NOTES ON CHAPTER 5 OF MILNE'S SHIMURA VARIETIES

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Abstract. This document is notes from a talk given in the Summer 2020 SHIMURA learning seminar on Shimura varieties, following Chapter 5 of Milne's notes on this subject [Mil17]. The goal of this talk was to define Shimura varieties in general and prove basic properties about them.

1. Preliminaries on reductive groups

In this section we set up some notation for reductive groups that will be used throughout the talk; along the way, we will explain what the notation means in examples, usually SL_n or GL_n. Let G be a reductive group over a field k. For us, the definition of *reductive*, which we won't recall here, includes the assumption that G be connected.

The abelian reductive groups are precisely the *algebraic tori*, those groups T that are isomorphic over the algebraic closure $\bar{k}$ of k to $\mathbb{G}_{\mathrm{m}}^n$, for some n.

Let Z denote the center of G. For example, the center of SL_n is the group μ_n. This example shows that even though G is connected, Z need not be connected. For this reason, we must sometimes work instead with the *connected center* Z°, the largest central subtorus of G.

A reductive group G is *semisimple* if its connected center is trivial. For example, SL_n and PGL_n are semisimple, but GL_n is not because its center is $\mathbb{G}_{\mathrm{m}}$, the scalar diagonal matrices.

A map $G \to H$ of reductive groups is an *isogeny* if it is (étale-locally) surjective and its kernel is finite. Examples of isogenies include any nontrivial multiplication $\mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}$ and the projection $\mathrm{SL}_n \to \mathrm{PGL}_n$. A word of warning: surjective group homomorphisms need not be surjective on k-points. For example, the map $\mathrm{SL}_n(k) \to \mathrm{PGL}_n(k)$ is generally not surjective because its cokernel is $k^\times/(k^\times)^n$.

The isogeny is *central* if, in addition, its kernel is contained in the center of the source. The central isogenies are the best-behaved for the structure theory of reductive groups. For example, there is an "exceptional", non-central isogeny $\mathrm{SO}_{2n} \to \mathrm{Sp}_{2n}$ when k has characteristic two. For this reason, in these notes, we will always take the word *isogeny* to mean *central isogeny*.

The notion of isogeny is extremely useful in the structure theory of reductive groups because it gives a classification of semisimple groups. Specifically, consider the category Isog of semisimple groups with morphisms the isogenies, and for each semisimple group G, the connected component Isog_G of Isog containing G. The semisimple group G is *simply-connected* if it is an initial object of Isog_G, and is *adjoint* if it is a final object of Isog_G.

One of the main results in the structure theory of reductive groups is that for any semisimple G, the category Isog_G does indeed have an initial object, called the *simply-connected*

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cover G^{sc} of G, and a final object, called the *adjoint quotient* G^{ad} of G. This result fails for non-semisimple groups: the group $\mathbb{G}_m$ admits infinitely many self-isogenies.

Given this result, we can easily describe the elements of Isog_G: they are the quotients of G^{sc} by subgroups of its center, a finite group. In particular, the adjoint group of G is just the quotient G/Z. For example, the group SL_n is simply-connected, its adjoint quotient is PGL_n, and the groups isogenous to SL_n are of the form SL_n/Γ where Γ is a subgroup of μ_n. For another example, the simply-connected cover of SO_n is the spin group Spin_n, and the adjoint group of SO_n is SO_n/μ_2 if n is even and SO_n otherwise.

An algebraic group is *simple* if it has no nontrivial normal algebraic subgroups. Simplicity of G does not imply simplicity of $G(k)$, as we have seen with $G = \text{PGL}_n$. A second main result in the structure theory of reductive groups is that every adjoint group G is a product of simple algebraic groups, which are in addition semisimple, and the simple factors are unique. By passing to the simply-connected cover, it follows that every simply-connected group is uniquely a product of *quasi-simple* simply-connected groups, that is, semisimple groups whose adjoint quotient is simple.

The *derived subgroup* G^{der} of G is the unique smooth connected subgroup of G such that $G^{\text{der}}(\bar{k}) = [G(\bar{k}), G(\bar{k})]$, the *commutator subgroup* of $G(\bar{k})$. The existence of such a group is proved in Brian Conrad's lecture notes on algebraic groups [Con, Corollary 16.4.1] and also SGA3 [Gro11, Exposé VI, Corollaire 7.10]. A reductive group is semisimple if and only if it equals its derived subgroup. In particular, PGL_n and SL_n are semisimple. Usually $G^{\text{der}}(k)$ is strictly larger than the commutator subgroup of $G(k)$. For example, the commutator subgroup of $\text{PGL}_n(k)$ is the image in $\text{PGL}_n(k)$ of the commutator subgroup of $\text{GL}_n(k)$, which is $\text{SL}_n(k)$, and we have already seen that in general the image of $\text{SL}_n(k)$ in $\text{PGL}_n(k)$ is a proper subgroup.

Although it is not true that every reductive group G is a product of a semisimple group and a torus – this already fails for GL_n – it *is* true up to isogeny: the multiplication map $G^{\text{der}} \times Z \to G$ is an isogeny.

There is a second way to canonically produce a torus from a reductive group G. The *abelianization* G^{ab} of G is the quotient G/G^{der}. Following Milne's notation, we will usually denote G^{ab} by T, and the quotient map $G \to T$ by ν. The restriction of ν to Z is an isogeny, and its kernel is the center of G^{der}. For example, in the case of GL_n, the map $\nu : \text{GL}_n \to T$ is the determinant, and the restriction of ν to Z is thus $\mathbb{G}_m \xrightarrow{n} \mathbb{G}_m$.

2. Definition of a Shimura datum

In this section, we define Shimura data following Deligne [Del79], explain their relationship to the connected Shimura data defined in the previous lecture, and explain why the second part of a Shimura datum is a disjoint union of Hermitian symmetric domains, after first reviewing some facts about the component groups of real reductive groups. Recall that the *Deligne torus* $\mathbb{S}$ is the Weil restriction of $\mathbb{G}_\text{m}$ from $\mathbb{C}$ to $\mathbb{R}$, so that in particular $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$, and that homomorphisms $\mathbb{S} \to \text{GL}(V)$ correspond to Hodge structures on V. In the previous lecture we phrased the definition of a connected Shimura datum using the circle U_1, but in this lecture we will instead use the Deligne torus. The definitions are related by the short exact sequence

$$1 \to \mathbb{G}_m \to \mathbb{S} \to U_1 \to 1,$$

where the map $\mathbb{S} \to U_1$ is $z \mapsto z/\bar{z}$.

Definition 1. A *Shimura datum* is a pair (G, X) consisting of a reductive group G over $\mathbb{Q}$ and a $G(\mathbb{R})$-conjugacy class X of homomorphisms $h : \mathbb{S} \to G_{\mathbb{R}}$ satisfying the following conditions.

SV1: For all $h \in X$, the Hodge structure on $\mathrm{Lie}(G)_{\mathbb{C}}$ defined by $\mathrm{ad} \circ h$ is of type
$$\{(-1, 1), (0, 0), (1, -1)\}.$$

SV2: For all $h \in X$, the automorphism $\mathrm{ad}(h(i))$ of $G^{\mathrm{ad}}_{\mathbb{R}}$ is a Cartan involution.

SV3: For all $h \in X$, every projection of h onto a simple factor of G^{ad} is nontrivial.

A *morphism of Shimura data* $(G, X) \to (G', X')$ is a homomorphism $G \to G'$ sending X to X'.

Remark 2. The axioms SV1, SV2, and SV3 are equivalent to the axioms for a connected Shimura datum given last time; in particular, our SV1 is equivalent to the earlier

SV1: for all $h \in X$, only the characters z, 1, and z^{-1} appear in the representation $(\mathrm{ad} \circ h, \mathrm{Lie}(G^{\mathrm{ad}})_{\mathbb{C}})$ of U_1.

Besides these superficial differences, the only change in the definition is that for a connected Shimura datum, we consider $G^{\mathrm{ad}}(\mathbb{R})^+$-conjugacy classes of homomorphisms to $G^{\mathrm{ad}}_{\mathbb{R}}$ whereas for a general Shimura datum we consider $G(\mathbb{R})$-conjugacy classes of homomorphisms to G. As we will see, this difference causes X to sometimes be disconnected.

Later, we will see that when G is simply-connected, the Lie group $G(\mathbb{R})$ is connected and the map $G(\mathbb{R}) \to G^{\mathrm{ad}}(\mathbb{R})^+$ is surjective. It follows that when G is simply-connected, the X in the definition of a (connected or general) Shimura datum is the same for the two definitions.

Example 3. Let $G = T$ be a torus. The only possibility for X is a singleton and the axioms SV1, SV2, and SV3 are always satisfied. Nonetheless, this Shimura datum is an important tool for studying the component sets of Shimura varieties.

Example 4. Let $G = \mathrm{GL}_2$ and let X be the conjugacy class of the homomorphism $h_0 : a + bi \mapsto \left[\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right]$. Since the image of this homomorphism, the circle group U_1, is its own centralizer in G, there is a bijection between X and $\mathrm{GL}_2(\mathbb{R})/U_1(\mathbb{R}) = \mathbb{H}^{\pm} := \mathbb{C} - \mathbb{R}$ given by sending gh_0g^{-1} to gi.

Example 5. In the previous example, the space X remains unchanged (up to isomorphism) if we replace GL_2 by PGL_2. At the same time, it is a classical fact that the group of isometries of the upper half plane is precisely "$\mathrm{PSL}_2(\mathbb{R})$", and as an algebraic group $\mathrm{PSL}_2 = \mathrm{PGL}_2$. The resolution to this apparent contradiction is that what the literature calls $\mathrm{PSL}_2(\mathbb{R})$, namely $\mathrm{SL}_2(\mathbb{R})/\mathbb{R}^{\times}$, is what we call the group $\mathrm{PGL}_2(\mathbb{R})^+$. In this sense, the notation "PSL_2" is incorrect and misleading.

Example 6. Let $G = \mathrm{SL}_2$ and let X be the conjugacy class $h_0'(z) = h_0(z/\bar{z})$, where h_0 is the homomorphism from Example 4. Then X is in bijection with the upper half-plane $\mathbb{H} \subset \mathbb{C}$.

This trio of examples generalizes in two ways, one due to Hilbert and the other to Siegel.

Example 7. There is a Shimura datum with group $G = \mathrm{GSp}_{2n}$ and $X = \mathbb{H}_n^+ \sqcup \mathbb{H}_n^-$, the Siegel half-spaces. We will study this Shimura datum next week. It generalizes the previous examples because $\mathrm{GSp}_2 = \mathrm{GL}_2$.

Example 8. Let F be a totally real field of degree n, let B be a (degree-two) quaternion algebra over F, and let $G = \underline{B}^{\times}$. Suppose that B splits at all real places of F, so that
$$B \otimes_F \mathbb{R} = \mathrm{M}_2(\mathbb{R})^n, \qquad G_{\mathbb{R}} = \mathrm{GL}_{2,\mathbb{R}}^n.$$

Taking self-products of Example 4 gives a Shimura datum (G, X) with $X \simeq (\mathbb{H}^{\pm})^n$, called a *Hilbert Shimura datum*.

Our next goal is to show that the conjugacy class X is not just a set, but has a canonical structure of a Hermitian symmetric domain. To exhibit this structure, we will relate the arbitrary Shimura data defined above to the connected Shimura data of the previous lecture.

2.1. **Connected components of real reductive groups.** By definition, a real reductive group G is connected; hence its group of real points is connected in the *Zariski* topology. This does not imply, however, that its group of real points is connected in the *analytic* topology, the topology for which $G(\mathbb{R})$ is a Lie group. The ultimate source of this discrepancy is the disconnectedness of $\mathbb{R}^\times$.

Example 9. The group of real points of the torus $\mathbb{G}_{\mathrm{m}}$ is $\mathbb{R}^\times$, and $\pi_0(\mathbb{R}^\times) = \mathbb{Z}/2\mathbb{Z}$. More generally, $\pi_0((\mathbb{R}^\times)^n) = (\mathbb{Z}/2\mathbb{Z})^2$. Later, this fact will tell us that for any reductive G the group $\pi_0(G(\mathbb{R}))$ is of the form $(Z/2\mathbb{Z})^n$ for some n.

Example 10. The group $\mathrm{GL}_n(\mathbb{R})$ has two connected components, and the determinant map $\mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}^\times$ exhibits the these components of $\mathrm{GL}_n(\mathbb{R})$ as the preimages of the two components of $\mathbb{R}^\times$. Similarly, $\pi_0(\mathrm{PGL}_2(\mathbb{R})) = \mathbb{Z}/2\mathbb{Z}$.

Example 11. Given natural numbers p and q, possible zero, let $n = p + q$ and let $\mathrm{O}(p, q)$ be the linear automorphisms of $\mathbb{R}^n$ preserving the hyperplane $x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_n^2$. The group $\mathrm{O}(p, q)$ is called the *indefinite orthogonal group* (at least when $pq \neq 0$). It is well known that $\mathrm{O}(p, q)$ is disconnected for the same reason as GL_n: the determinant map hits both components of $\mathbb{R}^\times$. To get a connected *algebraic* group we must take the special orthogonal group $\mathrm{SO}(p, q)$, the kernel of the determinant. When $p = 0$ or $q = 0$, the Lie group $\mathrm{SO}(p, q) = \mathrm{SO}(n)$ is connected. However, when $pq \neq 0$, it can be shown that $\mathrm{SO}(p, q)$ has two connected components; see for instance [FH91, Exercise 7.2].

Let $G(\mathbb{R})^+$ be the connected component of $G(\mathbb{R})$ (in the analytic topology) that contains the identity; it is a normal subgroup of $G(\mathbb{R})^+$.

The following proposition is of independent interest, and also explains why the two definitions of Shimura datum coincide for simply-connected groups.

Proposition 12.

(1) *Let $G \to H$ be a surjective homomorphism of algebraic groups over $\mathbb{R}$. The induced map $G(\mathbb{R})^+ \to H(\mathbb{R})^+$ is surjective.*

(2) *If G is a simply-connected algebraic group over $\mathbb{R}$ then the Lie group $G(\mathbb{R})$ is connected.*

In general, it is possible for Lie groups to have a countable infinity of connected components; for example, the kernel of the map to $\mathrm{SL}_2(\mathbb{R})$ from its universal cover, the metaplectic group, has kernel $\mathbb{Z}$. However, a theorem of Whitney shows that this defect cannot happen for the real points of algebraic groups, or more generally, smooth varieties.

Theorem 13 ([Whi57, Theorem 3])**.** *Let V be a smooth variety over $\mathbb{R}$. Then the topological space $V(\mathbb{R})$ has finitely many connected components.*

A theorem of Matsumoto describes the possible component groups of $G(\mathbb{R})$.

Theorem 14 ([Mat64, Théorème 1])**.** *Let G be a connected algebraic group and let $S \subseteq G$ be a maximal split torus. Then*

(1) *$G(\mathbb{R}) = G(\mathbb{R})^+ S(\mathbb{R})$; and*
(2) *the homomorphism $\pi_0(S(\mathbb{R})) \to \pi_0(G(\mathbb{R}))$ is surjective, so that $\pi_0(G(\mathbb{R}))$ is an elementary abelian 2-group.*

One final piece of notation. Given a real reductive group G, let $G(\mathbb{R})_+$ be the preimage of $G^{\mathrm{ad}}(\mathbb{R})^+$ under the map $G(\mathbb{R}) \to G^{\mathrm{ad}}(\mathbb{R})$. There is an evident containment $G(\mathbb{R})^+ \subseteq G(\mathbb{R})_+$, but in general it is a proper containment. For example,

$$(\mathbb{R}^\times \times \mathrm{GL}_n(\mathbb{R}))^+ = \mathbb{R}_{>0} \times \mathrm{GL}_n(\mathbb{R})^+, \qquad (\mathbb{R}^\times \times \mathrm{GL}_n(\mathbb{R}))_+ = \mathbb{R}^\times \times \mathrm{GL}_n(\mathbb{R})^+.$$

2.2. **Induced connected Shimura data.** Given a Shimura datum (G, X), we would like to construct a connected Shimura datum with group G^{der}. The challenge, then, is to construct the $G^{\mathrm{ad}}(\mathbb{R})^+$-conjugacy class X^{der} of homomorphisms $\mathbb{S} \to G^{\mathrm{der}}_{\mathbb{R}}$ out of the $G(\mathbb{R})$-conjugacy class X. As Examples 4 and 6 showed, we should not generally expect that $X = X^{\mathrm{der}}$; rather, X^{der} should be a connected component of X.

By composition with the adjoint quotient map, each homomorphism $h : \mathbb{S} \to G_{\mathbb{R}}$ gives rise to a homomorphism $\bar{h} : \mathbb{S} \to G^{\mathrm{ad}}_{\mathbb{R}}$. It is not so hard to show that this assignment is injective. The image $\overline{X}$ of X under $h \mapsto \bar{h}$ need not be a $G^{\mathrm{ad}}(\mathbb{R})_+$ conjugacy class, but it is acted on by this group. We will thus choose X^{der} to be some $G^{\mathrm{ad}}(\mathbb{R})^+$-orbit in $\overline{X}$. It is easy to see that the pair $(G^{\mathrm{ad}}, X^{\mathrm{ad},+})$ satisfies the axioms of a connected Shimura datum. In particular, $X^{\mathrm{ad},+}$ has a canonical structure of a Hermitian symmetric domain.

Let X^{ad} be the $G^{\mathrm{ad}}(\mathbb{R})$-conjugacy class of homomorphisms containing X^{der}. [Mil17, Proposition 4.9] shows that X^{der} is a connected component of X^{ad} and its stabilizer is $G^{\mathrm{ad}}(\mathbb{R})_+$. Hence X^{ad} has a canonical structure of a finite disjoint union of symmetric domains, and its set of components can be (noncanonically) identified with the elementary abelian 2-group $G^{\mathrm{ad}}(\mathbb{R})/G^{\mathrm{ad}}(\mathbb{R})^+$. Evidently the action of the group $G^{\mathrm{ad}}(\mathbb{R})$ on X^{ad} permutes the components.

Returning now to the original Shimura datum (G, X), we see that $\overline{X} \subseteq X^{\mathrm{ad}}$, or equivalently, the isomorphic set X, is a finite union of connected components of X^{ad}, each isomorphic to X^{der}. Every component has the same stabilizer, the preimage $G(\mathbb{R})_+$ of $G^{\mathrm{ad}}(\mathbb{R})^+$, and therefore the components set of X can be identified with the group $G(\mathbb{R})/G(\mathbb{R})_+$. There is an inclusion

$$\pi_0(X) \simeq G(\mathbb{R})/G(\mathbb{R})_+ \hookrightarrow G^{\mathrm{ad}}(\mathbb{R})/G^{\mathrm{ad}}(\mathbb{R})^+ \simeq \pi_0(X^{\mathrm{ad}})$$

but it is not in general an isomorphism, for instance, when $G = \mathrm{SL}_2$.

Although X^{der} was not obtained canonically from X because we choose a $G^{\mathrm{ad}}(\mathbb{R})^+$-orbit, the resulting connected Shimura datum does not depend on this choice up to isomorphism, and is therefore canonically attached to the original Shimura datum.

3. Definition of a Shimura variety

Let (G, X) be a Shimura datum and let K be a compact-open subset of $G(\mathbb{A}_{\mathrm{f}})$. The goal of this section is to show that the double coset space

$$\mathrm{Sh}_K(G, X) := G(\mathbb{Q})\backslash X \times G(\mathbb{A}_{\mathrm{f}})/K,$$

where $G(\mathbb{Q}) \times K^{\mathrm{op}}$ acts on $X \times G(\mathbb{A}_{\mathrm{f}})$ by $q(x, a)k \mapsto (qx, qak)$, has a canonical structure of a complex variety; a *Shimura variety* is any variety isomorphic to such an object. We will then assemble all the Shimura varieties for a fixed Shimura datum into a tower, and explain how this tower gives information about automorphic representations.

Remark 15. For comparison with global fields of positive characteristic, it is often useful to formulate the definitions to be uniform in all places of $\mathbb{Q}$, so that ∞ is not distinguished in any way. In the case of Shimura varieties, this formulation is the observation that

$$\mathrm{Sh}_K(G, X) \simeq G(\mathbb{Q})\backslash G(\mathbb{A})/(K_\infty \times K)$$

where K_∞ is the centralizer of X in $G(\mathbb{R})$.

3.1. **Variety structure.** The proof that $\mathrm{Sh}_K(G, X)$ is a variety has two steps. First, decompose the space into components that are quotients of Hermitian symmetric domains by (torsion-free) arithmetic groups, hence varieties. Second, show that there are finitely many connected components. In this subsection we will discuss only the first step, and leave the second to a later section, where we calculate the components set of $\mathrm{Sh}_K(G, X)$ more carefully.

In what follows, we will need the group $G(\mathbb{Q})_+ := G(\mathbb{Q}) \cap G(\mathbb{R})_+$.

Lemma 16. *Let (G, X) be a Shimura datum, let K be a compact open subgroup of $G(\mathbb{A}_\mathrm{f})$, let $C := G(\mathbb{Q})_+\backslash G(\mathbb{A}_\mathrm{f})/K$, and for $g \in C$, let $\Gamma_g := gKg^{-1} \cap G(\mathbb{Q})_+$.*

1. *The set C is finite.*
2. *The map $[x] \mapsto [x, g]$ is a homeomorphism $\coprod_{g \in C} \Gamma_g\backslash X^+ \to \mathrm{Sh}_K(G, X)$.*

Proof. (1) is a routine calculation. It requires the fact that the for every connected component X^+ of X, the natural map $G(\mathbb{Q})_+\backslash X^+ \times G(\mathbb{A}_\mathrm{f}) \to G(\mathbb{Q})\backslash X \times G(\mathbb{A}_\mathrm{f})$ is a bijection; this can be proved using the fact that $G(\mathbb{Q})$ is dense in $G(\mathbb{R})$. (2) can be proved using strong approximation when G^{der} is simply-connected, and the general case "is not much more difficult". □

When G^{der} is simply-connected, we can give an even better description of the components set of $\mathrm{Sh}_K(G, X)$, reducing it to the abelianization T. As we will see later, this means that the components set is a "zero-dimensional Shimura variety". Let $T(\mathbb{R})^\dagger$ be the image of the map $\nu : Z(\mathbb{R}) \to T(\mathbb{R})$, a finite-index subgroup, and let $T(\mathbb{Q})^\dagger := T(\mathbb{Q}) \cap T(\mathbb{R})^\dagger$.

Theorem 17. *Let (G, X) be a Shimura datum with G^{der} simply-connected and let*

$$C' := T(\mathbb{Q})^\dagger\backslash T(\mathbb{A}_\mathrm{f})/\nu(K).$$

Then for K sufficiently small, there is an isomorphism $\pi_0(\mathrm{Sh}_K(G, X)) \simeq C'$.

Proof. Milne's proof is quite long, and we will only point out the places in the proof where the simple-connectivity of $H := G^{\mathrm{der}}$ is used. First, it is used to ensure the vanishing of the Galois cohomology groups $\mathrm{H}^1(\mathbb{Q}_\ell, H)$. Second, it is used to ensure that the map

$$\mathrm{H}^1(\mathbb{Q}, H) \to \prod_{\ell \le \infty} \mathrm{H}^1(\mathbb{Q}_\ell, H)$$

is injective, or in other words, that the Hasse principle holds. □

Remark 18. Deligne gave a general description of the components set in his report on Shimura varieties [Del79, Résumé 2.1.16], but Milne considers only the case when G^{der} is simply-connected because that assumption simplifies the description.

Corollary 19. *If K is sufficiently small then $\mathrm{Sh}_K(G, X)$ has a canonical structure of a complex variety.*

Proof. We saw last time that the quotient $\Gamma_g\backslash X^+$ has a canonical structure of a complex variety when Γ_g is torsion-free, in particular neat; and when K is sufficiently small, each Γ_g will be neat. □

3.2. **Towers of Shimura varieties.** For applications to the theory of automorphic forms, one of the main motivations for Shimura varieties, it is more useful to packages the Shimura varieties $\mathrm{Sh}_K(G, X)$, for varying K, into a single object. In this subsection, we will explain how to make such a package, and how it relates to the theory of automorphic forms.

Definition 20. The *tower of Shimura varieties* attached to a Shimura datum (G, X) is the inverse system

$$\mathrm{Sh}(G, X) := (\mathrm{Sh}_K(G, X))_K$$

of Shimura varieties, indexed by some partially ordered set of sufficiently small compact-open subgroups K of $G(\mathbb{A}_f)$.

Here "K sufficiently small" means that $\mathrm{Sh}_K(G, X)$ is a variety, which holds, for instance, if each Γ_g is neat.

Remark 21. Implicit in the definition is the assertion that the transition maps are regular. That is, given $K' \leq K$ be compact open subgroups of $G(\mathbb{A}_f)$ such that $\mathrm{Sh}_K(G, X)$ and $\mathrm{Sh}_{K'}(G, X)$ are Shimura varieties, the induced map $\mathrm{Sh}_{K'}(G, X) \to \mathrm{Sh}_K(G, X)$ is regular.

The main purpose for assembling Shimura varieties into a tower is that towers of Shimura varieties are acted on by adelic groups. Specifically, for each $g \in G(\mathbb{A}_f)$ the map

$$\rho_K(g) : \mathrm{Sh}_K(G, X) \to \mathrm{Sh}_{g^{-1}Kg}(G, X)$$

defined by $[x, a] \mapsto [x, ag]$ is regular, as we mentioned earlier, and these maps assemble into a right action ρ of $G(\mathbb{A}_f)$ on the tower $\mathrm{Sh}(G, X)$. (Again, there is an assertion that the map $\rho_K(g)$ is a morphism of varieties.)

Remark 22. The cohomology of a tower of Shimura varieties is an inverse system of abelian groups (or vector spaces, depending on the coefficient field). Since cohomology is contravariant, this cohomology tower is equipped with a left action of $G(\mathbb{A}_f)$, and the limit of the system is thus a representation of $G(\mathbb{A}_f)$. This construction is of great importance for the Langlands correspondence.

Since the transition maps in a tower of Shimura varieties are finite, hence affine, the limit

$$S := \varprojlim_K \mathrm{Sh}_K(G, X)$$

of this inverse system exists in the category of $\mathbb{C}$-schemes. In general S is not finite type; however, it is locally Noetherian and regular and it admits a right action of $G(\mathbb{A}_f)$. The Shimura varieties in the tower can be recovered from the limit S because

$$\mathrm{Sh}_K(G, X) = S/K.$$

We can also describe the limit as a double coset space.

Theorem 23. *Let (G, X) be a Shimura datum and let $Z(\mathbb{Q})^-$ be the closure of $Z(\mathbb{Q})$ in $Z(\mathbb{A}_f)$. Then*

$$\varprojlim_K \mathrm{Sh}_K(G, X) = \frac{G(\mathbb{Q})}{Z(\mathbb{Q})} \Big\backslash X \times (G(\mathbb{A}_f)/Z(\mathbb{Q})^-.$$

If the axiom SV5 holds (defined below) then

$$\varprojlim_K \mathrm{Sh}_K(G, X) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f).$$

In the case where SV5 holds, we recognize the limit as the kind of space on which an adelic automorphic form is defined.

One final note: Deligne showed that the formation of the tower of Shimura varieties is functorial.

Theorem 24. *A morphism $(G, X) \to (G', X')$ of Shimura data induces a morphism*

$$\mathrm{Sh}(G, X) \to \mathrm{Sh}(G', X')$$

of towers of Shimura varieties, which is a closed embedding if $G \to G'$ is injective.

Remark 25. Milne has an example showing that the induced morphism of inverse systems need not preserve the level; the example uses Siegel and Hilbert varieties. Perhaps Attilio can discuss this example in his lecture.

4. Zero-dimensional Shimura varieties

In this section we study the simplest class of Shimura varieties, those whose reductive group is a torus T. In this case the Shimura varieties are just finite sets, but these sets are of independent interest because, as we will show, they are the component sets of Shimura varieties for simply-connected reductive groups.

We saw earlier that for any homomorphism $h : \mathbb{S} \to T$, the pair (T, h) is a Shimura datum. The corresponding Shimura variety for a compact-open subgroup $K \leq T(\mathbb{A}_{\mathrm{f}})$ is the finite set

$$T(\mathbb{Q})\backslash T(\mathbb{A}_{\mathrm{f}})/K.$$

Ono [Ono61] studied this group and called it the *class group* $\mathrm{Cl}(T)$ of the torus T; the terminology comes from the fact that the class group of the Weil restriction of $\mathbb{G}_{\mathrm{m}}$ from a number field F to $\mathbb{Q}$ is the class group of F. The finiteness of the class group is then a special case of the theorem that the Shimura variety above is finite.

In the application to connected components, we will need to slightly extend the definition of a zero-dimensional Shimura variety. Let Y be a finite set on which $T(\mathbb{R})/T(\mathbb{R})^{+}$ acts transitively. Define

$$\mathrm{Sh}_K(T, Y) := T(\mathbb{Q})\backslash Y \times T(\mathbb{A}_{\mathrm{f}})/K.$$

Lemma 26. *For $Y = T(\mathbb{R})/T(\mathbb{R})^{\dagger}$, the natural map $T(\mathbb{Q})^{\dagger}\backslash T(\mathbb{A}_{\mathrm{f}})/K \to T(\mathbb{Q})\backslash Y \times T(\mathbb{A}_{\mathrm{f}})/K$ is an isomorphism.*

Proof. Use the denseness of $T(\mathbb{Q})$ in $T(\mathbb{R})$. □

It can be shown that when G^{der} is simply-connected, the map $\nu : G(\mathbb{A}_{\mathrm{f}}) \to T(\mathbb{A}_{\mathrm{f}})$ maps compact-open subgroups to compact-open subgroups. It follows that when G^{der} is simply-connected, its components set is a zero-dimensional Shimura variety.

Corollary 27. *Let (G, X) be a Shimura datum with G^{der} simply-connected and let $Y = T(\mathbb{R})/T(\mathbb{R})^{\dagger}$. Then*

$$\pi_0(\mathrm{Sh}_K(G, X)) \simeq \mathrm{Sh}_{\nu(K)}(T, Y).$$

Example 28. Let $(G, X) = (\mathrm{GL}_2, \mathbb{H}^{\pm})$ and $K = K(N)$, so that $T = \mathbb{G}_{\mathrm{m}}$ and $Y = \{\pm 1\}$. Recall that

$$\mathbb{Q}_{>0}\backslash \mathbb{A}_{\mathrm{f}}^{\times} = \widehat{\mathbb{Z}}.$$

Since $\mathbb{Q}_{>0} = \mathbb{Q}^{\times}/\{\pm 1\}$, it follows that

$$\pi_0(\mathrm{Sh}_K(G, X)) = \mathbb{Q}^{\times}\backslash\{\pm 1\} \times \mathbb{A}_{\mathrm{f}}^{\times}/(1 + N\widehat{\mathbb{Z}}) = \widehat{\mathbb{Z}}^{\times}/(1 + N\widehat{\mathbb{Z}}) = (\mathbb{Z}/n\mathbb{Z})^{\times}.$$

5. Additional axioms

In this section we'll discuss four additional axioms that one might impose on a Shimura datum, and that simplify the theory; our main focus is on axiom SV5. We'll first review some terminology needed to state the axioms. A number field F is *totally real* if each of its infinite places is real, *totally real* if each of its infinite places is complex, and *complex multiplication* (or *CM*) if it is a quadratic totally imaginary extension of a totally real field. Let (G, X) be a Shimura datum. The restriction of any element $h : \mathbb{S} \to G_{\mathbb{R}}$ of X to $\mathbb{G}_m \subseteq \mathbb{S}$ factors through the center of G. The composition of this factorization with the inversion map $\mathbb{G}_{\mathrm{m}} \xrightarrow{-1} \mathbb{G}_{\mathrm{m}}$ is called the *weight homomorphism*, denoted $w_X : \mathbb{G}_{\mathrm{m},\mathbb{R}} \to Z^{\circ}_{\mathbb{R}}$.

The four axioms are as follows.

SV2*: For all $h \in X$, the automorphism $\mathrm{ad}(h(i))$ is a Cartan involution of $G_{\mathbb{R}}/w_X(\mathbb{G}_m)$.
SV4: The weight homomorphism $\mathbb{G}_{\mathrm{m},\mathbb{R}} \to Z^{\circ}_{\mathbb{R}}$ is defined over $\mathbb{Q}$.
SV5: The group $Z(\mathbb{Q})$ is a discrete subgroup of $Z(\mathbb{A}_{\mathrm{f}})$.
SV6: The torus Z° splits over a CM-field.

When SV4 holds, for every representation of G on a $\mathbb{Q}$-vector space V, the Hodge structure induced on V by the Shimura datum is *rational*, meaning it is defined over $\mathbb{Q}$.

5.1. **Criterion for SV5.** In this subsection we discuss a criterion for SV5 to be satisfied. A k-torus T is *anisotropic* if every homomorphism $T \to \mathbb{G}_{\mathrm{m}}$ is trivial. The torus T contains a largest anisotropic torus, which we will denote by T^{a}. It can be shown that for k a local field, T is anisotropic if and only if $T(k)$ is compact.

Theorem 29. *Let T be a torus over $\mathbb{Q}$. The following are equivalent.*

1. *$T(\mathbb{Q})$ is discrete in $T(\mathbb{A}_{\mathrm{f}})$. (SV5 for $T = Z$.)*
2. *$T(\mathbb{Z})$ is discrete in $T(\mathbb{A}_{\mathrm{f}})$.*
3. *$T(\mathbb{Z})$ is finite.*
4. *$T^{\mathrm{a}}(\mathbb{R})$ is compact.*

Proof. The equivalence of (1), (2), and (3) follows from a theorem of Serre that every finite-index subgroup of $T(\mathbb{Z})$ contains a congruence subgroup, and the equivalence of (1) and (4) follows from a theorem of Ono. □

Corollary 30. *A Shimura datum (G, X) satisfies SV5 if and only if the largest anisotropic subtorus of Z remains anisotropic over $\mathbb{R}$.*

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