TALK 2: TOROIDAL COMPACTIFICATIONS.

Last time... $\Gamma\backslash X^+ \hookrightarrow \underbrace{(\Gamma\backslash X^+)^*}_{\text{highly singular in general}}$ minimal comp.

Today...

$$\begin{tikzcd} & (\Gamma\backslash X^+)^{tor} \arrow[d] \\ \Gamma\backslash X^+ \arrow[ur, hook, "\text{o.i.}"] \arrow[r, hook, "\text{o.i.}"'] & (\Gamma\backslash X^+)^* \end{tikzcd}$$

$(\Gamma\backslash X^+)^{tor}$: smooth
boundary has codim 1
<u>not</u> canonical (depends on some choices)

$(\Gamma\backslash X^+)^*$: singular
high codim. of boundary
canonical

<u>Warning.</u> The theory of toroidal comp. is difficult & technical.
All I'd try to do is give you an idea how this works in the "simplest" nontrivial example.

<u>Plan.</u> 1. Motivational (trivial) example: $G = SL_2$, i.e. $\Gamma\backslash\mathcal{H}^*$ as a toroidal comp.

2. Non-trivial example: $G = \mathrm{Res}_{F/\mathbb{Q}} SL_{2,F}$

Why not Siegel? When $n=2$, it's 3-dim'l and has 0 & 1-dim'l bdy comp.
$\Rightarrow$ it's harder...

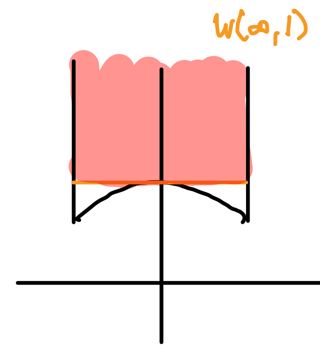
3. Brief Siegel example.

1. <u>Motivational (trivial) example:</u> $G = SL_2 \rightsquigarrow \Gamma\backslash\mathcal{H}$, $\Gamma = GL_2(\mathbb{Z})$

Nbhd of ∞ in $\mathcal{H}$ is: $W(\infty, r) = \{ z \mid \mathrm{Im}\, r > 0 \}$, $r > 0$.

Now, $B = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \leq SL_2(\mathbb{Z})$ stabilizes ∞. Moreover:

$$\begin{tikzcd} B\backslash\mathcal{H} \arrow[r] & \Gamma\backslash\mathcal{H} \\ B\backslash W(\infty,1) \arrow[u, hook] \arrow[r, "\cong"'] & \Gamma\backslash W(\infty,1) \arrow[u, hook] \end{tikzcd}$$



fund. domain for $\Gamma\backslash\mathcal{H}$

So... need to compactify $B\backslash W(\infty,1) \hookrightarrow \underbrace{B\backslash \mathcal{H}}$.

work with this...

Now...

$$\begin{tikzcd} B\backslash\mathcal{H} \arrow[r, hook] \arrow[d] & B\backslash\mathbb{C} \arrow[d, "e"] & z \arrow[d, mapsto] \\ U^* \arrow[r, hook] & \mathbb{C}^* & e(z) = e^{2\pi i z} \end{tikzcd}$$

$U^* = \{z \in \mathbb{C}^* \mid 0 < |z| < e^{-2\pi}\}$

$B = \{\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z}\}$ acts by $z \mapsto z+n$

But $\mathbb{C}^*$ is easy to "compactify":
$$\begin{tikzcd} \mathbb{C}^* \arrow[r, hook] & \mathbb{C} \\ U^* \arrow[u, phantom, "\cup" rotate=90] \arrow[r, hook] & U \arrow[u, phantom, "\cup" rotate=90] \end{tikzcd}$$
"torus embedding"

So...

$$\begin{tikzcd} & B\backslash W(\infty,1) \arrow[r, hook] & B\backslash \mathcal{H} \arrow[r, "e"] & U^* \arrow[r, hook] & U \arrow[dll, bend left] \\ \Gamma\backslash W(\infty,1) \arrow[ur, "\cong"] \arrow[r, hook] & \Gamma\backslash\mathcal{H} \end{tikzcd}$$

nbhd of ∞

glue $\Gamma\backslash\mathcal{H}$ to U along $\Gamma\backslash W(\infty,1)$

How to come up with $\mathbb{C}^* \hookrightarrow \mathbb{C}$ algebraically?

Note: $B \cong \mathbb{Z}$ & $(\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{R}) = \mathbb{R} \supseteq \overline{\mathbb{R}_+}$

character gp of $\mathbb{Z}$ $\cong \mathbb{Z}$ $\rightsquigarrow$ look at the alg. torus ass. to it

$$\begin{tikzcd} S = \mathrm{Spec}\left(\mathbb{Z}[x^n \mid n \in \mathbb{Z}] \big/ \begin{smallmatrix} x^{n_1}x^{n_2} = x^{n_1+n_2} \\ x^0 = 1 \end{smallmatrix}\right) \arrow[d, hook] \arrow[r, phantom, "\cong"] & \mathbb{C}^* \arrow[d, hook] \\ S_{\overline{\mathbb{R}_+}} = \mathrm{Spec}\left(\mathbb{Z}[x^n \mid n \text{ s.t. } n \in \overline{\mathbb{R}_+} \cap \mathbb{Z}]\right) \arrow[r, phantom, "\cong"] & \mathbb{C} \end{tikzcd}$$

$\mathbb{Z}[x^n \mid n \in \mathbb{N}]$

2. <u>Example:</u> Hilbert modular surfaces

<u>Recall.</u> $G = \mathrm{Res}_{F/\mathbb{Q}} SL_{2,F} \rightsquigarrow \Gamma \backslash \mathcal{H} \times \mathcal{H}, \quad \Gamma = SL_2(\mathcal{O}_F)$

(Assume F has narrow class number 1.)

Then $(\Gamma \backslash \mathcal{H} \times \mathcal{H})^* = \Gamma \backslash \mathcal{H} \times \mathcal{H} \cup \{\infty\}$ but ∞ is singular...

Nbhd of ∞: $W(\infty, r) = \{(z_1, z_2) \mid \mathrm{Im}\, z_1 \cdot \mathrm{Im}\, z_2 > r\}$

$\Rightarrow$ again, for $r \gg 0$,

$$B \backslash W(\infty, r) \xrightarrow{\cong} \Gamma \backslash W(\infty, r)$$

but here B is slightly more complicated:

$$B = \{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in SL_2(\mathcal{O}_F)\} / Z \cong \{\begin{pmatrix} \varepsilon & \mu \\ 0 & 1 \end{pmatrix} \mid \begin{array}{l} \varepsilon \in \mathcal{O}_F^\times \\ \mu \in \mathcal{O}_F \end{array}\} \cong M \rtimes V$$

(center acts trivially)

$M = \mathcal{O}_F, \; V = \mathcal{O}_F^\times$

Look at $M \backslash W(\infty, r) \hookrightarrow M \backslash \mathcal{H}^2 \hookrightarrow M \backslash \mathbb{C}^2$ first.

For μ_1, μ_2 = basis of M, have

$$\varphi_{\mu_1, \mu_2} : M \backslash \mathbb{C}^2 \longrightarrow \mathbb{C}^* \times \mathbb{C}^*$$
$$z \bmod M \longmapsto (u, v) \quad \text{where}$$

$$\underline{e}(z_j) = u^{\sigma_j(\mu_1)} v^{\sigma_j(\mu_2)}$$

$$\sigma_1, \sigma_2 : F \hookrightarrow \mathbb{R}$$

If ν_1, ν_2 another basis $\rightsquigarrow$ some compatibility...

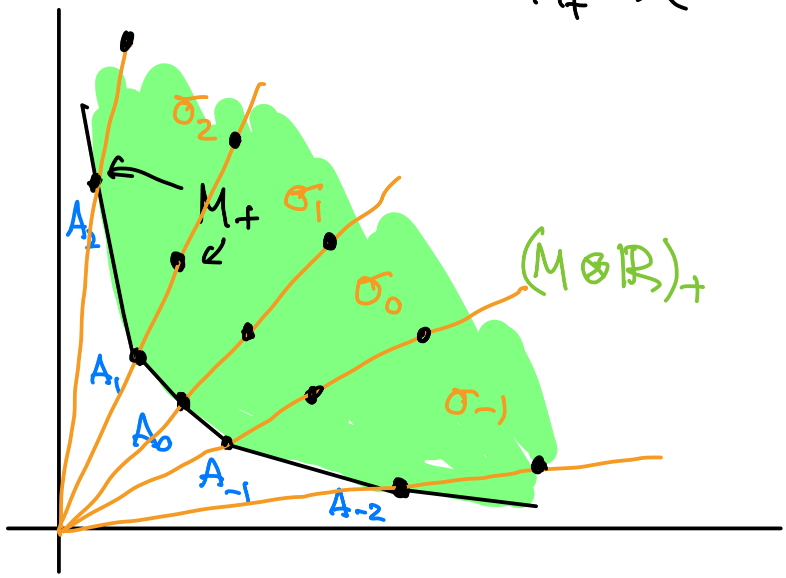
Moreover, can clearly use $\mathbb{C}^* \times \mathbb{C}^* \hookrightarrow \mathbb{C} \times \mathbb{C}$ as "compactification"

$\Rightarrow$ glue along different choices of bases.

How to choose basis?

$M \otimes \mathbb{R} \cong \mathbb{R}^2$

$M_+ \hookrightarrow (M \otimes \mathbb{R})_+ \cong (\mathbb{R}_+)^2$



$\forall k \in \mathbb{Z}$, $\{A_{k-1}, A_k\}$ basis of M

& $A_{k-1} + A_k = b_k A_{k+1}$ for $b_k \in \mathbb{Z}$

$\forall \sigma_k$, take a copy $(\mathbb{C}^2)_{\sigma_k}$ of $\mathbb{C}^2$

$$\rightsquigarrow \quad M\backslash\mathbb{C}^2 \xrightarrow{\varphi_{A_{k-1},A_k}} \mathbb{C}^* \times \mathbb{C}^* \longrightarrow (\mathbb{C}^2)_{\sigma_k} \qquad (u_k, v_k)$$

$$\begin{array}{cc} (\mathbb{C}^2)_{\sigma_k} & (u_k, v_k) \\ \downarrow & \downarrow \\ (\mathbb{C}^2)_{\sigma_{k+1}} & (u_k^{b_k} v_k, 1/u_k) \end{array}$$

Note: $(\mathbb{C}^2)_{\sigma_k} \cap (\mathbb{C}^2)_{\sigma_{k+1}} = \{u_k \neq 0\}$, $(\mathbb{C}^2)_{\sigma_k} \cap (\mathbb{C}^2)_{\sigma_{k+2}} = \{u_k, v_k \neq 0\}$
$\cong \underline{\mathbb{C}^* \times \mathbb{C}^*}$

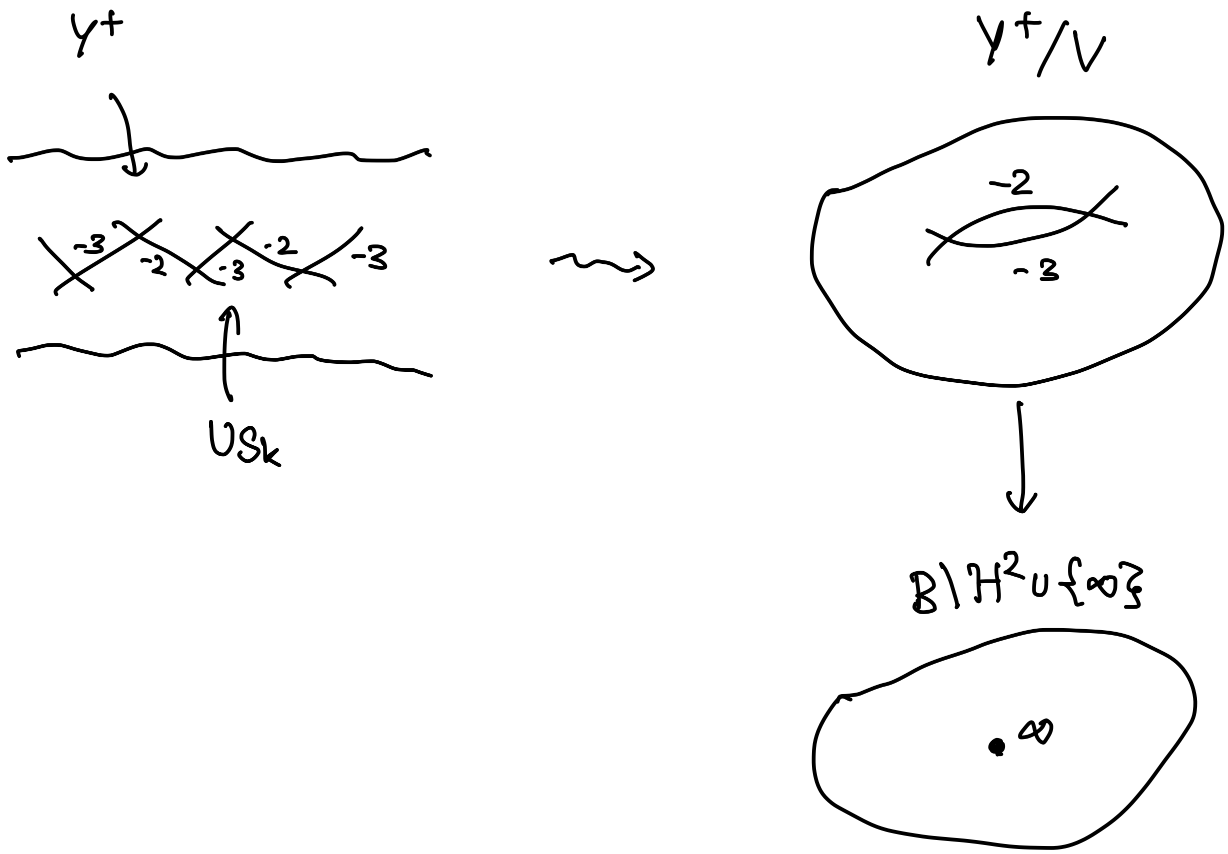
$\rightsquigarrow$ Glue: $\bigcup_k (\mathbb{C}^2)_{\sigma_k} = Y$

$$\Rightarrow \text{get} \quad \begin{array}{ccc} M\backslash\mathbb{C}^2 & \hookrightarrow & Y \\ \cup & & \cup \\ M\backslash\mathcal{H}^2 & \hookrightarrow & Y^+ = M\backslash\mathcal{H}^2 \cup \underbrace{\bigcup_{k\in\mathbb{Z}} S_k}_{\text{boundary}} \end{array}$$

$S_k = \{u_k = 0\}$

Fact. $V = \mathcal{O}_F^{\times} \curvearrowright Y^+$ freely & properly discontinuously

$\rightsquigarrow \quad V\backslash Y^+ \cong B\backslash\mathcal{H}^2 \cup \bigcup_{\text{finite}} S_k$ smooth model of nbhd of cusp



fiber above cusp $\cong$ circle of $\mathbb{P}^1$'s

What are $\mathbb{C}^* \times \mathbb{C}^* \hookrightarrow (\mathbb{C}^2)_{\sigma_k}$ algebraically?

$N = M^\vee$ = character group of $M = \mathcal{O}_F$

$$\begin{array}{ccc} \rightsquigarrow S = \mathrm{Spec}\left(\mathbb{Z}[X^n \mid n \in N] \right. & \Big/ \begin{matrix} X^{n_1}X^{n_2} = X^{n_1+n_2} \\ X^0 = 1 \end{matrix} & \left. \right) \cong \mathbb{C}^* \times \mathbb{C}^* \\ \downarrow & \downarrow & \downarrow \\ S_\sigma = \mathrm{Spec}\left(\mathbb{Z}[X^n \mid n \text{ s.t. } n(m) \geq 0 \ \forall m \in \sigma \right) & & \cong (\mathbb{C}^2)_\sigma \end{array}$$

Note: $S_{\sigma_k} \cap S_{\sigma_{k+1}} = S_{\overline{\sigma_k} \cap \overline{\sigma_{k+1}}}$ etc

$\rightsquigarrow$ can glue $\bigcup_{\sigma \in \Sigma} S_\sigma$, Σ = "polyhedral cone decomp." of $(M \otimes \mathbb{R})_+$

In general: a toroidal decomp. depends on a choice of "polyhedral cone decomp." of something...

③ Quick example: Siegel threefold.

$\Gamma = Sp(2g, \mathbb{Z})$

$\rightsquigarrow S_g = \{ Y \in \mathrm{Sym}_g(\mathbb{R}) \} \supseteq S_g^+ = \{ Y > 0 \}$

so that $H_g = S_g + i \cdot S_g^+$

$\rightsquigarrow$ "polyhedral cone decomp." of $\overline{S_g^+}$

$\underline{g=2}$

$$\sigma_0 = \{0\} \quad \leftrightsquigarrow \quad X_0 \cong \Gamma \backslash \mathcal{H}_2$$

$$\cap$$

$$\sigma_1 = \left\{ \begin{pmatrix} & \\ & \lambda \end{pmatrix} : \lambda \geq 0 \right\} \quad \leftrightsquigarrow \quad X_1 \cong \Gamma \backslash \mathcal{H}_1 \times \mathbb{P}^1$$

$$\cap$$

$$\sigma_2 = \left\{ \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} : \lambda_1, \lambda_2 \geq 0 \right\} \quad \leftrightsquigarrow \quad X_2 \cong \mathbb{C}$$

$$\cap$$

$$\sigma_3 = \left\{ \begin{pmatrix} \lambda_1 + \lambda_3 & -\lambda_3 \\ -\lambda_3 & \lambda_2 + \lambda_3 \end{pmatrix} : \lambda_1, \lambda_2, \lambda_3 \geq 0 \right\} \leftrightsquigarrow X_3 = \{*\}$$

$(\rightsquigarrow$ look at $\Sigma = GL(2, \mathbb{Z}) \cdot \{\sigma_0, \sigma_1, \sigma_2, \sigma_3\})$