli of abelian varieties over finite fields.

It is thus natural to ask if we could extend the method to more general classes of Shimura varieties, relating them to certain moduli problems which behave as good as $\mathcal{M}_K$.

We notice that the moduli problem in the last section considers the data of abelian varieties, polarizations, and level structures. We could in fact add one more structure, namely the endomorphism. This leads to the *PEL-type Shimura varieties*, where the word "PEL" stands for *polarizations, endomorphisms, and level structures.*

2.1. Algebras with involutions.

We briefly introduce results about algebra with involutions.

Let B be a semisimple algebra over $\mathbb{Q}$. An *involution* $*$ on B is defined as a $\mathbb{Q}$-linear automorphism of B, such that $** = \mathrm{id}$ and $(ab)^* = b^* a^*$.

Over the algebraic closure, we have the following known classification:

Lemma 2.1. *Let* $(B, *)$ *be a* $\mathbb{Q}$*-algebra with an involution. Then the base field extension* $(B \otimes \overline{\mathbb{Q}}, *)$ *over the algebraic closure* $\overline{\mathbb{Q}}$ *is isomorphic to a finite product of the following three types:*

C $(\mathrm{M}_n, b^* = b^t)$;
A $(\mathrm{M}_n \otimes \mathrm{M}_n, (a,b)^* = (b^t, a^t))$;
BD $(\mathrm{M}_n, b^* = J_n b^t J^t)$.

[1] Recall that to give a polarization of an abelian variety A, it suffices to specify an ample line bundle on A or an element in the Nèron-Severi group $\mathrm{NS}(A)$. Here we regard λ as an element in the base extension $\mathrm{NS}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$, so it makes sense to consider a rational multiple of it.

For a simple $\mathbb{Q}$-algebra with involution $(B,*)$, it is called type (C), (A), or (BD) if the base change $(B\otimes_F\overline{\mathbb{Q}},*)$ is a product of objects of the given type, where F is the maximal $*$-invariant subfield in the center of B.

A $\mathbb{Q}$-algebra with involution $(B,*)$ is called *positive* if it satisfies

$$\mathrm{Tr}_{B/\mathbb{Q}}(b^*b)>0,\ \forall b\in B.$$

Given an algebra with involution $(B,*)$, a *symplectic* $(B,*)$*-module* is a symplectic space (V,ψ) where V admits a $\mathbb{Q}$-linear B-action such that

$$\psi(bu,v)=\psi(u,b^*v),\ \forall b\in B,\ u,v\in V.$$

2.2. **PEL datum.** Fix a tuple $(B,*;V,\psi)$, where $(B,*)$ is a simple $\mathbb{Q}$-algebra with involution, and (V,ψ) is a symplectic $(B,*)$-module of dimension $2n$ over $\mathbb{Q}$.

Consider the following group, such that for any $\mathbb{Q}$-algebra R we have

$$G(R)=\{g\in \mathrm{GL}_B(V)(R)\mid \psi(gu,gv)=\nu(g)\cdot\psi(u,v),\ for\ \nu(g)\in R^\times\}.$$

By definition, G is a subgroup of $\mathrm{GSp}(\psi)$.

The PEL datum is then defined to be certain pairs (G,X), for X being a $G(\mathbb{R})$-conjugacy class $G(\mathbb{R})\cdot h_0$ for a homomorphism $h_0:\mathbb{S}\to G_{\mathbb{R}}$. Moreover, we want our choice of h_0 to be compatible with the symplectic space (V,ψ), in a way that $2\pi i\cdot\psi$ is a polarization of the Hodge structure (V,h_0).

We first give three examples of such pairs.

Example 2.2. (C) Suppose $(B,*)=(\mathbb{Q},\mathrm{id})$, which is of type (C) by the Lemma 2.1. Then the symplectic space $(V,\psi)=(\mathbb{Q}^{2n},J_n)$ in the last subsection together with the h_0 gives the datum $(\mathrm{GSp}_{2n},X(\psi)=\mathcal{H}^\pm)$, which is the Siegel Shimura datum.

(A) Suppose $(B,*)=(E,*)$, where E is an imaginary quadratic extension of $\mathbb{Q}$ and $*$ is the complex conjugation. We fix an isomorphism $E\otimes_{\mathbb{Q}}\mathbb{R}\to\mathbb{C}$. Let $a\geq b$ be two non-negative integers. Then we may choose the symplectic $(E,*)$-module (V,ψ) to be the $\mathbb{Q}$-vector space $V=E^{a+b}$ of dimension $2(a+b)$, with the E-linear alternate form

$$J_{a,b}=\begin{pmatrix} & & I_b\\ & \epsilon I_{a-b} & \\ -I_b & & \end{pmatrix},$$

where ϵ is the element in $\mathcal{O}_E$ such that $-i\epsilon\in\mathbb{R}_{>0}$. The associated group scheme for $(E,*;V,\psi)$ is given by

$$\mathrm{GU}_{a,b}(R)=\{(g,r)\in \mathrm{GL}_{a+b}(V)(R)\times R^\times\mid {}^t\overline{g}J_{a,b}g=r\cdot J_{a,b}\}.$$

Its adjoint group is the indefinite unitary group $U_{a,b}$ of signs (a,b).

We may choose h_0 to be the homomorphism $\mathbb{C}^\times\to\mathrm{GL}(V)(\mathbb{R})$ such that $h_0(i)$ is given by $\begin{pmatrix} & & -I_b\\ & -iI_{a-b} & \\ I_b & & \end{pmatrix}$. Note that as an element in $\mathrm{GL}(V)(\mathbb{R})$ the matrix $h_0(i)$ is well defined by our choice of the isomorphism $E\otimes_{\mathbb{Q}}\mathbb{R}\cong\mathbb{C}$. Then the resulted Hermitian symmetric domain is $\mathcal{H}^\pm_{a,b}$, which is sublocus of $\mathrm{M}_b(\mathbb{C})\times\mathrm{M}_{a-b,b}(\mathbb{C})$ that satisfies some unitary positivity condition analogous to $\mathcal{H}^\pm_n$. This is a Shimura datum, and is often called the *unitary Shimura datum.*

(BD) Suppose B is a quaternion algebra over $\mathbb{Q}$ with $B\otimes_{\mathbb{Q}}\mathbb{R}\cong\mathbb{H}$. The algebra B is equipped with a canonical involution $*$. We let $V=B^{2n}$, and let ψ be the alternate form defined by the matrix J_n. Take $h_0:\mathbb{S}\to\mathrm{GL}(V)_{\mathbb{R}}$ to be the homomorphism such that $h_0(i)$ is $\begin{pmatrix} & -I_n\\ I_n & \end{pmatrix}$. Then the derived group of $G(\mathbb{R})$ is isomorphic to the intersection $\mathrm{SO}_{2n}(\mathbb{C})\cap U_{n,n}(\mathbb{C})$. However, the pair $(G,G(\mathbb{R})\cdot h_0)$ is not a Shimura datum, as the group G is not connected.

It turns out that for type (C) and (A), we could always find a Hodge structure $h_0:\mathbb{S}\to G_{\mathbb{R}}$ that makes the pair $(G,G(\mathbb{R})\cdot h_0)$ a Shimura datum.

Proposition 2.3. *Let $(B,*;V,\psi)$ be a tuple, such that $(B,*)$ is simple algebra with involution of type (C) or (A) that satisfies the positivity condition. Then there exists a homomorphism $h:\mathbb{S}\to G_{\mathbb{R}}$, which is unique up to conjugation by an element of $G(\mathbb{R})$, such that the Hodge structure (V,h) has type $\{(-1,0),\ (0,-1)\}$, and $2\pi i\cdot\psi$ is a polarization of (V,h). Moreover, $(G,G(\mathbb{R})\cdot h)$ is a Shimura datum.*

We call the Shimura datum $(G,G(\mathbb{R})\cdot h)$ above the *(simple) PEL Shimura datum of type (C) or (A).*
Now we can consider the moduli problem associated to the PEL Shimura datum.

Theorem 2.4. *Let $(B,*;V,\psi)$ be a simple positive algebra with involution in type (C) or (A) as above, and (G,X) be the associated PEL Shimura datum. Then for an open compact subgroup K of $G(\mathbb{A}_f)$, the corresponding Shimura variety $\mathrm{Sh}_K(G,X)(\mathbb{C})$ parametrizes the isomorphism classes of tuples $((A,i),\lambda,\eta K)$, where*

- *A is a complex abelian variety;*
- *a $\mathbb{Q}^\times$-multiple of λ is a polarization of the abelian variety A;*
- *i is a $\mathbb{Q}$-homomorphism $B\to\mathrm{End}(A)\otimes_{\mathbb{Z}}\mathbb{Q}$;*
- *ηK is a K-orbit of $B\otimes\mathbb{A}_f$-linear isomorphism $V(\mathbb{A}_f)\to V_f(A)$, sending ψ to a $\mathbb{A}_f^\times$-multiple of e_λ,*

such that it satisfies the following condition:

There exists a B-linear isomorphism $a:V\to\mathrm{H}_1(A,\mathbb{Q})$ sending ψ to a $\mathbb{Q}^\times$-multiple of λ, and ${}^a h_A$ is an element in X, where h_A is the canonical rational Hodge structure on the homology group $\mathrm{H}_1(A,\mathbb{Q})$.

Remark 2.5. The result is true for a slightly more general types of PEL datum: as long as the PEL datum (G,X) associated to $(B,*;V,\psi)$, which is not necessarily of type (C) or (A), is Shimura in the sense of [Mil06, Section 5]. This includes the case for certain type (B) PEL data.

Remark 2.6. As mentioned in the Example 2.2, (c), the automorphic group associated to a PEL-datum of type (BD) can fail to be connected, which disobeys the definitions of the Shimura datum (cf. [Mil06, Section 5]). More precisely, the condition SV3 is not satisfied here; namely there exists a $\mathbb{Q}$-factor of G^{ad} on which the projection of a Hodge structure h is trivial. We refer the reader to the discussion after the Example 5.1.3.6 in [Lan10] for details.

3. Hodge-type Shimura varieties

In this section, we introduce the Hodge-type Shimura varieties.
First we recall from the last section that given a symplectic $(B,*)$-module (V,ψ), the group scheme G associated to the tuple $(B,*;V,\psi)$ naturally admits a closed embedding into the symplectic group $\mathrm{GSp}(\psi)$. Moreover, as the Hodge structure h for G for a PEL datum is required to be compatible with the symplectic structure ψ, we get an induced morphism of data

$$(G,G(\mathbb{R})\cdot h)\longrightarrow(\mathrm{GSp}(\psi),X(\psi)).$$

The above provides a prototype for the Hodge-type Shimura varieties. We now give the precise definition.

Definition 3.1. A Shimura datum (G,X) is called of *Hodge-type* if it admits a *Siegel embedding*

$$(G,X)\longrightarrow(\mathrm{GSp}(\psi),X(\psi)).$$

Namely as a $\mathbb{Q}$-algebraic group G admits a closed immersion into $\mathrm{GSp}(\psi)$ for some symplectic space (V,ψ), which carries the $G(\mathbb{R})$-conjugay class X injectively into $X(\psi)$.
The associated Shimura variety is called *Hodge-type Shimura variety.*

By the definition above, a typical example of the Hodge-type Shimura variety is a PEL Shimura variety.
Similar to the discussion for the PEL Shimura variety, we want to give a moduli interpretation for the Hodge-type Shimura varieties. To do so, we first recall a classical result of Chevalley describing subgroups of $\mathrm{GL}(V)$ in terms of stabilizer of tensors.

Proposition 3.2. *Given a faithful self-dual representation $G\to\mathrm{GL}(V)$ of G, there exists a finite set T of tensors of V such that G is the subgroup of $\mathrm{GL}(V)$ that acts on each element in T via a scalar multiplication.*

Here we recall that a *tensor* of a $\mathbb{Q}$-vector space is a multilinear map $t : V \times \cdots \times V \to \mathbb{Q}$, or an element in $(V^{\otimes_{\mathbb{Q}} r})^\vee$. When V is equipped with an action by G, by twisting the right side with the Tate twist of weight $-2s$, we get the notion of *Hodge tensors* of weight $-2s$:

$$t : V \times \cdots \times V \longrightarrow \mathbb{Q}(s);$$
$$t(gv_1, \ldots, gv_r) = \nu(g)^s \cdot t(v_1, \ldots, v_r).$$

Granting the above result on algebraic groups, we can now formulate a moduli problem for the Hodge-type Shimura varieties.

Theorem 3.3. *Let (G, X) be a Hodge-type Shimura datum, and let K be an open compact subgroup of $G(\mathbb{A}_f)$. Let t_i be the defining Hodge tensors associated to $G \to \mathrm{GL}(V)$. Then there exists a bijection between $\mathrm{Sh}_K(G, X)(\mathbb{C})$ and the set $\mathcal{M}_K$, where the latter is the isomorphism classes of tuples $(A, (s_i)_{0 \le i \le m}, \eta K)$, where*

- *A is a complex abelian variety;*
- *a $\mathbb{Q}^\times$-multiple of s_0 is a polarization of the abelian variety A;*
- *s_i for $1 \le i \le m$ are Hodge tensors of $\mathrm{H}_1(A, \mathbb{Q})$.*
- *ηK is a K-orbit of $\mathbb{A}_f$-linear isomorphisms $V(\mathbb{A}_f) \to V_f(A)$ sending ψ onto an $\mathbb{A}_f^\times$-multiple of s_0, and each t_i onto s_i,*

which satisfies a condition that

There exists an isomorphism $a : V \to \mathrm{H}_1(A, \mathbb{Q})$ sending ψ to a $\mathbb{Q}^\times$-multiple of s_0, t_i to s_i for $1 \le i \le m$, and ${}^a h$ is an element in X.

Example 3.4. Recall a *Hodge cycle* for an abelian variety A is an element s in the intersection $\mathrm{H}^{2r}(A, \mathbb{Q}) \cap \mathrm{H}^{r,r}$. Since the cohomology of an abelian varieties satisfy

$$\mathrm{H}^m(A, \mathbb{Q}) = \wedge^m \mathrm{H}^1(A, \mathbb{Q}),$$

we may rewrite the m-th cohomology group as

$$\mathrm{H}^m(A, \mathbb{Q}) = \mathrm{Hom}(\wedge^m W, \mathbb{Q}),$$

where $W = \mathrm{H}_1(A, \mathbb{Q})$ is the first homology. So the Hodge cycle s induces a multilinear map

$$s : W^{\otimes 2r} \longrightarrow \wedge^{2r} W \to \mathbb{Q}.$$

Moreover, as the element r is living in the intersection $\mathrm{H}^{2r}(A, \mathbb{Q}) \cap \mathrm{H}^r(A(\mathbb{C}), \Omega_A^r)$, the above map is a morphism of Hodge structure onto $\mathbb{Q}(r)$. So we get a Hodge tensor for the homology group $W = \mathrm{H}_1(A, \mathbb{Q})$.

Example 3.5. By the definition, it is clear that a PEL Shimura variety is Hodge. In particular, it is possible to describe the endomorphism condition into conditions for Hodge tensors.

4. Abelian motives and abelian type Shimura varieties

At last, we introduce the abelian type Shimura varieties.

Again, the motivation is to extend the moduli of abelian varieties into a slightly more general situation. This time, we are not just consider one single abelian variety at a time, but also allows disjoint unions, duals, and fiber products of a finite amount of abelian varieties. Translating this into their first cohomology, then we get not just the rational/integral Hodge structure coming from the first cohomology $\mathrm{H}_1(A, \mathbb{Q})$ of a single abelian variety, but allowing more linear algebraic operations on Hodge structures. The resulting collection of objects, is the category of *abelian motives.*

Precisely, we have the following definitions.

Definition 4.1. An *abelian motive* over $\mathbb{C}$ is a triple (V, e, m), where we have

- V is a variety over $\mathbb{C}$ whose connected components are all abelian varieties;
- e is an idempotent in $\mathrm{End}(\mathrm{H}^*(A, \mathbb{Q}))$;
- m is an integer.

There exists a natural way to associate a rational Hodge structure from an abelian motive, by

$$A = (V, e, m) \longmapsto \mathrm{H}(A) := e\mathrm{H}^*(A, \mathbb{Q})(m).$$

Now we define the abelian type Shimura datum.

Definition 4.2. (i) A connected Shimura datum (G, X^+) is called of *abelian type* if there exists connected Hodge-type Shimura data (H_i, X_i^+), together with an isogeny $\prod H_i \to H$ carrying $\prod X_i^+$ to X^+.

(ii) A Shimura datum (G, X) is called of *abelian type* if (G^{der}, X^+) is of abelian type.

With the help of the Riemann's theorem and a little bit use of linear algebra, we get the following moduli interpretation of abelian type Shimura varieties.

Theorem 4.3. *Let (G, X) be an abelian type Shimura datum, and let K be an open compact subgroup of $G(\mathbb{A}_f)$. Assume there exists a symplectic space (V, ψ) such that G admits a faithful action on V with $g\psi = \nu(g)^m\psi$ for a fixed $m \in \mathbb{Z}$, and let t_i be the Hodge tensors for V defining G as a subgroup of $\mathrm{GL}(V)$.*[2]

Then there exists a bijection between $\mathrm{Sh}_K(G, X)(\mathbb{C})$ and the set $\mathcal{M}_K$, where the latter is the isomorphism classes of tuples $(A, (s_i)_{0\le i\le m}, \eta K)$, where

- *A is an abelian motive;*
- *$\pm s_0$ is a polarization of the $\mathrm{H}(A)$;*
- *s_i for $1 \le i \le m$ are Hodge tensors of $\mathrm{H}(A)$.*
- *ηK is a K-orbit of $\mathbb{A}_f$-linear isomorphisms $V(\mathbb{A}_f) \to V_f(A)$ sending ψ onto an $\mathbb{A}_f^\times$-multiple of s_0, and each t_i onto s_i,*

which satisfies a condition that

> *There exists an isomorphism $a : V \to \mathrm{H}(A)$ sending ψ to a $\mathbb{Q}^\times$-multiple of s_0, t_i to s_i for $1 \le i \le m$, and ${}^a h$ is an element in X.*

References

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[Lan10] K. Lan. An example-based introduction to Shimura varieties. 2010. http://www-users.math.umn.edu/~kwlan/articles/intro-sh-ex.pdf.

[2]Note that when $m = 1$, we get the closed immersion from G into $\mathrm{GSp}(\psi)$. In general, the condition only implies that the derived subgroup G^{der} admits a closed immersion into $\mathrm{Sp}(\psi)$.