Goal: realizing Hermitian Symmetric Domains (HSD) as moduli space of Hodge structures.

Classical Hodge Theory Review

X - smooth proj var/$\mathbb{C}$

$V := H^k(X, \mathbb{R})$ singular cohomology.

$V_{\mathbb{C}} := V \otimes \mathbb{C} = H^k(X, \mathbb{C})$ is equipped with a complex conjugation.

- Hodge decomp: $V_{\mathbb{C}}$ has decomp as $\mathbb{C}$-vect spaces

$$V_{\mathbb{C}} = \bigoplus_{p,q} V^{p,q}, \text{ s.t. } \overline{V^{p,q}} = V^{q,p}, \text{ called } \underline{\text{Hodge decomp}}.$$

Also, $V^{p,q} = 0$ if $p+q \neq k$

- Riemann Hodge Bilinear Relation:

Fix a Kähler form $\omega \in H^{1,1}(X)$, then we can talk about $V_{prim} = H^k(X, \mathbb{R})_{prim}$. For $\alpha \in H^{p,q}(X)_{prim} - \{0\}$,

$$i^{p-q}(-1)^{\binom{k}{2}} \int_X \alpha \wedge \bar{\alpha} \wedge \omega^{\dim_{\mathbb{C}} X - k} > 0$$

Hodge Structure in General

Def. A Hodge structure is an $\mathbb{R}$-vect sp. V together with a

Hodge decomposition $V_{\mathbb{C}} := V \otimes \mathbb{C} = \bigoplus_{p,q \in \mathbb{Z}} V^{p,q}$ s.t. $\overline{V^{p,q}} = V^{q,p}$

A Hodge structure is pure of weight k if $V^{p,q} = 0 \ \forall p+q \neq k$

We can encode the Hodge structure on V by an action $h: \mathbb{C}^{\times} \to GL(V_{\mathbb{C}})$ defined as: Let $z \in \mathbb{C}^{\times}$ act on $V^{p,q}$ by scalar $z^{-p} \bar{z}^{-q}$, i.e.

$$h(z)v = z^{-p}\bar{z}^{-q}v \quad \text{for } v \in V^{p,q}$$

Rmk 1. $\overline{h(z)v} = h(z)\bar{v}$ (even when z is not real!)

So h is really a real representation: $h(z)V \subseteq V$, making V a real rep of $\mathbb{C}^\times$, and $V_{\mathbb{C}}$ is its complexification.

Rmk 2. $\mathbb{C}^\times$ should be treated as a real alg gp

$$\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m = \{(x_1, y_1, x_2, y_2) : (x_1+iy_1)(x_2+iy_2)=1\}$$

Weil restriction of scalars

$$= \left\{(x_1, y_1, x_2, y_2) : \begin{array}{l} x_1x_2 - y_1y_2 = 1 \\ x_1y_2 + y_1x_2 = 0 \end{array}\right\}$$

and the rep h is a homomorphism of real alg gp $\mathbb{S} \to GL(V)$.

$\mathbb{S}$ is called Deligne torus and $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$.

The point is, complex conjugation should act trivially on $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$.

Rmk 3. Conversely, any real rep $h : \mathbb{C}^* \to GL(V)$ is given by a unique Hodge structure on V.

As an upshot, a Hodge structure can be given as a pair (V, h). This is a purely real datum!

Hodge Tensor

Def. Let (V,h) be a Hodge structure of weight n.

A tensor $t : V^{\otimes r} = V \otimes \cdots \otimes V \to \mathbb{R}$ is a Hodge tensor if one of the following equivalent condition holds:

- $t(h(z)v_1, \cdots, h(z)v_r) = (z\bar{z})^{-nr/2} t(v_1, \cdots, v_r) \quad \forall z \in \mathbb{C}^\times$
- If $v_1, \cdots, v_r$ satisfy $v_i \in V^{p_i, q_i}$ $(p_i + q_i = n)$, then $t(v_1, \cdots, v_r) = 0$ unless $\sum p_i = \sum q_i \left(= \frac{nr}{2}\right)$
  (i.e. $v_1 \otimes \cdots \otimes v_r$ is of type $\left(\frac{nr}{2}, \frac{nr}{2}\right)$)

Polarization

Def. Let (V, h) be a Hodge structure of wt n.
A polarization of (V, h) is a Hodge tensor

$$\psi : V \otimes V \to \mathbb{R}$$

(i.e. $\psi_{\mathbb{C}} : V_{\mathbb{C}} \otimes V_{\mathbb{C}} \to \mathbb{C}$ are built up from $\mathbb{C}$-bilinear maps

$$V^{p,q} \times V^{q,p} \to \mathbb{C} \quad)$$

s.t. $\psi_C(u, v) := (2\pi i)^n \psi(u, h(i) v)$ is symmetric and positive definite.

Rmk. $h(i)$ acts as i^{q-p} on $V^{p,q}$. It is called Weil operator.

Problem: ψ_C is not real, unless we remove $(2\pi i)^n$.
What is going on?

E.g. (taking $(2\pi i)^n$ out of definition)
(X, ω) compact Kähler manifold of dim n.
$V = H^k(X, \mathbb{R})_{prim}$

$$\psi(u,v) := (-1)^{\binom{k}{2}} \int_X u \wedge v \wedge \omega^{n-k} \quad \text{for } u,v \in V \text{ or } V_{\mathbb{C}}.$$

Can check by Riemann-Hodge:

$\psi_C(u,\bar{u}) := \psi(u, h(i)\bar{u}) > 0$ for $u \in V^{p,q} - 0$

$\langle u,v\rangle := \psi_C(u,\bar{v})$ is Hermitian.

A family of Hodge structures

Fix real vector space V, a weight $n \in \mathbb{Z}$, and

a dimension vector $\underline{d}: \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}_{\geq 0}$ s.t.

- $d(p,q) = d(q,p)$
- $d(p,q) = 0$ if $p+q \neq n$
- $\sum_{p,q} d(p,q) = \dim V$

Let $S(\underline{d})$ = set of Hodge structures on V s.t. $\dim V^{p,q} = d(p,q)$.

Then $S(d) \subseteq \prod_{d(p,q)\neq 0} Gr_{d(p,q)} V_{\mathbb{C}}$ is a locally closed real subvar: Picking $h \in S(d)$ is the same as picking $d(p,q)$-dim subspaces $V^{p,q}$ of $V_{\mathbb{C}}$, satisfying the open condition that they are linearly indep, and the closed condition that $\overline{V^{p,q}} = V^{q,p}$.

Now, require more structures

- A nondegenerate bilinear form $\psi: V \otimes V \to \mathbb{R}$
- A family T of tensors on V

Define

$$S(d,\psi,T) := \left\{ h \in S(d) : \begin{array}{l} \psi \text{ is a polarization of } h \\ \text{each } t\in T \text{ is a Hodge tensor for } h \end{array} \right\}$$

Each $t \in T$ being a Hodge tensor is a closed condition on h

[Recall def: $t(h(v_1),\cdots,h(v_r)) = (z\bar{z})^{-nr/2}\, t(v_1,\cdots,v_r)$]

In addition, ψ_c being sym is a closed condition, and ψ_c being positive def is a topologically (not Zariski!) open condition for h.

Thus $S(d,\psi,T)$ is a locally closed subset of

$$\prod_{d(p,q)\neq 0} Gr_{d(p,q)} V_{\mathbb{C}}.$$

<u>Thm 2.14</u> Let S^+ be a conn component of $S(d,\psi,T)$

(a) S^+ has a natural complex manifold structure

(b) Under certain condition, S^+ is a Hermitian sym. domain (HSD). "variation of Hodge structure" "Griffiths transversity"

(c) Every irreducible HSD arises this way.

<u>Sketch of Proof.</u>

(a) Pick $h_0 \in S^+$

Observe that: $\forall z_0 \in \mathbb{C}^\times$, $h \in S^+$, let $\underset{\in GL(V)}{g = h(z_0)}$, then

$$gh_0g^{-1}(z) := g \cdot h_0(z) \cdot g^{-1} \text{ lies in } S(d,\psi,T).$$

Clearly it lies in $S(d)$. To show $gh_0g^{-1} \in S(d,T)$,

$$t\left(\left(g h_0(z) g^{-1} v_i\right)_i\right) = t\left(\left(h(z_0) h_0(z) h(z_0)^{-1} v_i\right)_i\right)$$

$$= |z_0 z z_0^{-1}|^{-nr} t((v_i)_i) = |z|^{-nr} t((v_i)_i) \qquad (*)$$

t is a Hodge tensor for both h and h_0

We omit the pf that ψ is a polarization of gh_0g^{-1}.

Now let $G \subseteq GL(V)$ be the smallest real alg subgp that contains $S_+(\mathbb{S}) = \{h(z_0) : \substack{h \in S_+ \\ z_0 \in \mathbb{C}^\times}\}$. Since $(*)$ is a polynomial equation in g, $G(\mathbb{R})$ acts on $S(d,\psi,T)$ as well, so $G^+(\mathbb{R})$ acts on S^+, i.e.

$$G^+(\mathbb{R}) \cdot h_0 \subseteq S^+$$

Conn component of identity

Deligne 1979 "because $\mathbb{S}$ is of multiplicative type"

$$\Rightarrow \quad G^+(\mathbb{R})\, h_0 = S^+ \qquad \text{[why?]}$$

$\therefore S^+ = G^+(\mathbb{R})/K_0$, $\quad K_0 =$ stabilizer of h_0.

It can be shown that

- K_0 is closed, so $G^+(\mathbb{R})/K_0$ is a real analytic mfd
- $G^+(\mathbb{R})/K_0$ has a natural almost-complex structure by working on Lie alg
- The almost-complex struct is integrable, giving S^+ a complex mfd structure

(b) Recall from Thm 1.21, to give a HSD is the same as to give (G, u), G real adjoint alg gp, $u: U_1 = S^1 \to G^{ad}$ satisfying (a)(b)(c) of Thm 1.21.

Now given $S(d, \psi, T)$, let G as above,

- G is real adjoint alg gp.
- Define $u: U_1 \to G^{ad}$

$$z \mapsto h_0(\sqrt{z})$$

- Well defined:

need to show $h_0(-1) = id$.

$$\Leftrightarrow h_0(-1) \in Z(G)$$

$h_0(-1)$ acts as $(-1)^{q-p}$ on $V^{p,q}$

$$= (-1)^{p+q}$$
$$= (-1)^n$$

$\therefore h_0(-1)$ acts as scalar $(-1)^n$ on V.

(c).

Prelimary on self-dual reductive gps

Given a reductive gp rep $G \to GL(V)$, define the dual rep $V^\vee$ as

$\alpha \in V^\vee$, $(g \cdot \alpha)(v) = \alpha(g^{-1} \cdot v)$

If $V \cong V^\vee$ are isom as G-rep, we say V is self dual.

E.g. $O(n) \to GL(n)$, $Sp(2n) \to GL(2n)$.

For $O(n)$:

$O(n)$ acts on column matrices.

The dual rep. is on $V^\vee = \{1 \times n \text{ matrices}\}$ acted by

$$(g \cdot \alpha) v = \alpha g^{-1}$$

$$\therefore g \alpha = \alpha g^{-1}$$

Define
$$V \xrightarrow{\varphi} V^\vee$$
$$v \longmapsto v^T$$

Then

$$\begin{tikzcd}
v \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & v^T \arrow[ddd, mapsto] \\
& V \arrow[r, "\varphi"] \arrow[d, "g"'] & V^\vee \arrow[d, "g"] & \\
& V \arrow[r, "\varphi"] & V^\vee & \\
gv \arrow[rrr, mapsto] & & & (gv)^T = v^T g^{-1}
\end{tikzcd}$$

because $g^{-1} = g^T$.

Prop 2.1 For any faithful self dual rep $G \hookrightarrow GL(V)$, $\exists$ finite set T of tensors on V s.t. G is the subgp of $GL(V)$ fixing all $t \in T$.

Back to (c)

Given irred. HSD D, G is the adj gp

s.t. $G(\mathbb{R})^+ = \mathrm{Hol}(D)^+$.

- Choose a faithful self-dual rep $G \hookrightarrow GL(V)$ of G.

  [e.g. Take faithful $G \overset{\rho}{\hookrightarrow} GL(V)$, then $\rho \oplus \rho^\vee$ works]

So find (T, ψ) s.t. $G =$ set of $g \in GL(V)$ that fixes T

- Find nondegenerate bilinear ψ fixed by G.

Define

$$h_0 : \mathbb{S} = \mathbb{C}^\times \longrightarrow U_1 \xrightarrow{u_0} G \hookrightarrow GL(V).$$

$$z \longmapsto \frac{z}{\bar{z}}$$

It can be shown: h_0 defines a Hodge structure on V

s.t. all $t \in T$ are Hodge tensors

and ψ is polarization. Let $\underline{d}$ be the dim. vector determined by h_0.

Then: $S^+ =$ component of $S(d, T, \psi)$ containing h_0

is naturally identified with D.