TALK 4: CONNECTED SHIMURA VARIETIES (ANGUS, 5/27)

1. Connected Shimura datum
2a. Arithmetic subgroups
2b. Thm (Baily – Borel), Thm (Borel)
3. Connected Shimura variety

<u>Review.</u> (from talk 2):

$$\left\{\begin{array}{l}\text{HSD } D \\ \text{w/ } p \in D\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l} G \text{ adjoint } \mathbb{R}\text{-Lie group} \\ u: U_1 \longrightarrow G(\mathbb{R}) \\ \text{s.t. (a), (b), (c)}\end{array}\right\}$$

(a) $1, z, z^{-1}$ are all characters of reps $U_1 \longrightarrow GL(\text{Lie}\, G_{\mathbb{C}})$

(b) $\text{ad}(u(-1))$ is a Cartan involution

(c) $u(-1) \not\mapsto 1$ $\forall$ simple factors of G

$$\{\text{HSD } D\} \longleftrightarrow \left\{\begin{array}{l}(G, [u]) \\ [u] \text{ is } G(\mathbb{R})^+\text{-conj. class of } U_1 \to G(\mathbb{R})\end{array}\right\}$$

<u>Def</u> A <u>connected Shimura datum</u> is a pair (G, D) where

- G s.s. alg. gp $/\mathbb{Q}$
- D corresponds to $G^{ad}(\mathbb{R})^+$-conj. class of $u: U_1 \longrightarrow G(\mathbb{R})$

s.t. (SU1) $z, 1, z^{-1}$ are all characters of $\text{Ad}\, u$,
(SU2) $\text{ad}(u(-1))$ is a Cartan involution,
(SU3) G^{ad} has no $\mathbb{Q}$-factor H s.t. $H(\mathbb{R})$ is compact.

<u>Lemma 4.7.</u> $u: U_1 \longrightarrow H \rightsquigarrow u$ is trivial $\Leftrightarrow$ H is compact

<u>(SV3)</u> G^{ad} has no $\mathbb{Q}$-factor H s.t. $U \longrightarrow G^{ad} \twoheadrightarrow H$ is trivial

Then SV3 $\Leftrightarrow$ SU3 by the lemma! Hence (SU3) $\Leftrightarrow$ (c) above!

Example. (Siegel upper half plane).

$$\mathcal{H}_n = \{\text{symm. } n\times n \text{ matrices } X+iY \text{ s.t. } Y \text{ pos. definite}\}$$
$$\circlearrowright$$
$$Sp_{2n}(\mathbb{R}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL_{2n}(\mathbb{R}) \;\middle|\; \begin{pmatrix} A & B \\ C & D \end{pmatrix}^t \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \right\}$$

with the action defined by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z = (AZ+B)\cdot(CZ+D)^{-1}.$$

(when $n=1$, this is the standard action $SL_2(\mathbb{R}) \circlearrowright \mathcal{H}$ $\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$)

The matrix $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ is an involution fixing iI_n and we can define

$$h_0 : \mathbb{U}_1 \longrightarrow Sp_{2n}(\mathbb{R})$$
$$x+iy \longmapsto \begin{pmatrix} xI_n & -yI_n \\ yI_n & xI_n \end{pmatrix}$$

$\rightsquigarrow$ $(Sp_{2n}, \mathcal{H}_n)$ is a connected Shimura datum.

2a. Arithmetic subgroups.

Def. G alg. gp $/\mathbb{Q}$

$\Gamma \subseteq G(\mathbb{Q})$ is arithmetic if $\exists\, G \hookrightarrow GL_n$ s.t. image of Γ is commensurable w/ $G(\mathbb{Q}) \cap GL_n(\mathbb{Z})$

(commensurable: $\Gamma \cap GL_n(\mathbb{Z})$ has finite index in both Γ & $GL_n(\mathbb{Z}) \cap G(\mathbb{Q})$)

Def. Fix $G \hookrightarrow GL_n$. Define: $\Gamma(N) = G(\mathbb{Q}) \cap \{g \equiv Id \mod (N)\}$

A subgroup $\Gamma \leqslant G(\mathbb{Q})$ is congruence if it contains $\Gamma(N)$ for some N and $[\Gamma : \Gamma(N)] < \infty$.

Prop 3.2. $G \longrightarrow G'$, $\Gamma \subseteq G$ arithmetic $\Rightarrow$ image of Γ is arithmetic

Thm 3.3. G reductive $/\mathbb{Q}$, $\Gamma \subseteq G(\mathbb{Q})$ arithmetic

$\Gamma \backslash G(\mathbb{R})$ has finite volume $\Leftrightarrow$ $\mathrm{Hom}(G, \mathbb{G}_m) = 0$.

⚠ Not all arithmetic subgroups are congruence! See appendix in Milne.

Fact. G simply connected, non-split, $\neq SL_2$ $\Rightarrow$ yes.

Neat subgroups.

- V v.sp. $\rightsquigarrow$ $g \in \mathrm{Aut}(V)$ is neat if $\langle \text{eigenvalues of } g \rangle \subseteq \mathbb{C}^\times$ is torsion-free
- $g \in G(\mathbb{Q})$ is neat if $\exists$ faithful rep. V s.t. g is neat on V
- $\Gamma \subseteq G(\mathbb{Q})$ is neat if all $g \in \Gamma$ are neat

2b. Theorems of Baily–Borel & Borel.

Thm (Baily–Borel).

$D = HSD$, $\Gamma \subseteq \mathrm{Hol}\, D^+$ torsion-free
$\Rightarrow$ $D(\Gamma) := \Gamma \backslash D$ has a compactification $D(\Gamma)^*$ s.t.
$D(\Gamma)^*$ is a projective variety $/\mathbb{C}$.

Rough idea: use "automorphic forms" as global section of an ample line bundle.

Ex. $\Gamma \backslash \mathfrak{h} \rightsquigarrow X(\Gamma)$ modular curve $= \mathrm{Proj}\left(\bigoplus_{k \geq 0} S_k(\Gamma)\right)$
($\bigoplus_{k \geq 0} S_k(\Gamma)$: modular forms)

Cor. $D(\Gamma)$ is quasi-projective.

Note that for $\Gamma \subseteq \Gamma'$, we have a map $D(\Gamma') \longrightarrow D(\Gamma)$. It is regular by the foll. thm.

Thm (Borel). If V is a quasi-proj. variety s.t. $f : V^{an} \longrightarrow D(\Gamma)^{an}$ is holomorphic
$\Rightarrow$ $f : V \longrightarrow D(\Gamma)$ is regular.

Idea: Use Picard's Big Theorem.

3. Connected Shimura varieties.

Let (G, D) be a connected Shimura datum.

Def. A connected Shimura variety relative to (G, D) is $D(\Gamma)$ for some Γ arithmetic, torsion-free.

The connected Sh. var. attached to (G, D) is $Sh^\circ(G, D) = \{D(\Gamma)\}_\Gamma$
which is an inverse system.

Examples • $(SL_2, \mathbb{Q})$ ⇝ Y_Γ open modular curve

• F totally real, B/F quaternion algebra

$$G/\mathbb{Q} = \ker(\mathrm{Nm} : B^\times \longrightarrow F^\times)$$

$$\rightsquigarrow G(\mathbb{R}) = \mathbb{H}^1 \times \cdots \times \mathbb{H}^1 \times SL_2(\mathbb{R}) \times \cdots \times SL_2(\mathbb{R})$$

$$\mathbb{H}^1 = \ker(\mathrm{Nm} : \mathbb{H}^\times \longrightarrow \mathbb{R}^\times)$$

⇝ Hilbert modular variety

4. Adelic description and double cosets.

G red. alg. $/\mathbb{Q}$

$\mathbb{A}_{fin}$ = ring of finite adeles $/\mathbb{Q}$, $\mathbb{A}_{fin} = \prod'(\mathbb{Q}_\ell, \mathbb{Z}_\ell)$ (restricted direct product)

If G has a model $/\mathbb{Z}_\ell$ ⇝ $G(\mathbb{Z}_\ell)$ makes sense, $G(\mathbb{Q}_\ell)$ always makes sense.
(OK outside of fin. many ℓ)

$$G(\mathbb{A}_{fin}) = \prod{}'(G(\mathbb{Q}_\ell), G(\mathbb{Z}_\ell))$$
(restricted direct product)

Prop. 4.1. Suppose $K \leq G(\mathbb{A}_{fin})$ compact, open subgroup. Then $K \cap GL(\mathbb{Q})$ is congruence.

Prop. 4.18. $$\Gamma\backslash\mathcal{D} \xrightarrow{\;\cong\;} G(\mathbb{Q})\backslash\mathcal{D} \times G(\mathbb{A}_{fin})/K$$
(G s.c., Γ congruence)
$$x \longmapsto [x, 1]$$

Rmk. The s.c. is needed b/c the proof uses strong approximation which requires s.c.

Strong approx. Thm. G alg. $/\mathbb{Q}$, G s.c., s.s., non-compact type. Then: $G(\mathbb{Q})$ is dense in $G(\mathbb{A}_f)$ (embedded diagonally).

Then: $$\varprojlim_{\Gamma} \Gamma\backslash\mathcal{D} = \varprojlim_{K} G(\mathbb{Q})\backslash\mathcal{D} \times G(\mathbb{A}_f)/K$$
$$= G(\mathbb{Q})\backslash\mathcal{D} \times G(\mathbb{A}_f)$$