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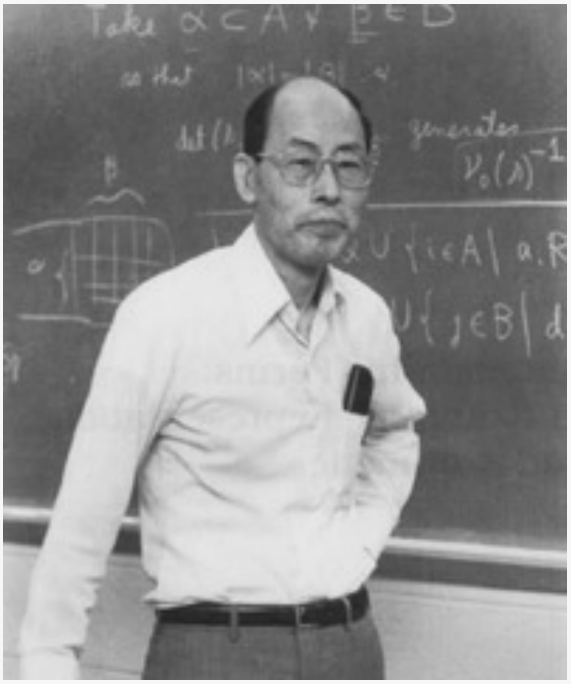


TALK 2: HERMITIAN SYMMETRIC DOMAINS (ANDY G., 5/13)

Milne defines manifolds, metrics, etc: we omit that here.

Hermitian forms.

V v.sp./$\mathbb{R}$, $J \in \mathrm{End}(V)$ s.t. $J^2 = -1$

A Herm. form on (V, J) is: $(\cdot,\cdot) : V \times V \longrightarrow \mathbb{C}$ s.t.

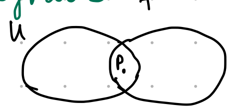
$$(Ju, v) = i(u,v)$$
$$(u,v) = \overline{(v,u)}$$

$\rightsquigarrow$ $\mathrm{Re}((\cdot,\cdot))$ associated real form.

$u \longmapsto (u,u)$ ass. quadratic form.

Given a real manifold:

- an almost complex structure on M is a choice of J for every $T_p(M)$ in a continuously varying way,
- it is integrable if



$TM|_U \cong U \times V_1$, $TM|_V \cong V \times V_2$ $\rightsquigarrow$ induced map $TM|_{U \times V} \longrightarrow TM|_{U \times V}$ preserves complex structure

- a hermitian metric on M is a metric g s.t.

$$g(Ju, Jv) = g(u,v).$$

Symmetric spaces.

- A manifold is homogeneous if its automorphism group acts transitively.
- A manifold is symmetric if $\exists$ $p \in M$, $i \in \mathrm{Aut}(M)$ s.t. $i^2 = \mathrm{Id}$ & p is isolated fixed point.
- A manifold is a Hermitian symmetric space if it is a Hermitian manifold which is symmetric.

Note: for a Herm. manifold: $\mathrm{Is}(M,g) = \mathrm{Is}(M^{\infty}, g) \cap \mathrm{Hol}(M)$.

Examples. (1) $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$

- $\dfrac{dx\,dy}{y^2}$ defines a Hermitian metric $\left\{ \dfrac{(dx)^2 + (dy)^2}{y^2} \right.$ actual metric?

- $PGL_2(\mathbb{R}) = \mathrm{Aut}(\mathbb{H})$

- $z \longmapsto -\frac{1}{z}$ is the automorphism making it symmetric ($p = i$)

  $i \longmapsto -\frac{1}{i} = i$

(2) $M = \mathbb{P}^1 = S^2 \subseteq \mathbb{R}^3$

for the group $SO(3) \ni \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}$

(3) $\mathbb{C}/\Lambda$

3 main types of symmetric spaces.

- Non-compact type $\left\{\begin{array}{l}\text{simply-connected} \\ \text{negative curvature}\end{array}\right.$
- Compact type $\left\{\begin{array}{l}\text{simply-connected} \\ \text{positive curvature}\end{array}\right.$
- Euclidean $\left\{\ 0 \text{ curvature}\right.$

A Hermitian symmetric domain is a Hermitian symmetric space of non-compact type.

FACT. Suppose M is a HSD. Then
$Is(M,g)^+$ is naturally a Lie group, adjoint, non-compact.

Example. $\mathcal{H}_g = \left\{\begin{array}{l}\text{symmetric complex } g\times g \text{ matrices} \\ X+iY \text{ s.t. } Y \text{ is pos.-def.}\end{array}\right\} \subseteq \mathbb{C}^{\frac{g(g+1)}{2}}$

$$\rightsquigarrow \mathrm{Aut}(\mathcal{H}_g) = Sp_{2g}(\mathbb{R})$$

A domain is a conn. open subset of $\mathbb{C}^n$.

Thm. Every bounded domain has a canonical metric of negative curvature.

(Pf is omitted.)

Hence every bdd symm. domain is a HSD.
Every HSD can be embedded in $\mathbb{C}^n$ as a bounded domain.

Example. $\mathcal{H} \cong \underset{\text{(disc)}}{\mathbb{D}} \subseteq \mathbb{C}$

For any symm. space S, $Is(M,g)$ is a Lie group and
$S \cong Is(M,g)^+/K_p \qquad K_p = $ stabilizer of p.

Note that: $Is(M,g) \begin{array}{l}\subseteq Is(M^\infty, g) \\ \subseteq Hol(M)\end{array}$ and ...

Fact. $Is(M,g)^+ = Is(M^\infty, g)^+ = Hol(M)^+$.

Thm. (M,g) is an HSD
Then $Hol(M)^+$ is an algebraic Lie group:
there is an alg. lie group $G \subseteq GL(\mathrm{Lie}(Hol(M)^+))$ s.t. $G(\mathbb{R}) = Hol(M)^+$.

$$\begin{array}{ccc}\{\text{HSD } D\} & \longrightarrow & \left\{\begin{array}{l}\text{real adjoint} \\ \text{lie group}\end{array}\right\}. \\ D & \longmapsto & Hol(D)^+\end{array}$$

Now, we want to think about going back.
Let D be an HSD with $p \in D$.

<u>Thm.</u> $\exists!\ u_p : U_1 \longrightarrow \mathrm{Hol}(D)$
s.t. $u_p(z)$ fixes p & acts like z on $T_p(D)$.

<u>Pf.</u> (See p. 14–15 in Milne.)

<u>Prop.</u> For D, D' with $p \in D, p' \in D'$ and
$\alpha : T_p(D) \longrightarrow T_{p'}(D')$
preserves sectional curvature,
α extends to an isometry. //

Then use this to show that $\cdot z$ extends to an aut. $\square$

A *Cartan involution* of a connected alg. gp $G/\mathbb{R}$ is an involution θ s.t. $G^\theta = \{ g \in G(\mathbb{C}) \mid g = \theta(\bar{g}) \}$
is compact.

This can always be realized as $G \longrightarrow GL(V)$
$T \longmapsto (T^{-1})^T$.

<u>Thm.</u> Cartan involution exists $\iff$ G is reductive
Moreover, all Cartan involutions are conjugate.

<u>Prop.</u> Id is a Cartan involution $\iff$ $G(\mathbb{R})$ is compact.

<u>Note:</u> U_1 is compact & abelian $\Rightarrow$ · its complex irreps are characters $z \longmapsto z^n$ for $n \in \mathbb{Z}$.

· its real irreps are
$z = x + iy \longmapsto \begin{pmatrix} x & y \\ -x & y \end{pmatrix}^n$
& the trivial rep.

<u>Thm.</u> $D = \mathrm{HSD},\ p \in D \rightsquigarrow u_p : U_1 \longrightarrow \mathrm{Hol}(D)^+$
(a) We get a rep on $H = \mathrm{Lie}(\mathrm{Hol}(D)^+)$ given by $\mathrm{ad} \circ u_p$
Then: only irreps in this rep. are $-1, 1$, trivial.

(b) Conjugating by $u(-1)$ gives $\mathrm{Hol}(D)$ a Cartan involution.
(c) $u_p(-1)$ does not project to Id in any simple factor of G.

<u>Pf.</u> We know that $D \cong \mathrm{Hol}(D)^+ / K_p$.
$\Rightarrow T_p(D) \cong T_e(G) / T_e(K_p)$ where $G = \mathrm{Hol}(D)^+$.

$u_p(z)$ acts by z

How does $u_p(z)$ act?

$u_p(z)$ acts trivially
(b/c $u_p(z)$ commutes with K_p

This proves (a).

For (b), Milne cites Helgason.

For (c), if $u_p(-1)$ projects to Id, that factor is compact.
Since G is of non-compact type, that factor is also non-compact. ↯ $\square$

Corollary. If G is a real adjoint Lie group and $u : U_1 \longrightarrow G$ satisfies (a), (b), (c) above, then D = conjugates of u by elts of $G(\mathbb{R})^+$ **is** an HSD.

Pf. Let K_u be what fixes u when we act by conjugation. Then:

$\implies D = \left(G(\mathbb{R})^+ / K_u\right) \cdot u$ makes D into a smooth manifold.

Moreover,

$$T_u(D) = \underset{\substack{\uparrow \\ u \text{ acts by } \pm 1, 0}}{T_e(G(\mathbb{R})^+)} \Big/ \underset{\substack{\uparrow \\ u \text{ acts by } 0}}{T_e(K_u)}$$

$\implies T_u(D)$ can be made a $\mathbb{C}$-vector space (w/ an integrable $\mathbb{C}$-v.sp. structure)

$\implies D$ is a complex manifold.

Finally, since $u(-1)$ is Cartan, D gets a Hermitian metric. $\square$

CONCLUSION.

$$\left\{\begin{matrix}\text{HSD w/} \\ \text{point } p \in D\end{matrix}\right\} \overset{G}{\rightleftarrows} \left\{\begin{matrix}\text{Real adjoint Lie gps with} \\ u : U_1 \to G \\ \text{sat. (a), (b), (c)}\end{matrix}\right\}$$

$$D \longmapsto \mathrm{Hol}(D)^+$$

$$D = G/K_u \longleftarrow\!\shortmid (G, u)$$